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TRAVELLING WAVE SOLUTIONS OF NONLINEAR STRING VIBRATION EQUATION

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Abstract

In this paper, the modified extended tanh-function method is applied to the nonlinear string vibration equation. By solving the ordinary differential equation, we obtain the travelling wave solutions of nonlinear string vibration equation.

1. Introduction

The string vibration equation is one of the classical equations in mathematical physics. Exact solutions play important roles in many physical phenomena, such as optics, acoustic wave, hydrodynamics. And nonlinear phenomena is also exact in nature, such as non-Newtonian fluid, plasma physics, quantum field theory. So, solving the nonlinear evolution equations becomes an important work. Moreover, many significant method of solving nonlinear evolution equation has been

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established, including homogeneous balance method [2], tanh-function method [3-5], Darboux transform method [6], and Jacobi elliptic function [7]. These methods are applied effectively to solve the specific nonlinear evolution equation. The solutions include triangle function solution, shock wave solution. Some exact solutions of nonlinear string vibration equation have been found [8, 9], however, finding new form exact solutions is still a meaningful work.

Recently, Lv et al. [1] put forward the nonlinear string vibration equation in the form

$$u_{tt} - 2a^2 u_x u_{xx} - 2\mu u_{xxxx} = 0, \tag{1}$$

in which a^2, μ are constants. The authors explained its meaning in physics with variational principle and solved the equation with reductive perturbation method. Guo et al. [10] obtained several types of solitary solutions and periodic solutions for nonlinear vibrating string equation by using the trial function method. In this paper, we apply the modified extended tanh-function method to find the new form travelling wave solutions of the nonlinear string vibration equation.

2. The Travelling Wave Solution of Nonlinear Evolution Equation

2.1. Modified extended tanh-function method

Considering the following nonlinear evolution equation:

$$F(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{2}$$

in which u is an unknown function and F is a polynomial of u and its derivatives. In the following, we try to find the travelling wave solutions by using the modified extended tanh-function method [5]:

$$u(x, t) = u(\xi), \ \xi = x - ct, \tag{3}$$

where c is a constant to be determined later. If we substitute (3) in (2), then Equation (2) reduces to a nonlinear ordinary differential equation:

$$F(u, u', u'', u''', \cdots) = 0. \tag{4}$$

Suppose that the solution of (4) has the form

$$u(\xi) = a_0 + \sum_{i=1}^{m} (a_i \omega^i(\xi) + b_i \omega^{-i}(\xi)), \tag{5}$$

where a_i, b_i are constant to be determined later and m is the balancing number which can be determined by balancing the highest order derivatives terms with the highest power. In function (5), ω satisfies the Riccati function

$$\omega' = b + \omega^2, \tag{6}$$

where b is a constant. Equation (6) has solutions depending on:

$$\omega = \begin{cases} -\sqrt{-b}\tanh(-\sqrt{-b}\xi) \text{ or } -\sqrt{-b}\coth(-\sqrt{-b}\xi), & b < 0, \\ \sqrt{b}\tan(\sqrt{b}\xi) \text{ or } -\sqrt{b}\cot(\sqrt{b}\xi), & b > 0, \\ -\dfrac{1}{\xi}, & b = 0. \end{cases}$$

We substitute Equations (5) and (6) into Equation (4), we can get a polynomial in $\omega^i(\xi)(i = 0, \pm 1, \pm 2, \cdots)$. Setting all the coefficients equal to zero, we can get a set of over determined nonlinear algebraic equations. By solving these equations, we can construct a variety of exact solutions.

