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BAYESIAN PREDICTION FOR THE EXPONENTIATED GAMMA DISTRIBUTION BASED ON UNIFIED HYBRID CENSORED SCHEME

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Abstract

In this article, we will study the exponentiated Gamma distribution and the maximum likelihood function of it and we study one- and two-sample Bayesian prediction intervals based on unified hybrid censored data. Because one- and two-sample Bayesian predictive survival function can not be computed in closed-form so we will used the Markov Chain Monte Carlo method to obtain the approximate predictive survival function. Lastly, we used a real data set to find Bayesian prediction intervals.

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1. Introduction

Gamma distribution is the most popular model for analyzing skewed data. Gupta et al. [1] introduced the exponentiated gamma (EG) distribution. For details, see Bakoban [2] and Shawky and Bakoban [3]. The EG distribution has the cumulative distribution function (cdf):

$$F(x, \theta, \gamma) = (1 - (1 + \gamma x)e^{-\gamma x})^{\theta}, \tag{1.1}$$

the probability density function (pdf):

$$f(x, \theta, \gamma) = \theta\gamma^2 xe^{-\gamma x}[1 - (1 + \gamma x)e^{-\gamma x}]^{\theta-1}. \tag{1.2}$$

Epstein [5] considered a hybrid censored scheme (HCS), which is a combination of Type-I and Type-II censoring schemes. This schemes have been used in many practice and because of some disadvantages for this scheme. Chandrasekar et al. [7] suggested two new scheme, named as generalized Type-I HCS and generalized Type-II HCS. In generalized Type-I HCS, one fixes $k, r \in (1, 2, \ldots, n)$ and time $T \in (0, \infty)$ such that $k < r < n$. If the k-th failure occurs before time T, the experiment is terminated at $\min(x_{r:n}, T)$. If the k-th failure occurs after time T, the experiment is terminated at $x_{k:n}$. So, it is clear that this HCS modifies the Type-I HCS by allowing the experiment to continue after time T if very few failures had observed until that time. In generalized Type-II HCS, one fixes $r \in (1, 2, \ldots, n)$ and $T_1, T_2 \in (0, \infty)$ s.t $T_1 < T_2$. If the r-th failure occurs before time T_1, the experiment is terminated at T_1; if the r-th failure occurs between T_1 and T_2, the experiment is terminated at $X_{r:n}$; otherwise, the experiment is terminated at T_2. There are some problems for this scheme. For example, in the generalized Type-I HCS, because of only one pre-assigned time T, we cannot warranty r failures. In generalized Type-II HCS, there is a potentiality of not noting any failure at all or note very few failures until the pre-fixed time T_2, and so it has the same problem as the Type-I HCS. Balakrishnan et al. [8] introduced a mixture of generalized Type-I HCS and generalized Type-II HCS which is called the *unified hybrid censoring scheme* (UHCS), which can be described as follows, fix integers $r, k \in 1, 2, \ldots, n$ such that $k < r < n$ and time points $T_1, T_2 \in (0, \infty)$, where $T_1 < T_2$. If the k-th failure occurs before time T_1, the experiment is terminated at $\min(\max(x_{r:n}, T_1), T_2)$. If the k-th failure occurs between T_1 and T_2, the experiment is terminated at min $(x_{r:n}, T_2)$ and if the k-th failure occurs after time T_2, the experiment is terminated at $x_{k:n}$. Under this censoring scheme, we can warranty that the experiment would be ended at most in time T_2 with at least k failures and if not, we can warranty exactly k failures. Therefore, under this UHCS, we have the following six cases:

Case I: $0 < x_{k:n} < x_{r:n} < T_1 < T_2$ in which case we terminate at T_1,

Case II: $0 < x_{k:n} < T_1 < x_{r:n} < T_2$ in which case we terminate at $x_{r:n}$,

Case III: $0 < x_{k:n} < T_1 < T_2 < x_{r:n}$ in which case we terminate at T_2,

Case IV: $0 < T_1 < x_{k:n} < x_{r:n} < T_2$ in which case we terminate at $x_{r:n}$,

Case V: $0 < T_1 < x_{k:n} < T_2 < x_{r:n}$ in which case we terminate at T_2,

Case VI: $0 < T_1 < T_2 < x_{k:n} < x_{r:n}$ in which case we terminate at $x_{k:n}$.

