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SUBAUTOMATA OF A GENERAL FUZZY AUTOMATON

MOHAMMAD HORRY

Department of Mathematics
Faculty of Chamran
Technical and Vocational University (TVU)
Kerman Branch
Kerman
Iran
e-mail: mhori@tvu.ac.ir
mohhorry@yahoo.com

Abstract

In this paper, we first define the notions of a successor of the state q in a max-min general fuzzy automaton and subautomaton of a max-min general fuzzy automaton. Then define a fuzzy subsystem of a max-min general fuzzy automaton. Finally, we derive the relationships between them.

1. Introduction and Preliminaries

The concept of fuzzy automata was introduced by Wee in 1967 [10].

Let $\sum$ be a non-empty finite set of symbols (alternatively called characters), called the alphabet. A string (or word) over $\sum$ is any finite

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sequence of symbols from $\sum$. Λ denotes the empty (or null) string and $\sum^*$ the closure of the alphabet, i.e., the set of all words over $\sum$. The length $\ell(x)$ of the word $x \in \sum^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\widetilde{P}(X)$ will denote the set of all fuzzy sets on X and $P(X)$ will denote the set of all subsets on X.

A deterministic finite-state automaton is a five-tuple denoted as $A = (Q, \sum, f, T, s)$, where Q is a finite set of states, $\sum$ is a finite set of input symbols, the function f from $Q \times \sum$ into Q is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.

A word $x = x_1x_2 \ldots x_n \in \sum^*$ is said to be accepted by A if there exist states $q_0, q_1, \ldots, q_n$ satisfying

(1) $q_0 = s$,

(2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, \ldots, n$,

(3) $q_n \in T$.

The empty word is accepted by A if and only if $s \in T$.

A nondeterministic finite-state automaton is a five-tuple denoted as $A = (Q, \sum, f, T, s)$, where Q is a finite set of states, $\sum$ is a finite set of input symbols, the function f from $Q \times \sum$ into $P(Q)$ is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.

A fuzzy finite-state automaton (FFA) is a six-tuple $\widetilde{F} = (Q, \sum, R, Z, \delta, \omega)$, where Q is a finite set of states, $\sum$ is a finite set of input symbols, R is the initial state of $\widetilde{F}$, Z is a finite set of output symbols,

$\delta : Q \times \sum \times Q \to [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0, 1]$ and $\omega : Q \to Z$ is the output function. Associated with each fuzzy transition, there is a membership value in $[0, 1]$ called the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

We use $\delta(q_i, a_k, q_j)$ to refer both to a transition and its weight in the sense that whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition, and otherwise, specifies the transition itself. The set of all transitions of $\widetilde{F}$ will be denoted by Δ. The above definition is generally accepted as a formal definition of a fuzzy finite-state automaton [2, 3, 6, 7, 8, 9].

In 2004, Doostfatemeh and Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [1].

In this paper, by using [1] and [11], we define the notions of a successor of the state q and subautomaton of a max-min general fuzzy automaton. Then define a fuzzy subsystem of a max-min general fuzzy automaton. Finally, we derive the relationships between them.

Definition 1.1 ([1])**.** A general fuzzy automaton (GFA) is an eight-tuple machine $\widetilde{F} = (Q, \sum, \widetilde{R}, Z, \widetilde{\delta}, \omega, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,

(ii) $\sum$ is a finite set of input symbols, $\sum = \{a_1, a_2, \ldots, a_m\}$,

(iii) $\widetilde{R}$ is the set of fuzzy start states, $\widetilde{R} \in \widetilde{P}(Q)$,

(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,

(v) $\omega : Q \to Z$ is the output function,

(vi) $\widetilde{\delta} : (Q \times [0, 1]) \times \sum \times Q \to [0, 1]$ is the augmented transition function,

(vii) $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is the membership assignment function,

(viii) $F_2 : [0, 1]^* \to [0, 1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$ has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \widetilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value (mv) of the state q_j at time $t + 1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The usual options for the function $F_1(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the fuzzy set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \widetilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \sum, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 1.2 ([1]) (Multi-membership resolution)**.** If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to it:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i.

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))],$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$ is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 1.3 ([11]). Let $\widetilde{F} = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata as $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ such that:

$$\widetilde{\delta}^* : Q_{act} \times \sum^* \times Q \to [0, 1],$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and for all $i \geq 0$,

$$\widetilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & \text{otherwise.} \end{cases}$$

Also, if the input at time t_i be u_i, where $u_i \in \sum$, $\forall 1 \leq i \leq n$, then

$$\widetilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \widetilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p),$$

$$\widetilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\widetilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \widetilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p)),$$

and recursively

$$\widetilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee \{\widetilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \widetilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots \wedge \widetilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\}.$$

If $q \in Q_{act}(t_i)$, we should write q belongs to an element of Q_{act}. Hereafter, we simply denote it as: $q \in Q_{act}$.

2. Subautomata of a General Fuzzy Automaton

Definition 2.1. Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and $p, q \in Q$. Then q is called a successor of p if there exists $x \in \sum^*$ such that $\widetilde{\delta}^*((p, \mu^t(p)), x, q) > 0$.