2.2. Application of this method to nonlinear evolution equation

Substitute Equation (3) into Equation (1), we get the following equation:

$$c^2 u'' - 2a^2 u' u'' - 2\mu u'''' = 0. \tag{7}$$

Then, we use the balancing procedure between $u_\xi u_{\xi\xi}$ with $u_{\xi\xi\xi\xi}$, $2m + 3 = m + 4 \Rightarrow m = 1$, so the solution has the form

$$u(\xi) = a_0 + a_1\omega + b_1\omega^{-1}, \tag{8}$$

where a_0, a_1, b_1 are constants and not all equal to zero. Substituting Equation (8) into Equation (7), we obtain a system of algebraic equations for $a_0, a_1, b_1,$ and b.

$$-4a^2b^2a_1^2 + 4a^2ba_1b_1 - 32a_1b^2\mu + 2a_1bc^2 = 0, \tag{9}$$

$$-4a^2a_1^2 - 48a_1\mu = 0, \tag{10}$$

$$-8a^2a_1^2b + 4a^2a_1b_1 - 80a_1b\mu + 2bb_1c^2 = 0, \tag{11}$$

$$-4a^2a_1b^2b_1 + 4a^2bb_1^2 - 32b^2b_1\mu + 2bb_1c^2 = 0, \tag{12}$$

$$-4a^2a_1b^3b_1 + 8a^2b^2b_1^2 - 80b^3b_1\mu + 2b^2b_1c^2 = 0, \tag{13}$$

$$4a^2b^3b_1^2 - 48b^4b_1\mu = 0. \tag{14}$$

By using a modem computer algebra system, say, Maple, we get the results as follows:

Case 1.

$$a_1 = -\frac{12\mu}{a^2},\ b_1 = 0,\ b = -\frac{c^2}{8\mu},\ a_0 = a_0.$$

If $b < 0$, we obtain the solitary wave solutions

$$u(\xi) = a_0 + \frac{12\mu}{a^2}\sqrt{\frac{c^2}{8\mu}}\tanh\left(\sqrt{\frac{c^2}{8\mu}}\xi\right), \tag{15}$$

$$u(\xi) = a_0 + \frac{12\mu}{a^2}\sqrt{\frac{c^2}{8\mu}}\coth\left(\sqrt{\frac{c^2}{8\mu}}\xi\right). \tag{16}$$

If $b > 0$, we obtain the solitary wave solutions

$$u(\xi) = a_0 - \frac{12\mu}{a^2}\sqrt{-\frac{c^2}{8\mu}}\tan(\sqrt{-\frac{c^2}{8\mu}}\xi), \tag{17}$$

$$u(\xi) = a_0 + \frac{12\mu}{a^2}\sqrt{-\frac{c^2}{8\mu}}\cot(\sqrt{-\frac{c^2}{8\mu}}\xi). \tag{18}$$

If $b = 0$, we obtain the solitary wave solution

$$u(\xi) = a_0 + \frac{12\mu}{a^2\xi}. \tag{19}$$

Case 2.

$$a_1 = -\frac{12\mu}{a^2},\ b_1 = -\frac{3c^2}{8a^2},\ b = -\frac{c^2}{32\mu},\ a_0 = a_0.$$

If $b < 0$, we obtain the solitary wave solutions

$$u(\xi) = a_0 + \frac{12\mu}{a^2}\sqrt{\frac{c^2}{32\mu}}\tanh(\sqrt{\frac{c^2}{32\mu}}\xi) + \frac{3c^2}{8a^2}\left[\sqrt{\frac{c^2}{32\mu}}\tanh(\sqrt{\frac{c^2}{32\mu}}\xi)\right], \tag{20}$$

$$u(\xi) = a_0 + \frac{12\mu}{a^2}\sqrt{\frac{c^2}{32\mu}}\coth(\sqrt{\frac{c^2}{32\mu}}\xi) + \frac{3c^2}{8a^2}\left[\sqrt{\frac{c^2}{32\mu}}\coth(\sqrt{\frac{c^2}{32\mu}}\xi)\right]^{-1}. \tag{21}$$