Let $\underline{X} = X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ be an UHCS observed from a life test, including n units taken from a population with $F(x)$ and $f(x)$ given in Equations (1.1) and (1.2), then the likelihood function of θ, γ for six cases is given by

$$L(\underline{x}; \theta; \gamma) = \frac{n!}{(n-D)!}\left[\prod_{i=1}^{D} f(x_i)\right][1 - F(c)]^{n-D}, \tag{1.3}$$

where D is the failures number up to time c.

$$(D,\, c) = \begin{cases} (r_1,\, T_1) & \text{for Case I,} \\ (r,\, x_{r:n}) & \text{for Case II and Case IV,} \\ (r_2,\, T_2) & \text{for Case III and Case V,} \\ (k,\, x_{k:n}) & \text{for Case VI,} \end{cases} \tag{1.4}$$

where r_1 and r_2 indicate the failures number that happen before time points T_1 and T_2, respectively. In the Bayesian approach, we assume that the parameters θ and γ are independently distribution as gamma a_1, b_1 and gamma a_2, b_2 priors, respectively. Then, the prior of θ and γ becomes

$$\pi_1(\theta) \propto (\theta^{a_1-1} e^{-b_1\theta}), \qquad \theta > 0,$$

$$\pi_2(\gamma) \propto (\gamma^{a_2-1} e^{-\gamma b_2}), \qquad \gamma > 0,$$

where a_1, a_2, b_1 and $b_2 > 0$. The joint prior distribution for θ and γ is

$$\pi(\theta,\, \gamma) \propto \theta^{a_1-1} \gamma^{a_2-1} e^{-(b_1\theta + b_2\gamma)}. \tag{1.5}$$

From (1.3) and (1.5), we obtain the joint posterior density function as follows:

$$\pi^*(\theta,\, \gamma|\underline{x}) \propto \theta^{D+a_1-1} \gamma^{a_2+2D-1} e^{-\gamma(b_2 + \sum_{i=1}^{D} x_i)}$$

$$e^{-\theta b_1 + (\theta-1)\sum_{i=1}^{D} \ln[1-(1+\gamma x_i)e^{-\gamma x_i}]} [1 - (1 - (1 + \gamma c)e^{-\gamma c})^{\theta}]^{n-D}. \tag{1.6}$$

2. Bayesian Prediction Intervals

Prediction is very important in statistical inference like statistical estimation and can be categorized into two types. The first type, the one-sample prediction and in this case the variable to be predicted comes from the same system of variables experientied and is therefore connected with the experientied data. In the second type, referred to as the two-sample prediction problem, the variable to be predicted comes from another independent future sample, which can be described them as follows:

2.1. One-sample Bayesian prediction

In this section, we progress a general technique for deriving the interval predictions for the l-th future order statistic $X_{l:n}$ for EG distribution, where $D < l \leq n.$ For more details about Bayesian prediction, see, for example, Shafay [10, 11]. The conditional density function of $X_{l:n}$ based on UHCS $\underline{X} = X_{1:n} < X_{2:n} < \ldots < X_{D:n}$, is as follows:

$$g(x_l|\underline{X}) = \begin{cases} g_1(x_l|\underline{X}) & \text{if } (D,\ c) = (r_1,\ T_1) \text{ for Case I,} \\ g_2(x_l|\underline{X}) & \text{if } (D,\ c) = (r,\ X_{r:n}) \text{ for Case II and Case IV,} \\ g_3(x_l|\underline{X}) & \text{if } (D,\ c) = (r_2,\ T_2) \text{ for Case III and Case V,} \\ g_4(x_l|\underline{X}) & \text{if } (D,\ c) = (k,\ X_{k:n}) \text{ for Case VI.} \end{cases} \tag{2.1}$$

From these equations and (1.2), (1.1) and as seen in Mohie El-Din et al. [12] and Ghazal and Hasaballah [15], we can get the conditional density function of $X_{l:n}$, in the six cases, as follows, for $x_l > T_1$,

$$g_1(x_l|\underline{X}) = \sum_{d=r}^{l-1} \sum_{v=0}^{l-d-1} \sum_{a=0}^{n-l} k_1 \theta \gamma^2 x_l e^{-\gamma x_l} [1 - (1 + \gamma x_l) e^{-\gamma x_l}]^{\theta(l-d-v+a)-1}$$

$$\times [1 - (1 + \gamma T_1) e^{-\gamma T_1}]^{\theta(v+d)} \varphi_j(T_1), \tag{2.2}$$

$$\varphi_j(T_1) = \frac{1}{\sum_{j=r}^{l-1} \binom{n}{j} [1 - (1 + \gamma T_1) e^{-\gamma T_1}]^{\theta d} [1 - (1 - (1 + \gamma T_1) e^{-\gamma T_1})^{\theta}]^{(n-j)}};$$

For $x_l > x_r$,

$$g_2(x_l|x_r) = \sum_{v=0}^{l-r-1}\sum_{a=0}^{n-l} k_2\theta\gamma^2 x_l e^{-\gamma x_l}$$

$$\times \frac{[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-r-v+a)-1}[1-(1+\gamma x_r)e^{-\gamma x_r}]^{\theta v}}{[1-(1-(1+\gamma x_r)e^{-\gamma x_r})^{\theta}]^{n-r}}; \tag{2.3}$$