We denoted $S(p)$ the set of all successors of p. If $T \subseteq Q$, then set of all successors of T denoted by $S(T)$, is defined by the set $S(T) = \bigcup \{S(p) : p \in T\}$.

Example 2.2. Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, where

$$Q = \{q_0, q_1, q_2\}, \sum = \{a, b\}, Q_{act}(t_0) = \widetilde{R} = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\},$$

$F_1(\mu, \delta) = \text{Min}(\mu, \delta)$, $Z = \emptyset$, ω and F_2 are not applicable, $\delta(q_0, a, q_1) = 0.4$, $\delta(q_0, b, q_2) = 0.5$, $\delta(q_1, a, q_2) = 0.3$, $\delta(q_2, a, q_2) = 0.2$.

If we choose the input string $x = aa \ldots a$, then

$$Q_{act}(t_1) = \{(q_1, \mu^{t_1}(q_1))\}, \qquad Q_{act}(t_i) = \{(q_2, \mu^{t_i}(q_2))\}, \forall i \geq 2,$$

$$\mu^{t_1}(q_1) = \widetilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = F_1(1, 0.4) = 0.4,$$

$$\mu^{t_2}(q_2) = \widetilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = F_1(0.4, 0.3) = 0.3,$$

$$\mu^{t_3}(q_2) = \widetilde{\delta}((q_2, \mu^{t_2}(q_2)), a, q_2) = F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_2)) = F_1(0.3, 0.2) = 0.2,$$

$$\mu^{t_4}(q_2) = \widetilde{\delta}((q_2, \mu^{t_3}(q_2)), a, q_2) = F_1(\mu^{t_3}(q_2), \delta(q_2, a, q_2)) = F_1(0.2, 0.2) = 0.2,$$

$$\mu^{t_i}(q_2) = 0.2, \forall i \geq 5,$$

$$\widetilde{\delta}^*((q_0, \mu^{t_0}(q_0)), \Lambda, q_0) = 1 \Rightarrow q_0 \in S(q_0),$$

$$\widetilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a, q_1) = 0.4 \Rightarrow q_1 \in S(q_0),$$

$$\widetilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_2) = 0.4 \wedge 0.3 = 0.3 \Rightarrow q_2 \in S(q_0),$$

$$\widetilde{\delta}^*((q_1, \mu^{t_1}(q_1)), a, q_2) = 0.3 \Rightarrow q_2 \in S(q_1).$$

Definition 2.3. Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and $\widetilde{F}^{*'} = (T, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^{*'}, F_1, F_2)$. Then $\widetilde{F}^{*'}$ is called a subautomaton of $\widetilde{F}^*$ if $S(T) \subseteq T$, where $\widetilde{\delta}^{*'} = \widetilde{\delta}^*|_{(Q_{act} \cap T) \times \sum^* \times T}$.

Definition 2.4. Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. λ is called a fuzzy subsystem of $\widetilde{F}^*$, if

$$\lambda(q) \geq \lambda(p) \wedge \widetilde{\delta}((p, \mu^t(p)), a, q), \forall p, q \in Q, a \in \sum.$$

Example 2.5. In Example 2.2, let $\lambda(q_0) = 0.3$, $\lambda(q_1) = 0.4$ and $\lambda(q_2)$ $\lambda(q_2) = 0.5$. If we choose the input string $x = aa \ldots a$, then

$Q_{act}(t_1) = \{(q_1, \mu^{t_1}(q_1))\}$, $\quad Q_{act}(t_i) = \{(q_2, \mu^{t_i}(q_2))\}, \forall i \geq 2,$

$\mu^{t_1}(q_1) = \widetilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = F_1(1, 0.4) = 0.4,$

$\mu^{t_2}(q_2) = \widetilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = F_1(0.4, 0.3) = 0.3,$

$\mu^{t_3}(q_2) = \widetilde{\delta}((q_2, \mu^{t_2}(q_2)), a, q_2) = F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_2)) = F_1(0.3, 0.2) = 0.2,$

$\mu^{t_4}(q_2) = \widetilde{\delta}((q_2, \mu^{t_3}(q_2)), a, q_2) = F_1(\mu^{t_3}(q_2), \delta(q_2, a, q_2)) = F_1(0.2, 0.2) = 0.2,$

$\mu^{t_i}(q_2) = 0.2, \forall i \geq 5,$

$\lambda(q_0) \geq \lambda(q_0) \wedge \widetilde{\delta}^*((q_0, \mu^{t_0}(q_0)), \Lambda, q_0) = 0.3 \wedge 1 = 0.3,$

$\lambda(q_1) \geq \lambda(q_0) \wedge \widetilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a, q_1) = 0.3 \wedge 0.4 = 0.3,$

$\lambda(q_2) \geq \lambda(q_1) \wedge \widetilde{\delta}^*((q_1, \mu^{t_1}(q_1)), a, q_2) = 0.4 \wedge 0.3 = 0.3.$

Thus λ is a fuzzy subsystem of $\widetilde{F}^*$.