If $b > 0$, we obtain the solitary wave solutions

$$u(\xi) = a_0 - \frac{12\mu}{a^2}\sqrt{-\frac{c^2}{32\mu}}\tan(\sqrt{-\frac{c^2}{32\mu}}\xi) - \frac{3c^2}{8a^2}\left[\sqrt{-\frac{c^2}{32\mu}}\tan(\sqrt{-\frac{c^2}{32\mu}}\xi)\right]^{-1}, \tag{22}$$

$$u(\xi) = a_0 + \frac{12\mu}{a^2}\sqrt{-\frac{c^2}{32\mu}}\cot(\sqrt{-\frac{c^2}{32\mu}}\xi) + \frac{3c^2}{8a^2}\left[\sqrt{-\frac{c^2}{32\mu}}\cot(\sqrt{-\frac{c^2}{32\mu}}\xi)\right]^{-1}. \tag{23}$$

If $b = 0$, we obtain the solitary wave solution

$$u(\xi) = a_0 + \frac{12\mu}{a^2\xi} + \frac{3c^2}{8a^2}\xi. \tag{24}$$

Case 3.

$$a_1 = 0,\ b_1 = -\frac{3c^2}{2a^2},\ b = -\frac{c^2}{8\mu},\ a_0 = a_0.$$

If $b < 0$, we obtain the solitary wave solutions

$$u(\xi) = a_0 + \frac{3c^2}{2a^2}\left[\sqrt{\frac{c^2}{8\mu}}\tanh\left(\sqrt{\frac{c^2}{8\mu}}\xi\right)\right]^{-1}, \tag{25}$$

$$u(\xi) = a_0 + \frac{3c^2}{2a^2}\left[\sqrt{\frac{c^2}{8\mu}}\coth\left(\sqrt{\frac{c^2}{8\mu}}\xi\right)\right]^{-1}. \tag{26}$$

If $b > 0$, we obtain the solitary wave solutions

$$u(\xi) = a_0 - \frac{3c^2}{2a^2}\left[\sqrt{-\frac{c^2}{8\mu}}\tan\left(\sqrt{-\frac{c^2}{8\mu}}\xi\right)\right]^{-1}, \tag{27}$$

$$u(\xi) = a_0 + \frac{3c^2}{2a^2}\left[\sqrt{-\frac{c^2}{8\mu}}\cot\left(\sqrt{-\frac{c^2}{8\mu}}\xi\right)\right]^{-1}. \tag{28}$$

If $b = 0$, we obtain the trivial wave solution

$$u(\xi) = a_0 + \frac{3c^2}{2a^2}\xi. \tag{29}$$

Case 4.