For $x_l > T_2$,

$$g_3(x_l|\underline{X}) = \sum_{d=r}^{l-1}\sum_{v=0}^{l-d-1}\sum_{a=0}^{n-l} k_3\theta\gamma^2 x_l e^{-\gamma x_l}[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-d-v+a)-1}$$

$$\times [1-(1+\gamma T_2)e^{-\gamma T_2}]^{\theta(v+d)}\varphi_j(T_2), \tag{2.4}$$

$$\varphi_j(T_2) = \frac{1}{\sum_{j=r}^{l-1}\binom{n}{j}[1-(1+\gamma T_2)e^{-\gamma T_2}]^{\theta d}[1-(1-(1+\gamma T_2)e^{-\gamma T_2})^{\theta}]^{(n-j)}};$$

and for $x_l > x_k$,

$$g_4(x_l|x_k) = \sum_{v=0}^{l-k-1}\sum_{a=0}^{n-l} k_4\theta\gamma^2 x_l e^{-\gamma x_l}$$

$$\times \frac{[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-k-v+a)-1}[1-(1+\gamma x_k)e^{-\gamma x_k}]^{\theta v}}{[1-(1-(1+\gamma x_k)e^{-\gamma x_k})^{\theta}]^{n-k}}. \tag{2.5}$$

From (1.6) and (2.1), we obtain the Bayesian predictive density function of $X_{l:n}$, given the UHCS as follows:

$$g^*(x_l|\underline{X}) = \begin{cases} g_1^*(x_l|\underline{X}) & \text{if } (D, c) = (r_1, T_1) \text{ for Case I,} \\ g_2^*(x_l|\underline{X}) & \text{if } (D, c) = (r, X_{r:n}) \text{ for Case II and Case IV,} \\ g_3^*(x_l|\underline{X}) & \text{if } (D, c) = (r_2, T_2) \text{ for Case III and Case V,} \\ g_4^*(x_l|\underline{X}) & \text{if } (D, c) = (k, X_{k:n}) \text{ for Case VI,} \end{cases} \tag{2.6}$$

$$g^*(x_l|\underline{X}) = \int_0^\infty \int_0^\infty g(x_l|\underline{X})\pi^*(\theta,\ \gamma|\underline{x})d\theta d\gamma, \tag{2.7}$$

where, for $x_l > T_1$,

$$g_1^*(x_l|\underline{X}) = \sum_{d=r}^{l-1}\sum_{v=0}^{l-d-1}\sum_{a=0}^{n-l} k_1\theta\gamma^2 x_l e^{-\gamma x_l}[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-d-v+a)-1}$$

$$\times[1-(1+\gamma T_1)e^{-\gamma T_1}]^{\theta(v+d)}\varphi_j(T_1)\pi^*(\theta,\ \gamma|\underline{x})d\theta d\gamma, \tag{2.8}$$

where $\underline{X} = (x_1, \ldots, x_{r_1})$. For $x_l > x_r$,

$$g_2^*(x_l|\underline{X}) = \sum_{v=0}^{l-r-1}\sum_{a=0}^{n-l} k_2\theta\gamma^2 x_l e^{-\gamma x_l}\pi^*(\theta,\ \gamma|\underline{x})$$

$$\times\frac{[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-r-v+a)-1}[1-(1+\gamma x_r)e^{-\gamma x_r}]^{\theta v}}{[1-(1-(1+\gamma x_r)e^{-\gamma x_r})^{\theta}]^{n-r}}d\theta d\gamma, \tag{2.9}$$

where $\underline{X} = (x_1, \ldots, x_r)$. For $x_l > T_2$,

$$g_3^*(x_l|\underline{X}) = \sum_{d=r}^{l-1}\sum_{v=0}^{l-d-1}\sum_{a=0}^{n-l} k_3\theta\gamma^2 x_l e^{-\gamma x_l}[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-d-v+a)-1}$$

$$\times[1-(1+\gamma T_2)e^{-\gamma T_2}]^{\theta(v+d)}\varphi_j(T_2)\pi^*(\theta,\ \gamma|\underline{x})d\theta d\gamma, \tag{2.10}$$

with $\underline{X} = (x_1, \ldots, x_{r_1})$, and for $x_l > x_k$,

$$g_4^*(x_l|\underline{X}) = \sum_{v=0}^{l-k-1}\sum_{a=0}^{n-l} k_2\theta\gamma^2 x_l e^{-\gamma x_l}\pi^*(\theta,\ \gamma|\underline{x})$$

$$\times\frac{[1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-k-v+a)-1}[1-(1+\gamma x_k)e^{-\gamma x_k}]^{\theta v}}{[1-(1-(1+\gamma x_k)e^{-\gamma x_k})^{\theta}]^{n-k}}d\theta d\gamma, \tag{2.11}$$

with $\underline{X} = (x_1, \ldots, x_k)$. From (2.6), for the l-th future UHCS we obtain the predictive survival function $p(x_l > t|x) = \overline{G}^*(t|x)$, for $t \geq 0$, as follows:

$$\overline{G}^*(t|\underline{X}) = \begin{cases} \overline{G}_1^*(t|\underline{X}) & \text{if } (D, c) = (r_1, T_1) \text{ for Case I,} \\ \overline{G}_2^*(t|\underline{X}) & \text{if } (D, c) = (r, X_{r:n}) \text{ for Case II and Case IV,} \\ \overline{G}_3^*(t|\underline{X}) & \text{if } (D, c) = (r_2, T_2) \text{ for Case III and Case V,} \\ \overline{G}_4^*(t|\underline{X}) & \text{if } (D, c) = (k, X_{k:n}) \text{ for Case VI,} \end{cases} \tag{2.12}$$

