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MEROMORPHIC SOLUTIONS OF NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

GUOZHEN REN and LINGYUN GAO

Department of Mathematics
Jinan University
Guangzhou, Guangdong 510632
P. R. China
e-mail: 1104832904@qq.com

Abstract

Using Nevanlinna theory of the value distribution of meromorphic functions and complex differential equations theory, we investigate the properties of one type of non-linear differential-difference equation with rational solution and the existence of transcendental meromorphic solutions of another type of equation, and obtain some results.

1. Introduction and Main Results

We use the standard symbols and fundamental results of Nevanlinna theory [1, 14]. A function $a(z)(\not\equiv 0, \infty)$ is called a small function with respect to $w(z)$, if $T(r, a) = S(r, w)$, where $S(r, w)$ denotes any quantity satisfying $S(r, w) = o(T(r, w))$ with $r \to \infty$ outside of a possible

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exceptional set of finite logarithmic measure. For a meromorphic function $w(z)$, the order and the hyper-order of w is respectively defined by

$$\rho(w) = \limsup_{r\to\infty} \frac{\log T(r, w)}{\log r},$$

$$\sigma(w) = \limsup_{r\to\infty} \frac{\log\log T(r, w)}{\log r}.$$

Moreover, we use the notation $\deg_w(P)$ for the degree of $P(z, w(z))$ with respect to w.

Recently, some papers focused on complex difference equations of Nevanlinna's theory and difference analogue. Many authors ([2-9]) have discussed the existence and the growth of meromorphic solutions of difference equations and obtained many results.

In 2017, Halburd and Korhonen [10] investigated the properties of meromorphic solutions of non-linear differential-difference equation

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{1}$$

where $a(z)$ is a rational function, $P(z, w(z))$ is a polynomial in w having rational coefficients in z, $Q(z, w(z))$ is a monic polynomial in w, $P(z, w)$ and $Q(z, w)$ are co-prime.

They obtained the following result:

Theorem A ([10]). *Let $w(z)$ be a non-rational solution of* (1). *If $\sigma(w) < 1$, then*

$$\deg_w(P) = \deg_w(Q) + 1 \le 3,$$

or the degree of $R(z, w)$ as a rational function in w is either 0 *or* 1.

In 2018, Liu et al. [11] considered the following non-linear differential-difference equation:

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{2}$$

where $R(z, w(z))$ is rational function in $w(z)$, $a(z)$ and $c(z)$ are non-zero rational functions. $P(z, w(z))$ and $Q(z, w(z))$ are defined in (1). They have considered the case that (2) admits a rational solution where all the coefficients of (2) are constants. A result was obtained as follows.

Theorem B ([11]). *Let* $w(z) = \dfrac{M(z)}{N(z)}$ *be a non-constant rational solution of* (2), $M(z)$ *and* $N(z)$ *be polynomials as follows*:

$$M(z) = a_m z^m + \cdots + a_1 z + a_0, \quad N(z) = b_n z^n + \cdots + b_1 z + b_0,$$

and let $h := P(z, 0)$, $g := P'(z, 0)$, $e := Q(z, 0)$ *in* (2), *we have*

(1) *if* $\deg_w(P) = 0$, *then* $m = n$;

(2) *if* $\deg_w(P) = 3$, $\deg_w(Q) = 2$, *then* $m \geq n$ *except that* $n = m + 1$ *provided that* $h = 0$, $e \neq 0$ *and* $g \neq 0$.

In this paper, we mainly consider the existence problem of meromorphic solutions of the Equations (3) and (4) as follows:

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} + b(z)\frac{w''(z)}{w(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{3}$$

$$c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} + b(z)\frac{w''(z)}{w(z)} = d(z), \tag{4}$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ are non-zero rational functions, $P(z, w(z)) = \sum_{i=0}^{l} c_i w^{l-i}(z)$ is a polynomial in w, $Q(z, w(z)) = \sum_{j=0}^{l} c_j w^{l-j}(z)$ is a monic polynomial in w, and they have no common roots, $\{c_i(z)\}$, $\{d_j(z)\}$ are rational functions in z.

Moreover, we mainly consider the case that all the coefficients of (3) are constants.

