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HOMOCLINIC ORBIT FOR A FOURTH-ORDER SYSTEM

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Abstract

In this paper, homoclinic orbit of a fourth-order system will be concerned, several assumptions are made to prove a fourth-order system has a homoclinic orbit q emanating from 0 where $q \in C^4(\mathbf{R}, \mathbf{R}^n)$. Under these assumptions, we can find the homoclinic orbit of the system by defining a function $I_k(q)$ via the Mountain Pass Theorem.

1. Introduction and Main Results

In this paper, we will deal with the existence of homoclinic orbit of the fourth-order system:

$$-q^{(4)} + Cq'' - L(t)q + V_q'(t, q) = 0, \tag{1.1}$$

where $q \in \mathbf{R}^n$, C is positive definite symmetric matrix, L and V satisfy:

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(L) $L(t)$ is a continuous T-periodic matrix valued function and is a positive definite symmetric for all $t \in [0, T]$.

(V_1) $V \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ is T-periodic in t.

(V_2) There is a $\mu > 2$ such that $0 < \mu V(t, q) \le (q, V_q'(t, q))$ for all $q \in \mathbf{R}^n \setminus \{0\}$.

(V_3) $V(t, q) = o(|q|^2)$ as $q \to 0$ uniformly for $t \in [0, T]$.

We say that a solution $q(t)$ of (1.1) is a homoclinic orbit if $q(t) \in C^4(\mathbf{R}, \mathbf{R}^n)$ such that $q(t) \to 0$ and $q'(t) \to 0$ as $|t| \to \infty$.

Theorem 1.1. *Assume* (L), (V_1)–(V_3) *hold. Then* (1.1) *has a nonzero homoclinic orbit.*

This paper is motivated by Rabinowitz [1], who has showed that under (L), (V_1)–(V_3), the system

$$q'' - L(t)q + V_q'(t, q) = 0 \tag{1.2}$$

has a nontrivial solution $q \in C^2(\mathbf{R}, \mathbf{R}^n)$ satisfying $|q(t)| + |q'(t)| \to 0$ as $|t| \to \infty$.

In Section 2, we will prove Theorem 1.1 using methods in [1].

2. Proof of Theorem 1.1

Similar to [1], for each $k \in N$, let $E_k = W_{2kT}^{2,2}(\mathbf{R}, \mathbf{R}^n)$ with the norm:

$$\|q\|_k^2 = \int_{-kT}^{kT} [\|q''(t)\|^2 + (Cq'(t), q'(t)) + (L(t)q(t), q(t))]dt,$$

and let

$$I_k(q) = \int_{-kT}^{kT} \left[\frac{1}{2} |q''(t)|^2 + \frac{1}{2} (Cq'(t),\ q'(t)) + \frac{1}{2} (L(t)q(t),\ q(t)) - V(t,\ q) \right] dt$$

$$= \frac{1}{2} \|q\|_k^2 - \int_{-kT}^{kT} V(t,\ q) dt. \tag{2.1}$$

It is easy to check $I_k \in C^1(E_k,\ \mathbf{R})$ and

$$I_k(q')u = \int_{-kT}^{kT} \left[(q'(t), u''(t)) + (Cq'(t), u'(t)) + (L(t)q(t), u(t)) - (V_q'(t, q(t), u(t))) \right] dt.$$

Lemma 2.1. *If* $I_k(q') = 0$, *then* $q \in C^4(\mathbf{R}, \mathbf{R}^n)$ *and* $q = q(t)$ *is a solution of* (1.1).

Proof. If q is a critical point of I_k, then

$$\int_{-kT}^{kT} \left[(q''(t),\ h''(t)) + (Cq'(t),\ h'(t)) + (L(t)q(t) - V_q'(t,\ q(t)),\ h(t)) \right] dt = 0,$$

for all $h \in E_k$. Letting $\delta(t) = L(t)q(t) - V_q'(t, q(t))$ means that $\int_{-kT}^{kT} \delta(t) dt = 0$ since $\mathbf{R}^n \subset E_k$. So there is $x \in E_k$ satisfying $x''(t) = \delta(t)$ and

$$\int_{-kT}^{kT} ((q''(t) - Cq(t) + x(t)),\ h(t)) dt = 0, \quad \forall h \in E_k.$$

Thus, there is a constant $\xi \in \mathbf{R}^n$ such that $q''(t) - Cq(t) + x(t) = \xi$ a.e. $x \in [-kT,\ kT]$. So

$$q^{(4)}(t) - Cq''(t) + L(t)q(t) - V_q'(t,\ q(t)) = 0,$$

since $\delta(t) = L(t)q(t) - V_q'(t,\ q(t))$ is continuous. The proof is complete. □

In order to prove Theorem 1.1 as in [1], we will use the Mountain Pass Lemma.