where

$$\overline{G}^*(x_l|\underline{X}) = \int_0^\infty \int_0^\infty u(x_l|\underline{X}) \pi^*(\theta, \gamma|\underline{x}) d\theta d\gamma, \qquad i = 1, 2, 3, 4, \tag{2.13}$$

and

$$u_i(t|\underline{X}) = \int_0^t g_i(x_l|\underline{X}) dx_l, \qquad i = 1, 2, 3, 4, \text{ for } \quad t \geq 0,$$

is given by

$$u(t|\underline{X}) = \begin{cases} u_1(t|\underline{X}) & \text{if } (D, c) = (r_1, T_1) \text{ for Case I,} \\ u_2(t|\underline{X}) & \text{if } (D, c) = (r, X_{r:n}) \text{ for Case II and Case IV,} \\ u_3(t|\underline{X}) & \text{if } (D, c) = (r_2, T_2) \text{ for Case III and Case V,} \\ u_4(t|\underline{X}) & \text{if } (D, c) = (k, X_{k:n}) \text{ for Case VI,} \end{cases} \tag{2.14}$$

$$u_1(t|\underline{X}) = \sum_{d=r}^{l-1} \sum_{v=0}^{l-d-1} \sum_{a=0}^{n-l} \times \frac{k_1 [1 - (1 - (1 + \gamma t) e^{-\gamma t})^{\theta(l-d-v+a)}][1 - (1 + \gamma T_1) e^{-\gamma T_1}]^{\theta(v+d)} \varphi_j(T_1)}{(l - d - v + a)}, \tag{2.15}$$

$$u_2(x_l|x_r) = \sum_{v=0}^{l-r-1}\sum_{a=0}^{n-l}$$

$$\times \frac{k_2[1-(1-(1+\gamma x_l)e^{-\gamma x_l}]^{\theta(l-r-v+a)}][1-(1+\gamma x_r)e^{-\gamma x_r}]^{\theta v}}{[1-(1-(1+\gamma x_r)e^{-\gamma x_r})^{\theta}]^{n-r}(l-r-v+a)},$$

$$(2.16)$$

$$u_3(t|\underline{X}) = \sum_{d=r}^{l-1}\sum_{v=0}^{l-d-1}\sum_{a=0}^{n-l}$$

$$\times \frac{k_3[1-(1-(1+\gamma t)e^{-\gamma t})^{\theta(l-d-v+a)}][1-(1+\gamma T_2)e^{-\gamma T_2}]^{\theta(v+d)}\varphi_j(T_2)}{(l-d-v+a)},$$

$$(2.17)$$

and

$$u_4(x_l|x_k) = \sum_{v=0}^{l-k-1}\sum_{a=0}^{n-l}$$

$$\times \frac{k_2[1-(1-(1+\gamma x_l)e^{-\gamma x_l})]^{\theta(l-k-v+a)}[1-(1+\gamma x_k)e^{-\gamma x_k}]^{\theta v}}{[1-(1-(1+\gamma x_k)e^{-\gamma x_k})^{\theta}]^{n-k}(l-k-v+a)}.$$

$$(2.18)$$

Then, the Bayesian predictive interval can be obtained by solving the following two equations:

$$\overline{G}^*(L_{X_{l:n}}|\underline{X}) = 1-\frac{\tau}{2} \quad \text{and} \quad \overline{G}^*(U_{X_{l:n}}|\underline{X}) = \frac{\tau}{2}, \tag{2.19}$$

where $\overline{G}^*(L_{X_{l:n}}|\underline{X})$ is given as in (2.12), and $L_{X_{l:n}}$ and $U_{X_{l:n}}$ indicate the lower and upper bounds, respectively.

2.2. Two-sample Bayesian prediction

Let $Z_{1:m} \leq Z_{2:m} \leq \ldots \leq Z_{m:m}$ be the order statistics from a new random sample of size m from the same population, see, Ghazal and Hasaballah [15] and Shafay and Balakrishnan [9, 13]. The marginal density function of the l-th order statistic from a sample of size m from a continuous distribution with cdf $F(x)$ and pdf $f(x)$ is given by

$$g_{Z_{l:m}}(Z_l|\theta) = \frac{m!}{(l-1)!\,(m-l)!}[F_{Z_l}]^{l-1}[1-F_{Z_l}]^{m-l}f(Z_l)$$

$$= \sum_{a=0}^{m-l}\frac{(-1)^a\binom{m-l}{a}m!}{(l-1)!\,(m-l)!}[F_{Z_l}]^{l+a-1}f(Z_l), \tag{2.20}$$

where $Z_l > 0$ and $1 \leq l \leq m.$

Upon substituting (1.1) and (1.2) in (2.20), the marginal density function of $Z_{l:m}$, becomes