Let $c_l = P(z, 0)$, $c_{l-1} = P'(z, 0)$, $d_l = Q(z, 0)$.d_l and c_l can not vanish at the same time. We will prove

Theorem 1. *Let* $w(z) = \dfrac{M(z)}{N(z)}$ *be a non-constant rational solution of* (3), $M(z)$ *and* $N(z)$ *be polynomials as follows*:

$$M(z) = a_m z^m + \cdots + a_1 z + a_0, \quad N(z) = b_n z^n + \cdots + b_1 z + b_0.$$

We have

(1) *if* $\deg_w(P) = 0$, *then* $m = n$;

(2) *if* $\deg_w(P) = l$, $\deg_w(Q) = l - 1 (1 \le l \le 7)$, *then* $m \ge n$, *particularly*, *if* $c_l = 0$, $c_{l-1} \ne 0$ *and* $d_l \ne 0$, *then* $n = m + 1$.

Theorem 2. *If* $a(z)w'(z) + b(z)w''(z) \ne 0$, *then* (4) *has no transcendental entire solutions*.

Theorem 3. *Suppose that* $w(z)$ *is a transcendental meromorphic solution of the Equation* (4), *then* $T(r, w) = N(r, w) + S(r, w)$.

2. Some Lemmas

We will use the following lemmas in our proofs of the above theorems.

Lemma 1 ([12]). *Let* w *be a meromorphic function and* $R(z, w)$ *be a rational function in* w *and meromorphic in* z. *If all the coefficients of* $R(z, w)$ *are small compared to* w, *then*

$$T(z, R(z, w)) = \deg_w(R) T(r, w) + S(r, w).$$

Lemma 2 ([13]). *Let* $T : [0, \infty) \to [0, \infty)$ *be a non-decreasing continuous function and let* $s \in (0, \infty)$. *If the hyper-order of* T *is strictly less than one*, *i.e.*,

$$\limsup_{r \to \infty} \frac{\log \log T(r)}{\log r} = \sigma < 1,$$

and $\delta \in (0, 1-\sigma)$, *then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 3 ([13])**.** *Let w be a non-constant meromorphic function and* $c \in C$. *If w is of finite order, then*

$$m\left(r, \frac{w(z+c)}{w(z)}\right) = O\left(\frac{\log r}{r} T(r, w)\right),$$

for all r outside of a set E satisfying

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} \frac{dt}{t}}{\log r} = 0,$$

i.e., outside of a set E of zero logarithmic density. If $\sigma(w) = \sigma < 1$ *and* $\varepsilon > 0$, *then*

$$m\left(r, \frac{w(z+c)}{w(z)}\right) = o\left(\frac{T(r, w)}{r^{1-\sigma-\varepsilon}}\right),$$

for all r outside of a set of finite logarithmic measure.

3. Proof of Theorem 1

Taking Nevanlinna characteristic function of both sides of (3) and applying Lemma 1, we have

$$T\left(r, c(z)w(z+1) + a(z)\frac{w'(z)}{w(z)} + b(z)\frac{w''(z)}{w(z)}\right) = T(r, R(z, w))$$

$$= \deg_w(R)T(r, w(z)) + O(\log r).$$

Using Lemmas 2 and 3, and using the logarithmic derivative lemma [12], it follows that

$$\begin{aligned}\deg_w(R)T(r, w(z)) &\le T(r, w(z+1)) + T(r, \frac{w'(z)}{w(z)}) + T(r, \frac{w''(z)}{w(z)}) + O(\log r)\\ &\le T(r, w) + N(r, \frac{w'(z)}{w(z)}) + N(r, \frac{w''(z)}{w(z)}) + S(r, w)\\ &\le T(r, w) + \left(\overline{N}(r, w(z)) + \overline{N}(r, \frac{1}{w(z)})\right)\\ &\quad + 2\left(\overline{N}(r, w(z)) + \overline{N}(r, \frac{1}{w(z)})\right) + S(r, w)\\ &\le 7T(r, w) + S(r, w).\end{aligned}$$

Thus, we have

$$(\deg_w(R) - 7)T(r, w(z)) \le S(r, w),$$

which implies that $\deg_w(R) < 7$, i.e.,

$$\deg_w(P) \le 7,\ \deg_w(Q) \le 7.$$

We will discuss the following two cases.