$$a_1 = -\frac{12\mu}{a^2},\ b_1 = -\frac{c^2}{2a^2},\ b = 0,\ a_0 = a_0,$$

we obtain the trivial wave solution,

$$u(\xi) = a_0 + \frac{12\mu}{a^2\xi} + \frac{c^2}{2a^2}\xi.$$

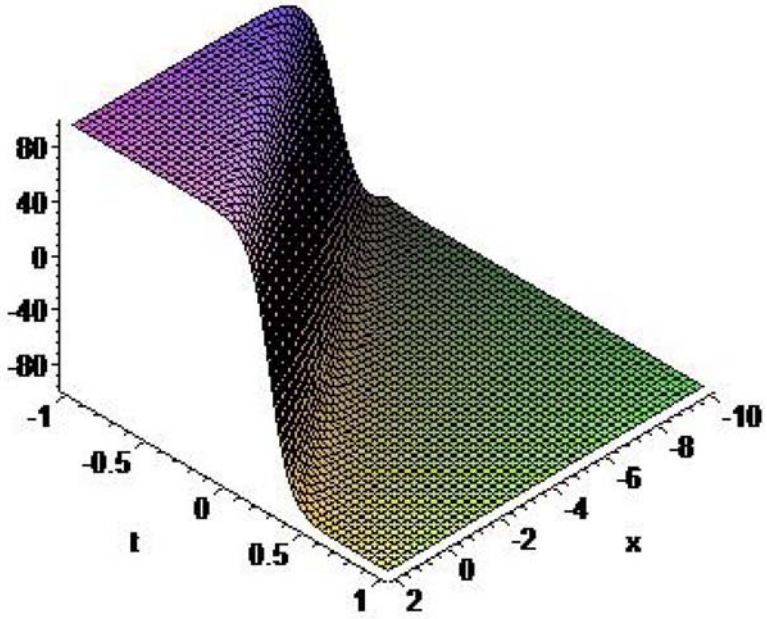

Figure 1. Solution (15) if $a_0 = 0$, $a = 1$, $\mu = c = 8$.

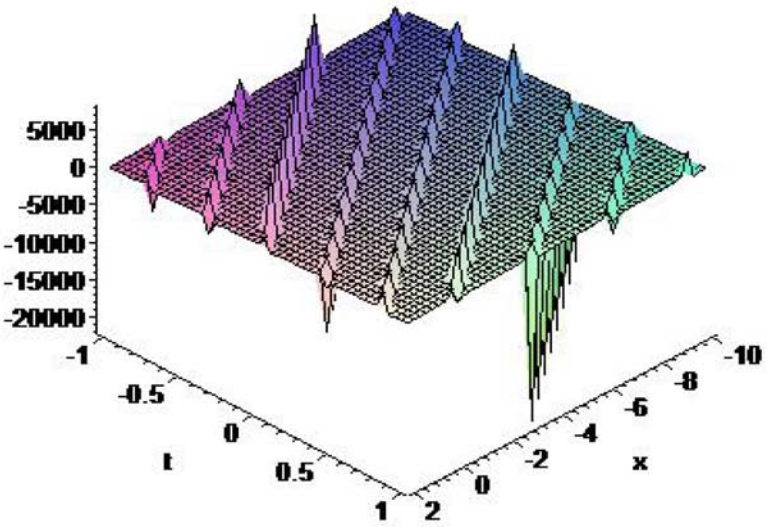

Figure 2. Solution (17) if $a_0 = 0$, $a = 1$, $\mu = -8$, $c = 8$.

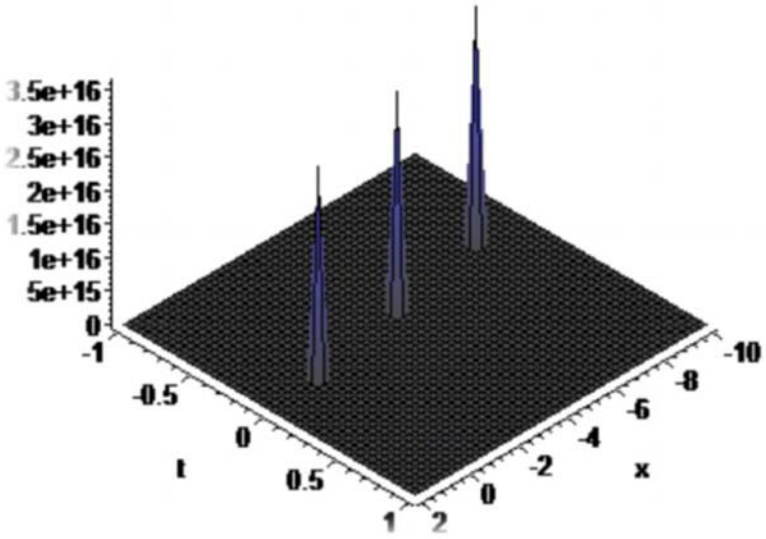

Figure 3. Solution (20) if $a_0 = 0$, $a = 1$, $\mu = c = 32$.