$$g_{Z_{l:m}}(Z_l|\theta, \gamma) = \sum_{a=0}^{m-l}\frac{(-1)^a\binom{m-l}{a}m!}{(l-1)!\,(m-l)!}$$

$$\times[1-(1+\gamma Z_l)e^{-\gamma Z_l}]^{\theta(l+q-1)}\theta\gamma^2 Z_l e^{-\gamma Z_l}[1-(1+\gamma Z_l)e^{-\gamma Z_l}]^{\theta-1}, \tag{2.21}$$

$$g_{Z_{l:m}}(Z_l|\theta, \gamma) = \sum_{a=0}^{m-l}\frac{(-1)^a\binom{m-l}{a}m!}{(l-1)!\,(m-l)!}\theta\gamma^2 Z_l e^{-\gamma Z_l}[1-(1+\gamma Z_l)e^{-\gamma Z_l}]^{\theta(l+a)-1}.$$

The Bayesian predictive density function is

$$g^*(Z_l|\underline{X}) = \int_0^\infty\int_0^\infty g(Z_l|\underline{X})\pi^*(\theta, \gamma|\underline{X})d\theta d\gamma,$$

$$g^{*}_{Z_{l:m}}(Z_l|\theta,\gamma)=\sum_{a=0}^{m-l}\frac{(-1)^a\binom{m-l}{a}m!}{(l-1)!\,(m-l)!}$$

$$\times\int_0^\infty\int_0^\infty \theta\gamma^2 Z_l e^{-\gamma Z_l}[1-(1+\gamma Z_l)e^{-\gamma Z_l}]^{\theta(l+a)-1}\times\pi^*(\theta,\gamma|\underline{X})d\theta d\gamma. \quad (2.22)$$

The predictive survival function $\overline{G}^{*}_{Z_{l:m}}(t|\underline{X})$, for $t\geq 0$,

$$\overline{G}^*(t|\underline{X})=\int_0^\infty\int_0^\infty u(Z_l|\underline{X})\pi^*(\theta,\gamma|\underline{X})d\theta d\gamma, \quad (2.23)$$

where

$$u(Z_l|\underline{X})=\int_t^\infty g(Z_l|\underline{X})dZ_l,$$

$$u(Z_l|\underline{X})=\sum_{a=0}^{m-l}\frac{(-1)^a\binom{m-l}{a}m!}{(l-1)!\,(m-l)!\,(l+a)!}[1-(1-(1+\gamma Z_l)e^{-\gamma Z_l}]^{\theta(l+a)}]. \quad (2.24)$$

Then, the Bayesian predictive bounds of a two-sided equi-tailed $(1-\tau)100\%$ for $Z_{l:m}$, $1\leq l\leq m$, can be obtained by solving the following two equations:

$$\overline{G}^*(L_{Z_{l:n}}|\underline{X})=1-\frac{\tau}{2}\quad\text{and}\quad\overline{G}^*(U_{Z_{l:n}}|\underline{X})=\frac{\tau}{2}, \quad (2.25)$$

where $\overline{G}^*(L_{Z_{l:n}}|\underline{X})$ is given as in (2.23), and $L_{Z_{l:n}}$ and $U_{Z_{l:n}}$ indicate the lower and upper. bounds, respectively. It is evident that is not possible to compute (2.12) and (2.23) analytically. Then, we use MCMC method to calculate the Bayesian prediction intervals.

3. MCMC Method

We consider the MCMC to find (θ,γ) from the posterior density function (1.6). The Metropolis-Hastings-within-Gibbs sampling is given as follow:

Algorithm

(1) Begin with initial $(\theta^{(0)}, \gamma^{(0)})$, $M =$ burn-in.

(2) Set $l = 1$.

(3) Generate $\gamma^{(l)}$ from Gamma $(D + a_2, b_2 + \sum_{i=1}^{D} x_i)$.

(4) Using Metropolis-Hastings, see Metropolis et al. [14], generate $\theta^{(l)}$ from $\pi_2^*(\theta|\gamma, x)$ with the $N(\theta^{(l-1)}, \sigma)$ proposal distribution, where σ^2 is the variance of γ obtained using variance-covariance matrix.

(5) Calculate θ^l and γ^l.

(6) Set $l = l + 1$.

(7) Repeat steps (3) – (6) N times.

(8) In case of one-sample Bayesian prediction, the approximate value of

$$\int_0^\infty \int_0^\infty u(x_l|\underline{X})\pi^*(\theta, \gamma|\underline{X})d\theta d\gamma = \frac{1}{N - M} \sum_{l=M+1}^{N} u(x_l|\underline{X}),$$

where

$$u(t|\underline{X}) = \begin{cases} u_1(t|\underline{X}) & \text{if } (D, c) = (r_1, T_1) \text{ for Case I,} \\ u_2(t|\underline{X}) & \text{if } (D, c) = (r, X_{r:n}) \text{ for Case II and Case IV,} \\ u_3(t|\underline{X}) & \text{if } (D, c) = (r_2, T_2) \text{ for Case III and Case V,} \\ u_4(t|\underline{X}) & \text{if } (D, c) = (k, X_{k:n}) \text{ for Case VI.} \end{cases}$$