Case 1: $\deg_w(P) = 0$ and $\deg_w(Q) = l (l = 0, \cdots, 7)$, we consider the following equation:

$$cw(z+1) + a\frac{w'(z)}{w(z)} + b\frac{w''(z)}{w(z)} = \frac{c_0}{\sum_{j=0}^{l} d_j w^{l-j}(z)}, \tag{5}$$

where $d_j (j = 1, \cdots, l)$ are constants, $a, b, c, c_0, d_0 = 1$ are non-zero constants. Substituting $w(z) = \frac{M(z)}{N(z)}$ into (5), we have

$$\frac{cM\overline{M}N^2 + aM'N^2\overline{N} - aMNN'\overline{N} + bM''N^2\overline{N} - bMNN''\overline{N} - 2bM'NN'\overline{N} + 2bM(N')^2\overline{N}}{MN^2\overline{N}}$$

$$= \frac{c_0 N^l}{d_0 M^l + d_l M^{l-1}N + d_2 M^{l-2}N^2 + \cdots + d_j M^{l-j}N^j + \cdots + d_l N^l}.$$

According to the above equation, it follows that

$$c\sum_{j=0}^{l} d_j M^{l-j+1}\overline{M}N^{j+2} + a\sum_{j=0}^{l} d_j M^{l-j}M'N^{j+2}\overline{N}$$

$$- a\sum_{j=0}^{l} d_j M^{l-j+1}N^{j+1}N'\overline{N} + b\sum_{j=0}^{l} d_j M^{l-j}M''N^{j+2}\overline{N}$$

$$- b\sum_{j=0}^{l} d_j M^{l-j+1}N^{j+1}N''\overline{N} - 2b\sum_{j=0}^{l} d_j M^{l-j}M'N^{j+1}N'\overline{N}$$

$$+ 2b\sum_{j=0}^{l} d_j M^{l-j+l}N^{j}(N')^2\overline{N} = c_0 MN^{l+2}\overline{N}. \tag{6}$$

Thus we see that the possible degrees of left-hand side of (6) are $(l-j+2)m+(j+2)n$, $(l-j+1)m+(j+3)n-1$ or $(l-j+1)m+(j+3)n-2 (j=0,\cdots,l)$. The coefficients of maximum degree terms are different, so there are no terms that can be cancelled. And we can see that the degree of right-hand side of (6) is $m+(l+3)n$.

If $m \geq n+1$, we have the maximum degree of left-hand side of (6) is $(l+2)m+2n$ and the degree of right-hand side of (6) is $m+(l+3)n$, then $m=n$, a contradiction. If $m < n-1$, we have

$$(l+2)m+2n \leq (l+1)m+3n-2 < (l+1)m+3n-1 < (l+1)m+3n$$

$$\leq lm+4n-2 < lm+4n-1 < lm+4n \leq \cdots \leq (l-j+1)m+(j+3)n$$

$$-2 < (l-j+1)m+(j+3)n-1 < (l-j+1)m+(j+3)n \leq \cdots \leq m$$

$$+(l+3)n-2 < m+(l+3)n-1 < m+(l+3)n.$$

(i) If $d_l \neq 0$, then the degree of left-hand side of (6) is $m+(l+3)n-1$, since $c_0 \neq 0$, the degree of right-hand side of (6) is $m+(l+3)n$, thus $m+(l+3)n-1 = m+(l+3)n$, which is impossible.

(ii) If $d_l = 0$ and $c_0 \neq 0$, in this case, the degree of right-hand side of (6) is $m + (l+3)n$, if $d_{l-1} \neq 0$, then the degree of left-hand side of (6) is $2m + (l+2)n - 1$, then $n = m - 1$, which is a contradiction with the assumption $m < n - 1$; if $d_{l-1} = 0$ and $d_{l-2} \neq 0$, then the degree of left-hand side of (6) is $3m + (l+1)n - 1$, then $2n = 2m - 1$, a contradiction.