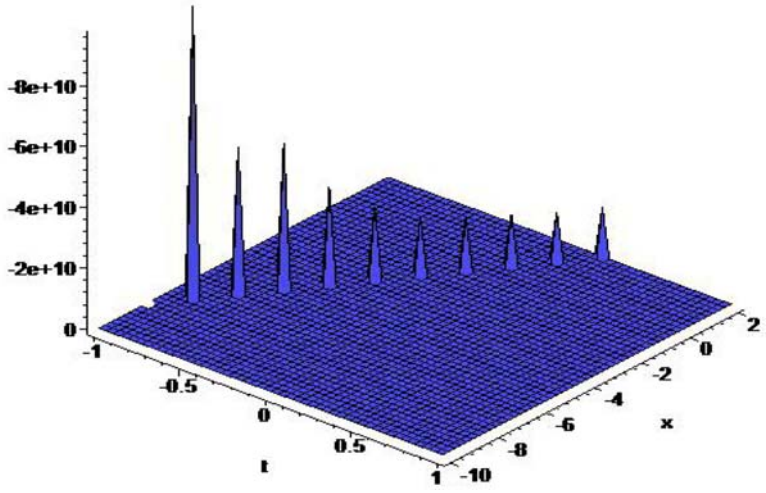

Figure 4. Solution (25) if $a_0 = 0$, $a = 1$, $\mu = c = 8$.

3. Conclusion

In this paper, by using the modified extended tanh-function, we obtain the travelling wave solution of nonlinear string vibration equation. Some of solutions are different from the previous ones in [8]-[10], we can get various of solitary solutions by choosing different special solutions of Riccati equation. We got 16 solutions by using expansion (8), if we use other expansions, we can get more solutions.

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References

[1] K. Z. Lv, X. R. Hong, Y. R. Shi and J. N. Sun, Variational principles and reductive perturbation approach to nonlinear vibrating string, Journal of Northwest Normal University (Natural Science Edition) 37(4) (2001), 35-39 (in Chinese).

[2] E. G. Fan and H. Q. Zhang, A note on the homogeneous balance method, Physics Letters A 246(5) (1998), 403-406.
DOI: https://doi.org/10.1016/S0375-9601(98)00547-7

[3] W. G. Ni and C. Q. Dai, Note on same result of different ansatz based on extended tanh-function method for nonlinear models, Applied Mathematics and Computation 270 (2015), 434-440.
DOI: https://doi.org/10.1016/j.amc.2015.08.052

[4] S. Bibi and S. T. Mohyud-Din, New traveling wave solutions of Drinefel'd-Sokolov-Wilson equation using tanh and extended tanh methods, Journal of the Egyptian Mathematical Society 22(3) (2014), 517-523.
DOI: https://doi.org/10.1016/j.joems.2013.12.006

[5] E. H. M. Zahran and M. M. A. Khater, Modified extended tanh-function method and its applications to the Bogoyavlenskii equation, Applied Mathematical Modelling 40(30 (2016), 1769-1775.
DOI: https://doi.org/10.1016/j.apm.2015.08.018

[6] B. Xia and Sirendaoreji Na, New type of Darboux transformation of the variable coefficients KdV equation, Journal of Inner Mongolia Normal University (Natural Science Edition) 35(2) (2006), 132-134 (in Chinese).

[7] S. K. Liu, Z. T. Fu, S. D. Liu and Q. Zhao, Expansion method about the Jacobi elliptic function and its applications to nonlinear wave equations, Acta Physica Sinica 50(11) (2001), 2068-2073 (in Chinese).

[8] J. M. Zhu and T. J. Xu, The new exact solutions to the equation of nonlinear vibrating string, Journal of Lishui University 10(5) (2004), 34-36 (in Chinese).

[9] Y. R. Shi, X. R. Hong, W. S. Duan, J. B. Zhao and K. Z. Lv, The exact solutions to the equation of nonlinear vibrating string, Journal of Northwest Normal University (Natural Science Edition) 38(2) (2002), 51-53 (in Chinese).

[10] P. Guo, Z. G. Chen and X. W. Sun, Several types of exact solutions to nonlinear vibrating string equation, College Physics 31(10) (2012), 13-14 (in Chinese).

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