(9) In case of two-sample Bayesian prediction, the approximate value of

$$\int_0^\infty \int_0^\infty u(Z_l|\underline{X})\pi^*(\theta, \gamma|\underline{X})d\theta d\gamma = \frac{1}{N - M} \sum_{l=M+1}^{N} u(Z_l|\underline{X}),$$

where

$$u(Z_l|\underline{X}) = \sum_{a=0}^{m-l} \frac{(-1)^a \binom{m-l}{a} m!}{(l-1)!\,(m-l)!} [1 - (1 - (1 + \gamma Z_l)e^{-\gamma Z_l})^{\theta(s+a)}].$$

4. Real Life Data

A real data set is taken from Singh et al. [4], these data represent the average monthly rainfall obtained from the Information System for Management of Water Resources from the State of São Paulo, including a period of 56 years from 1947 to 2003, for the month of November, which is represented as;

Table 4.1. Real data set

0.2	0.8	1.1	1.3	1.4	1.7	1.8	1.9	2.1	2.1	2.2	2.5	2.6	2.8
2.8	2.9	2.9	2.9	2.9	3.1	3.2	3.3	3.5	3.5	3.5	3.7	3.8	3.8
3.9	4	4.1	4.1	4.6	4.7	4.8	5	5.2	5.2	5.4	5.4	5.4	5.4
5.5	5.5	6.2	6.2	6.7	6.9	7.3	7.3	7.4	8.7	8.8	9.9	10.8	24.1

We plot the empirical SF and the fitted SF in the Figure 1. They show that EG distribution (θ, γ) provide a reasonable fit to the above real data.

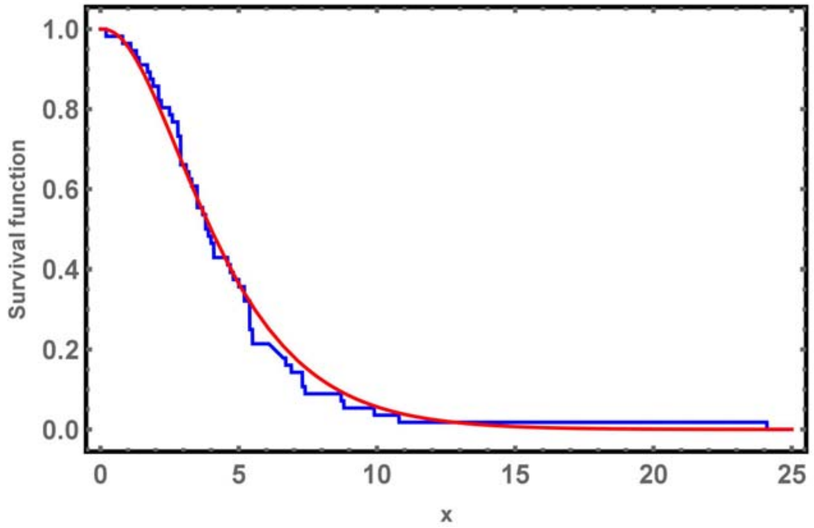

Figure 1. The empirical and fitted survival functions.

Now, we consider the case when the data are censored. We generate six artificially UHCS sets from the uncensored data set as

Case I: $T_1 = 8$, $T_2 = 10$, $k = 40$, $r = 43$. In this case: $D = 44$, $c = T_1 = 8$.

Case II: $T_1 = 5.5$, $T_2 = 6.9$, $k = 40$, $r = 45$. In this case: $D = 45$, $c = x_{D:n} = 6.2$.

Case III: $T_1 = 5.8$, $T_2 = 6$, $k = 41$, $r = 46$. In this case: $D = 47$, $c = T_2 = 6$.

Case IV: $T_1 = 5$, $T_2 = 7.3$, $k = 45$, $r = 46$. In this case: $D = 47$, $c = x_{r:n} = 6.9$.

Case V: $T_1 = 5$, $T_2 = 6.4$, $k = 40$, $r = 50$. In this case: $D = 49$, $c = T_2 = 6.4$.

Case VI: $T_1 = 5.5$, $T_2 = 6.5$, $k = 48$, $r = 50$. In this case: $D = 49$ $c = x_{k:n} = 7.3$.

Based on the above six UHCS, we used the outcomes in Section 2 to build 95% one-sample Bayesian prediction intervals for new order statistics $X_{l:n}$, from the same sample in addition to 95% two-sample Bayesian prediction intervals for new order statistics $Z_{l:m}$, where $l = 1, 2, \ldots, 20$, from a new unobserved sample with size $m = 20$ when the hyper parameters $(a_1 = a_2 = b_1 = b_2 = 0)$. Tables 4.2, 4.3, 4.4, 4.5, 4.6, and, 4.7 presents the outcomes for one-sample predictions, and Tables 4.8, 4.9, 4.10, 4.11, 4.12, and 4.13 presents the outcomes for two-sample predictions, where $\theta = 0.7$ and $\gamma = 0.3$ in all cases.