Similarly, if $d_j = 0 (j = 1, \cdots, l)$, we have $ln = lm - 1$ a contradiction again. If $m = n - 1$, then

$$(l+1)m + 3n - 2 < (l+2)m + 2n = (l+1)m + 3n - 1 = lm + 4n - 2$$

$$< (l+1)m + 3n = lm + 4n - 1 = (l-1)m + 5n - 2 < \cdots = (l-j+1)m$$

$$+ (j+3)n - 2 < (l-j+2)m + (j+2)n = (l-j+1)m + (j+3)n - 1$$

$$= (l-j)m + (j+4)n - 2 < (l-j+1)m + (j+3)n < \cdots = m + (l+3)n$$

$$- 2 < 2m + (l+2)n = m + (l+3)n - 1 < m + (l+3)n.$$

(i) If $d_l \neq 0$, then the degree of left-hand side of (6) is $m + (l+3)n - 1$, since $c_0 \neq 0$, the degree of right-hand side of (6) is $m + (l+3)n$, thus $m + (l+3)n - 1 = m + (l+3)n$, which is impossible.

(ii) If $d_l = 0$ and $d_{l-1} \neq 0$, then the degree of left-hand side of (6) is $2m + (l+2)n - 1$, then $n = m - 1$, which is a contradiction with the assumption $m = n - 1$; if $d_{l-1} = 0$ and $d_{l-2} \neq 0 (j = l)$, then the degree of left-hand side of (6) is $3m + (l+1)n - 1$, then $2n = 2m - 1$, a contradiction.

Similarly, if $d_j = 0 (j = 1, \cdots, l)$, we have $ln = lm - 1$ a contradiction again. Anyway, $m \leq n - 1$ is impossible.

Thus, we have $m = n$ if $\deg_w(P) = 0$ and $\deg_w(Q) = l (l = 0, \cdots, 7)$.

Case 2: $\deg_w(P) = l$ and $\deg_w(Q) = l - 1 (l = 1, \cdots, 7)$, we consider the following equation:

$$cw(z+1) + a\frac{w'(z)}{w(z)} + b\frac{w''(z)}{w(z)} = \frac{\sum_{i=0}^{l} c_i w^{l-i}(z)}{\sum_{j=1}^{l} d_j w^{l-j}(z)}, \tag{7}$$

where $c_i (i = 0, \cdots, l)$, $d_j (j = 2, \cdots, l)$ are constants and a, b, c, c_0, $d_1 = 1$ are non-zero constants. Substituting $w(z) = \frac{M(z)}{N(z)}$ into (7), we have

$$\frac{cM\overline{M}N^2 + aM'N^2\overline{N} - aMNN'\overline{N} + bM''N^2\overline{N} - bMNN''\overline{N} - 2bM'NN'\overline{N} + 2bM(N')^2\overline{N}}{MN^2\overline{N}}$$

$$= \frac{\sum_{i=0}^{l} c_i M^{l-i}N^i}{\sum_{j=1}^{l} d_j M^{l-j}N^j}.$$

According to the above equation, it follows that

$$\begin{aligned} &c\sum_{j=1}^{l} d_j M^{l-j+1}\overline{M}N^{j+2} + a\sum_{j=1}^{l} d_j M^{l-j}M'N^{j+2}\overline{N} \\ &- a\sum_{j=1}^{l} d_j M^{l-j+1}N^{j+1}N'\overline{N} + b\sum_{j=1}^{l} d_j M^{l-j}M''N^{j+2}\overline{N} \\ &- b\sum_{j=1}^{l} d_j M^{l-j+1}N^{j+1}N''\overline{N} - 2b\sum_{j=1}^{l} d_j M^{l-j}M'N^{j+1}N'\overline{N} \\ &+ 2b\sum_{j=1}^{l} d_j M^{l-j+1}N^j(N')^2\overline{N} = \sum_{i=0}^{l} c_i M^{l-i+1}N^{i+2}\overline{N}. \end{aligned} \tag{8}$$

Thus we see that the possible degrees of left-hand side of (8) are $(l-j+2)m+(j+2)n, (l-j+1)m+(j+3)n-1$ or $(l-j+1)m+(j+3)n-2(j=1,\cdots,l)$ and the possible degrees of right-hand side of (8) are $(l-i+1)m+(i+3)n,$ where $i=0,\cdots,l.$

If $m \geq n,$ since c, d_l are non-zero constants, the degree of both hand sides of (8) are equal, both are $(l+1)m+3n,$ this case may happen.