Proposition 2.1. *Let E be a real Banach space and $I \in C^1(E, \mathbf{R})$ satisfy (PS). (PS) says whenever $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$, the sequence u_m possesses a convergent subsequence.*

If further $I(0) = 0$,

(I_1) There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;

and

(I_2) There exists $e \in E \setminus \overline{B}_\rho(0)$ such that $I(e) \leq 0$, then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I_k(g(s)), \tag{2.2}$$

where

$$\Gamma = \{g \in C([0, 1], E) | g(0) = 0 \text{ and } g(1) = e\}, \tag{2.3}$$

$B_\rho(0)$ denotes an open ball of radius ρ about 0. □

We will obtain a critical point of I_k via the Mountain Pass Theorem. Moreover, the critical points of I_k in E_k are $2kT$ periodic solutions of (1.1). First we need to prove $I_k(q)$ satisfies (*PS*). Recall that we say I satisfied (*PS*) if that $I(u_m)$ is bounded and $I'(u_m) \to 0$ imply that $u_m \to u$ for some $u \in E$ along a subsequence.

Lemma 2.2. *$I_k(q)$ satisfies (PS).*

Proof. To verify (*PS*), define first

$$\|q\|_\beta = \left(\|q''\|_{L^2}^2 + C\|q'\|_{L^2}^2 + (L(t)q, q)_R + \beta\|q\|_{L^2}^2\right)^{\frac{1}{2}},$$

where $\beta > 0$. Thus $\|\cdot\|_\beta$ is equivalent to $\|\cdot\|_{E_k}$. Suppose q_m is a sequence in E_k, $|I(q_m)| \le K$ and $I'(q_m) \to 0$. For m large enough,

$$\begin{aligned} &|I'(q_m)q_m| \\ &= |\|q_m\|_\beta^2 - \beta\|q_m\|_{L^2}^2 - \int_{-kT}^{kT}(q_m, V_q'(t, q_m))d\tau| \\ &\le \|q_m\|_\beta. \end{aligned} \tag{2.4}$$

Therefore,

$$\left|\int_{-kT}^{kT}(q_m, V_q'(t, q_m))_R d\tau\right| \le \|q_m\|_\beta + \|q_m\|_\beta^2. \tag{2.5}$$

For $y \in R$ large enough, we have

$$|y|^2 \le \mu(y, V_q'(t, q_m)). \tag{2.6}$$

Hence, by (V_2) and (2.5)–(2.6) for a constant γ independent of m, we have

$$\begin{aligned} \int_{-kT}^{kT} V(t, q_m)d\tau &\le \frac{1}{\mu}\int_{-kT}^{kT}(q_m, V_q'(t, q_m))d\tau \\ &\le \frac{1}{\mu}\left(\|q_m\|_\beta + \|q_m\|_\beta^2\right), \\ K &\ge \frac{1}{2}\|q_m\|_\beta^2 - \frac{1}{2}\beta\|q_m\|_{L^2}^2 - \int_{-kT}^{kT} V(t, q_m)d\tau \\ &\ge \frac{1}{2}\|q_m\|_\beta^2 - \left(\frac{1}{2}\beta + \frac{1}{\mu}\right)\left(\|q_m\|_\beta + \|q_m\|_\beta^2\right) - \gamma. \end{aligned} \tag{2.7}$$

Choosing $\beta \in \left(0, 1 - \frac{2}{\mu}\right)$, (2.7) shows $\|q_m\|_{E_k}$ is uniform bound, and then there is a subsequence of q_m converges weakly in E_k and strongly in L^∞

to $q \in E_k$. The mean value of q_m converges to the mean value of $\overline{q}$. By (2.4), we have $I(q') = q - \phi(q)$, where $\phi(q)$ is compact, hence $I(q') \to 0$, then $q_m \to \phi(\overline{q})$, (PS) is verified. The proof is complete. □

Lemma 2.3. *If* (L) *and* (V_1)–(V_3) *hold, then* I_k *defined in* (2.2) *possesses a critical point* q_k *with the corresponding critical value* c_k.