Table 4.2. 95% One-sample Bayesian prediction bounds for $X_{l:n}$, $l = 45, \ldots, 56$ in Case I of UHCS

l	Lower	Upper	Width
45	8.0096	9.4493	1.4397
46	8.01755	9.9699	1.9523
47	8.0436	10.9459	2.9023
48	8.05711	11.564	3.506
49	8.1188	12.5663	4.4474
50	8.1774	13.7276	5.5498
51	8.4039	14.9645	6.5605
52	8.9332	16.4919	7.5586
53	9.2109	18.4038	9.1928
54	10.1486	20.8739	9.374
55	10.7904	24.9808	14.1904
56	12.4103	34.1567	21.7464

Table 4.3. 95% One-sample Bayesian prediction bounds for $X_{l:n}$, $l = 46, \ldots, 56$ in Case II of UHCS

l	Lower	Upper	Width
46	6.21129	7.8944	1.6831
47	6.7662	9.007	2.2416
48	7.4175	10.1238	2.2706
49	8.1566	11.3184	3.1618
50	8.9954	12.633	3.6376
51	9.8901	14.1904	4.2102
52	11.1838	16.0934	4.9095
53	12.6923	18.5401	5.8478
54	14.7015	21.9042	7.2072
55	17.813	27.444	9.6125
56	23.7789	40.5038	16.7746

Table 4.4. 95% One-sample Bayesian prediction bounds for $X_{l:n}$, $l = 48, \ldots, 56$ in Case III of UHCS

l	Lower	Upper	Width
48	6.0143	8.1430	2.1286
49	6.0596	9.2976	3.2379
50	6.1888	10.5849	4.3960
51	6.5431	12.052	5.9769
52	6.8708	13.7448	6.8739
53	7.5031	15.830	8.3269
54	8.1421	18.69	10.5479
55	9.1197	23.0354	13.9157
56	10.7583	32.9413	22.183

Table 4.5. 95% One-sample Bayesian prediction bounds for $X_{l:n}$, $l = 48, \ldots, 56$ in Case IV of UHCS

l	Lower	Upper	Width
48	6.9140	9.0011	2.08713
49	7.6106	10.3886	2.77801
50	8.4580	11.873	3.4149
51	9.4562	13.5255	4.0692
52	10.6637	15.5173	4.8535
53	12.1664	18.0149	5.8485
54	14.2098	21.4914	7.2816
55	17.3795	27.241	9.8614
56	24.1388	41.2441	17.1053

Table 4.6. 95% One-sample Bayesian prediction bounds for $X_{l:n}$, $l = 50, \ldots, 56$ in Case V of UHCS

l	Lower	Upper	Width
50	6.4188	9.1946	2.7758
51	6.6861	10.8356	4.1495
52	6.70947	12.7011	5.99159
53	7.1523	14.9844	7.8320
54	7.8282	18.0368	10.2086
55	8.8112	22.7305	13.9193
56	10.4413	33.0457	22.6045

Table 4.7. 95% One-sample Bayesian prediction bounds for $X_{l:n}$, $l = 50, \ldots, 56$ in Case VI of UHCS for $c = 7.3$ and $D = 49$

l	Lower	Upper	Width
50	7.3185	10.055	2.7365
51	8.2662	12.0016	3.7353
52	9.4775	14.1723	4.6947
53	10.9957	16.8021	5.8064
54	13.0575	20.4581	7.4005
55	16.2347	26.3739	10.1392
56	22.9031	40.5755	17.6724

Table 4.8. 95% Two-sample Bayesian prediction bounds for $Z_{l:m}$, $l = 1, \ldots, 20$ in Case I of UHCS

l	Lower	Upper	Width	l	Lower	Upper	Width
1	0.03752	2.0239	1.9863	11	2.5928	8.7128	6.12
2	0.1895	2.7925	2.6030	12	2.9617	9.4865	6.5248
3	0.3775	3.4216	3.0441	13	3.3803	10.4782	7.0978
4	0.6070	4.0273	3.4202	14	3.7919	11.5397	7.7478
5	0.8477	4.6570	3.8093	15	4.3162	12.6944	8.3781
6	1.0934	5.2353	4.1418	16	4.9013	14.284	9.3826
7	1.3685	5.8839	4.5153	17	5.5926	16.2269	10.6343
8	1.6477	6.6093	4.8616	18	6.4257	18.8443	12.4167
9	1.9104	7.2005	5.2504	19	7.5732	22.9337	15.3604
10	2.2577	7.9422	5.6848	20	9.2723	32.1685	22.8962