Next, we assume that $m < n$ in the following. We have

$$(l+1)m+3n < lm+4n < \cdots < (l-i+1)m+(i+3)n < \cdots < 2m+(l+2)n < m+(l+3)n.$$

(i) If $d_l \neq 0,$ then the degree of left-hand side of (8) is at most $m+(l+3)n-1,$ if $c_l \neq 0,$ the degree of right-hand side of (8) is at most $m+(l+3)n,$ thus $m+(l+3)n-1 = m+(l+3)n,$ which is impossible; if $c_l = 0$ and $c_{l-1} \neq 0,$ the degree of right-hand side of (8) is $2m+(l+2)n,$ then we have

$$m+(l+3)n-1 = 2m+(l+2)n.$$

So $n = m+1.$ If $c_l = c_{l-1} = 0,$ and $c_{l-2} \neq 0,$ by comparing the degrees of the two hand sides of (8), it yields $2n = 2m+1,$ which is a contradiction.

Similarly, if $c_i = 0(i=1,\cdots,l)$ and $c_0 \neq 0,$ then we have $ln = lm+1,$ a contradiction again.

(ii) If $d_l = 0$ and $c_l \neq 0,$ according to the above analysis, we can see that the degree of the right-hand side of (8) is at most $m+(l+3)n,$ if $d_{l-1} \neq 0,$ then the degree of left-hand side of (8) is $2m+(l+2)n-1,$ then $n = m-1,$ which is a contradiction with the assumption $m < n;$ if $d_{l-1} = 0$ and $d_{l-2} \neq 0,$ then the degree of left-hand side of (8) is $3m+(l+1)n-1,$ then $2n = 2m-1,$ a contradiction.

Similarly, if $d_j = 0 (j = 2, \cdots, l)$, since $d_1 = 1$, we have $(l-1)n = (l-1)m - 1$, a contraction again.

In a word, if $\deg_w(P) = l$ and $\deg_w(Q) = l-1$, then $m \geq n$. Specially, if $c_l = 0$, $c_{l-1} \neq 0$ and $d_l \neq 0$, we have $n = m+1$. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Suppose that $w(z)$ is a transcendental entire solution of (4). We consider the following equation:

$$c(z)w(z)w(z+1) = d(z)w(z) - a(z)w'(z) - b(z)w''(z). \tag{9}$$

If $w(z)$ is a transcendental entire solution of (9), then $w(z)$ has at most finitely many zeros.

In fact, if $w(z)$ has infinitely many multiple zeros, we can fide a zero z_0, such that

$$d(z_0)w(z_0) - a(z_0)w'(z_0) - b(z_0)w''(z_0) \neq 0,$$

but the left-hand side of (9) is equal to zero at z_0, which is a contradiction.

If $w(z)$ has infinitely many single zero, by the assume that $a(z)w'(z) + b(z)w''(z) \not\equiv 0$, we can fide a single zero z_1, such that

$$d(z_1)w(z_1) - a(z_1)w'(z_1) - b(z_1)w''(z_1) \neq 0,$$

but the left-hand side of (9) is equal to zero at z_1, a contradiction again. So $w(z)$ has at most finitely many zeros.

Thus, according to the Hadamard factorization theorem, we assume that

$$w(z) = q(z)e^{p(z)},$$

where $q(z)$ is non-zero polynomial and $p(z)$ is an entire function. Substituting into (9), we have

$$c(z)q(z+1)e^{p(z+1)}$$

$$= d(z) - a(z)\left[p'(z) + \frac{q'(z)}{q(z)}\right] - b(z)\left[q''(z) + [p'(z)]^2 + \frac{q''(z) + 2p'(z)q'(z)}{q(z)}\right]. \tag{10}$$

From (10), we can see that the order of growth of the left-hand side is always greater than the right-hand side, which is a contradiction. Therefore, (4) has no transcendental entire solutions. The proof of Theorem 2 is complete.

5. Proof of Theorem 3

Suppose that $w(z)$ is a transcendental meromorphic solution of (4). Rewrite (4) as

$$c(z)w(z) = \left(d(z) - a(z)\frac{w'(z)}{w(z)} - b(z)\frac{w''(z)}{w(z)}\right)\frac{w(z)}{w(z+1)}. \tag{11}$$

Taking proximity functions from both sides of (11), and using the logarithmic derivative lemma and Lemma 3, we have

$$m(r, w) = S(r, w).$$

Thus

$$T(r, w) = N(r, w) + S(r, w).$$

The proof of Theorem 3 is complete.

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