Proof. By (2.2), we have $I_k(0) = 0$. By (V_2), let $f(s) = V(t, s\xi)$; for any fixed point $\xi \in \mathbf{R}^n$, we have

$$s f'(s) \geq \mu f(s) = \mu V(t, s\xi).$$

For $s \geq 1$, $s f'(s) \geq \mu f(s)$. Integrate it in $[1, s]$, then $f(s) \geq s^\mu f(1)$. That is $V(t, s\xi) \geq s^\mu V(t, \xi)$, for this, we can get

$$V(t, \xi) \leq V\left(t, \frac{\xi}{|\xi|}\right)|\xi|^\mu, \text{ for } 0 < |\xi| \leq 1,$$

$$\geq V\left(t, \frac{\xi}{|\xi|}\right)|\xi|^\mu, \quad \text{for } |\xi| \geq 1. \tag{2.8}$$

Then there are constants $\alpha_k, \varepsilon_k > 0$ such that

$$\alpha_k \|q\|_{L^\mu[-kT, kT]} \leq \|q\|_{L^x[-kT, kT]} \leq \varepsilon_k \|q\|_k. \tag{2.9}$$

Moreover by (2.1) and (2.8), we have

$$I_k(\beta\varphi) = \frac{1}{2}\|\beta\varphi\|_k^2 - \int_{-kT}^{kT} V(t, \beta\varphi)dt \leq \frac{1}{2}\beta^2\|\varphi\|_k^2 - \int_{-kT}^{kT} V\left(t, \frac{\beta\varphi}{|\beta\varphi|}\right)|\beta\varphi|^\mu dt.$$

Then there are constants $a_1, a_2 > 0$ such that

$$I_k(\beta\varphi) \leq \frac{1}{2}\beta^2\|\varphi\|_k^2 - a_1|\beta|^\mu \int_{-kT}^{kT}|\varphi|dt + a_2, \tag{2.10}$$

for all $\beta \in R$ and $\varphi \in E_k \setminus \{0\}$. By (2.1) and (V_3), $e = e_k$ and any $e_k \in E_k \setminus \{0\}$ shows $I_k(e_k) \leq 0$. Consequently by Proposition 2.1, I_k

possesses a positive critical value c_k given by (2.2) and (2.3) with $E = E_k$ and $\Gamma = \Gamma_k$. Let q_k denote the corresponding critical point of I on E_k. The proof is complete. □

Remark. Note that $q_k \neq 0$ since $c_k > 0$.

Lemma 2.4. *The c_k and q_k obtained in Lemma* 2.3 *are independent of k.*

Proof. As in [1], we have $\varphi \in E_1 \setminus \{0\}$ such that

(i) $$\varphi(\pm T) = \varphi'(\pm T) = 0,$$

(ii) $$I_1(\varphi) \leq 0.$$

By (2.10), we can find ϕ satisfies (i), any sufficiently large multiple φ of ϕ satisfies (i) and (ii), by Lemma 2.3, let

$$\begin{aligned} e_k(t) &= \varphi(t) \quad |t| \leq T \\ &= 0 \qquad T \leq |t| \leq kT. \end{aligned}$$

Then by (2.8), we have $e_k \in E_k \setminus \{0\}$ and $I_k(e_k) = I_1(e_1) \leq 0$. Note also that $g_k(s) = se_k \in \Gamma_k$ for all $k \in N$ and $I_k(g_k(s)) = I_1(g_1(s))$. Therefore by (2.2),

$$c_k \leq \max_{s\in[0,1]} I_1(g_1(s)) \equiv M. \tag{2.11}$$

So c_k is independent of k and (2.11) leads to a priori bounds for q_k. Since $I'(q_k) = 0$, by (V_2),

$$\begin{aligned} c_k &= I_k(q_k) - \frac{1}{2} I_k'(q_k)q_k \\ &= \int_{-kT}^{kT} \left[\frac{1}{2}(q_k, V_q'(t, q_k) - V(t, q_k)\right]dt \\ &\geq \left(\frac{\mu}{2} - 1\right)\int_{-kT}^{kT} V(t, q_k)dt. \end{aligned} \tag{2.12}$$

Hence by (2.1) and (2.12), we have

$$\|q_k\|_k^2 = \int_{-kT}^{kT} \left[|q''(t)|^2 + (Cq'(t),\, q'(t)) + ((L(t)q(t),\, q(t))\right] dt$$

$$= 2I_k(q) + 2\int_{-kT}^{kT} V(t,\, q) dt$$

$$= 2c_k + \frac{4c_k}{\mu - 2}$$

$$\leq 2M\left(1 + \frac{2}{\mu - 2}\right) \equiv M_1. \tag{2.13}$$

By (2.9), we have a k-dependent bound on $\|q_k\|_{L^\infty[-kT, kT]}$. The proof is complete. □

Proof of Theorem 1.1. We can complete this theorem under the previous lemmas. By these lemmas, we obtain a critical point of I_k and we know the critical points of I_k in E_k are periodic solutions of (1.1).