Table 4.9. 95% Two-sample Bayesian prediction bounds for $Z_{l:m}$, $l = 1, \ldots, 20$ in Case II of UHCS

l	Lower	Upper	Width	l	Lower	Upper	Width
1	0.0321	1.9792	1.9471	11	2.5343	8.6221	6.0877
2	0.1825	2.7650	2.5824	12	2.9181	9.54	6.9218
3	0.3710	3.3903	3.0192	13	3.3172	10.4234	7.1061
4	0.5939	4.0490	3.45508	14	3.8025	11.5266	7.7239
5	0.7994	4.5993	3.7999	15	4.2545	12.7216	8.4670
6	1.0553	5.2357	4.1803	16	4.8470	14.2395	9.3924
7	1.3164	5.8453	4.5289	17	5.5276	16.1171	10.5895
8	1.6119	6.4922	4.8810	18	6.3891	18.8024	12.4132
9	1.9019	7.1398	5.2378	19	7.4985	22.7742	15.2757
10	2.2162	7.8694	5.6531	20	9.2481	32.2172	22.9691

Table 4.10. 95% Two-sample Bayesian prediction bounds for $Z_{l:m}$, $l = 1, \ldots, 20$ in Case III of UHCS

l	Lower	Upper	Width	l	Lower	Upper	Width
1	0.0295	1.9520	1.9224	11	2.5149	8.6325	6.1175
2	0.1634	2.7348	2.571	12	2.8553	9.4888	6.6335
3	0.3499	3.3809	3.0309	13	3.3009	10.5073	7.2063
4	0.5351	3.9702	3.4351	14	3.7148	11.5432	7.8283
5	0.7840	4.5988	3.8147	15	4.2502	12.7868	8.5365
6	1.0006	5.2235	4.2229	16	4.8171	14.2655	9.4484
7	1.2665	5.7974	4.9126	17	5.5529	16.3121	10.7592
8	1.5622	6.4749	4.9126	18	6.3668	18.9001	12.5332
9	1.8455	7.1284	5.2828	19	7.5073	23.0491	15.5418
10	2.1664	7.8437	5.6772	20	9.2535	32.4077	23.1542

Table 4.11. 95% Two-sample Bayesian prediction bounds for $Z_{l:m}$, $l = 1, \ldots, 20$ in Case IV of UHCS

l	Lower	Upper	Width	l	Lower	Upper	Width
1	0.0306	1.9729	1.9423	11	2.5687	8.7095	6.1407
2	0.1702	2.7444	2.5711	12	2.9231	9.6110	6.6878
3	0.3485	3.3918	3.0432	13	3.3553	10.5781	7.2227
4	0.5687	4.0187	3.4499	14	3.7995	11.7069	7.9073
5	0.8056	4.6423	3.8366	15	4.3211	12.969	8.6478
6	1.0246	5.2448	4.2202	16	4.9204	14.5788	9.6584
7	1.3117	5.8564	4.5447	17	5.5976	16.4748	10.8772
8	1.5734	6.5368	4.9634	18	6.4894	19.1861	12.6967
9	1.866	7.2298	5.3638	19	7.6147	23.3623	15.7475
10	2.2026	7.9627	5.7601	20	9.4983	33.0879	23.5816

Table 4.12. 95% Two-sample Bayesian prediction bounds for $Z_{l:m}$, $l = 1, \ldots, 20$ in Case V of UHCS

l	Lower	Upper	Width	l	Lower	Upper	Width
1	0.04517	1.9153	1.8908	11	2.5271	8.8947	6.3676
2	0.1515	2.7177	2.5661	12	2.9018	9.7598	6.8579
3	0.3219	3.3857	3.0637	13	3.3869	10.8214	7.4344
4	0.5227	3.9643	3.4416	14	3.7956	11.8855	8.0898
5	0.7536	4.6182	3.8646	15	4.3642	13.2361	8.8717
6	0.9990	5.2511	4.2520	16	4.9812	14.9223	9.94113
7	1.2586	5.9115	4.6529	17	5.6689	16.8134	11.1444
8	1.5695	6.5394	4.9698	18	6.5514	19.6224	13.0711
9	1.8383	7.2308	5.3924	19	7.7716	24.0399	16.2683
10	2.1729	8.0575	5.8845	20	9.6358	34.0353	24.3995

Table 4.13. 95% Two-sample Bayesian prediction bounds for $Z_{l:m}$, $l = 1, \ldots, 20$ in Case VI of UHCS

l	Lower	Upper	Width	l	Lower	Upper	Width
1	0.0245	1.9344	1.9098	11	2.5354	8.9041	6.3687
2	0.1484	2.6902	2.5416	12	2.9294	9.7825	6.8530
3	0.3380	3.4147	3.0767	13	3.3396	10.7514	7.4116
4	0.5253	3.9973	3.4719	14	3.8425	11.8935	8.0509
5	0.7570	4.6258	3.8688	15	4.4085	13.2562	8.8476
6	0.9860	5.2209	4.2349	16	5.0175	14.8725	9.8549
7	1.2686	5.2209	4.2349	17	5.6774	16.9404	11.263
8	1.5559	6.6067	5.0512	18	6.6121	19.7822	13.1701
9	1.8685	7.3072	5.4387	19	7.8278	23.9968	16.169
10	2.2132	8.0675	5.8543	20	9.6718	34.0038	24.332

5. Conclusion

From the above tables, in case one- and two-sample Bayesian prediction interval we notice that, when l increase the lower and upper increase.

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