As in [1], for $q_k \in E_k$, $t, \tau \in [-kT, kT]$

$$|q(t)| \leq q(\tau)| + \left|\int_\tau^t q'(s) ds\right|.$$

We have

$$|q(t)| \leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |q(t)| d\tau + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left|\int_\tau^t q'(s) ds\right| d\tau$$

$$\leq 2\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|q'(\tau)|^2 + |q(\tau)|^2\right) d\tau\right)^{\frac{1}{2}},$$

$$\leq c\|q\|_k, \tag{2.14}$$

where c depends on L. Hence

$$\|q\|_{L^\infty[-kT, kT]} \le c\|q\|_k. \tag{2.15}$$

And so, (2.13) and (2.15) imply

$$\|q_k\|_{L^\infty[-kT, kT]} \le cM_1^{\frac{1}{2}} \equiv M_2. \tag{2.16}$$

Similarly, (2.16) will also hold if we replace q_k with q_k'. Because $q = q_k(t)$ is a solution of (1.1), q_k is bounded in $C^4[a, b]$ for any $a < b$, and hence there is a subsequence of q_k converging in $C^4_{\text{loc}}(\mathbf{R}, \mathbf{R}^n)$ to a solution q of (1.1) satisfying

$$\int_{-\infty}^{\infty} \Big[|q''(t)|^2 + (Cq'(t), q'(t)) + (L(t)q(t), q(t))\Big]dt < \infty.$$

It follows that

$$|q(t)| + |q'(t)| \to 0 \text{ as } |t| \to \infty.$$

Now it is sufficient show $q(t) \neq 0$. By the inner product of (1.1) with q_k and then integration by parts shows:

$$\int_{-kT}^{kT} \Big(-q_k^{(4)} + Cq_k'' - L(t)q_k + V_q'(t, q_k), q_k\Big)d\tau = 0$$

$$\int_{-kT}^{kT} -|q_k''|^2 - (Cq_k', q_k') - (L(t)q_k, q_k) - V(t, q) + (V_q(t, q_k), q_k)d\tau = 0$$

$$\|q_k\|_k^2 = \int_{-kT}^{kT} (q_k, V_q'(t, q_k))dt. \tag{2.17}$$

As in [1], set

$$Y(0) = 0 \text{ and for } s > 0$$

$$Y(s) = \max_{\substack{t\in[0, T] \\ |\xi|\le s}} \frac{(\xi, V_q'(t, \xi))}{|\xi|^2}.$$

By (V_1)–(V_2), $Y(s) > 0$ if $s > 0$, $Y \in C(\mathbf{R}^+, \mathbf{R}^+)$, Y is monotone non-decreasing and $Y(s) \to \infty$ as $s \to \infty$. Also by the definition of Y

$$\frac{(q_k, V_q'(t, q_k(t))}{|q_k(t)|^2} \le Y(\|q_k\|_{L^\infty[-kT, kT]}) \quad \text{for all } t \in [-kT, kT]. \tag{2.18}$$

Hence by (2.17) and (2.18),

$$\|q_k\|_k^2 \le Y(\|q_k\|_{L^\infty[-kT, kT]}) \int_{-kT}^{kT} |q_k|^2 dt$$

$$\le l_0^{-1} Y(\|q_k\|_{L^\infty[-kT, kT]}) \|q_k\|_k^2,$$

where $l_0 > 0$ satisfies $L(t) \ge l_0 I_{2n}$, $\forall t$. And

$$Y(\|q_k\|_{L^\infty[-kT, kT]}) \ge l_0.$$

By the properties of Y, there is a $M_3 > 0$ such that

$$\|q_k\|_{L^\infty[-kT, kT]} \ge M_3. \tag{2.19}$$

Because L and V are T-periodic in t and $q_k(t)$ is a $2kT$-periodic solution, $q(t + jT)$, $j \in [-k, k] \cap Z$ is also a solution of (1.1), and we may assume that the maximum of $|q_k(t)|$ occurs in $[0, T]$. Therefore if $q_k(t) \to 0$ in C^1_{loc} along our subsequence,

$$\|q_k\|_{L^\infty[-kT, kT]} = \max_{t \in [0, T]} |q_k(t)| \to 0.$$

Contrary to (2.19). The proof is complete. □

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