Theorem 2.6. *Let* $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton and* λ *be a fuzzy subset on* Q. *Then* λ *is a fuzzy subsystem of* $\widetilde{F}^*$ *if and only if* $\lambda(q) \geq \lambda(p) \wedge \widetilde{\delta}^*((p, \mu^t(p)), x, q), \forall p, q \in Q, x \in \sum^*$.

Proof. Let λ be a fuzzy subsystem of $\widetilde{F}^*$ and $p, q \in Q, x \in \sum^*$. We prove the theorem by mathematical induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\lambda(p) \wedge \widetilde{\delta}^*((p, \mu^t(p)), x, q) = \lambda(p) \wedge 1 = \lambda(p) = \lambda(q)$. If $q \neq p$, then $\lambda(p) \wedge \widetilde{\delta}^*((p, \mu^t(p)), x, q) = \lambda(p) \wedge 0 = 0 \leq \lambda(q)$.

Assume that the theorem is true for all $u \in \sum^*$ such that $|u| = n - 1$, $n > 1$. Let $x = au$, where $a \in \sum$ and $|u| = n - 1$. Then we have

$$\begin{aligned}
\lambda(p) \wedge \widetilde{\delta}^*((p, \mu^t(p)), x, q) &= \lambda(p) \wedge \widetilde{\delta}^*((p, \mu^t(p)), au, q) \\
&= \lambda(p) \wedge \Big(\bigvee_{r \in Q} \widetilde{\delta}^*((p, \mu^t(p)), a, r) \wedge \widetilde{\delta}^*((r, \mu^t(r)), u, q) \Big) \\
&= \bigvee_{r \in Q} \{(\lambda(p) \wedge \widetilde{\delta}^*((p, \mu^t(p)), a, r)) \wedge \widetilde{\delta}^*((r, \mu^t(r)), u, q)\} \\
&\leq \bigvee_{r \in Q} (\lambda(r) \wedge \widetilde{\delta}^*((r, \mu^t(r)), u, q)) \leq \lambda(q).
\end{aligned}$$

The converse is trivial. □

Theorem 2.7. *Let* $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton. Then every constant fuzzy set* λ *on* Q *is a fuzzy subsystem of* $\widetilde{F}^*$.

Proof. Let λ is constant fuzzy set on Q. So for any $p, q \in Q$, we have $\lambda(p) = \lambda(q)$. Then for any $a \in \sum$, clearly $\lambda(q) = \lambda(p) \geq \lambda(p) \wedge \tilde{\delta}((p, \mu^t(p)), a, q)$. Therefore, λ is a fuzzy subsystem of $\tilde{F}^*$. □

Theorem 2.8. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* λ_1 *and* λ_2 *be fuzzy subsystems of* $\tilde{F}^*$. *Then* $\lambda_1 \cap \lambda_2$ *and* $\lambda_1 \cup \lambda_2$ *are fuzzy subsystems of* $\tilde{F}^*$.

Proof. Since λ_1 and λ_2 are fuzzy subsystems of $\tilde{F}^*$, for $p, q \in Q$, $x \in \sum^*$, we have $\lambda_1(q) \geq \lambda_1(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q)$ and $\lambda_2(q) \geq \lambda_2(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q)$. So

$$(\lambda_1 \cap \lambda_2)(q) = \lambda_1(q) \wedge \lambda_2(q) \geq (\lambda_1(p) \wedge \lambda_2(p)) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q).$$

Hence, $\lambda_1 \cap \lambda_2$ is a fuzzy subsystem of $\tilde{F}^*$. Also, we have

$$(\lambda_1 \cup \lambda_2)(q) = \lambda_1(q) \vee \lambda_2(q) \geq (\lambda_1(p) \vee \lambda_2(p)) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q).$$

Hence, $\lambda_1 \cup \lambda_2$ is a fuzzy subsystem of $\tilde{F}^*$. □

Theorem 2.9. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton and* λ *be a fuzzy subsystem of* $\tilde{F}^*$. *Then* $\tilde{F}^{*'} = (supp(\lambda), \sum, \tilde{R}, Z, \omega, \tilde{\delta}^{*'}, F_1, F_2)$ *is a subautomaton of* $\tilde{F}^*$, *where* $\tilde{\delta}^{*'} = \tilde{\delta}^*|_{(Q_{act} \cap supp(\lambda)) \times \sum^* \times supp(\lambda)}$.

Proof. Let $q \in S(supp(\lambda))$. Then $q \in S(p)$, for some $p \in supp(\lambda)$. So $\lambda(p) > 0$. Since $q \in S(p)$, then there exists $x \in \sum^*$ such that $\tilde{\delta}^*((p, \mu^t(p)), x, q) > 0$. Now, since λ is a fuzzy subsystem of $\tilde{F}^*$, then we have

$$\lambda(q) \geq \lambda(p) \wedge \tilde{\delta}((p, \mu^t(p)), x, q) > 0.$$

Thus $q \in supp(\lambda)$. Therefore $S(supp(\lambda)) \subseteq supp(\lambda)$. Hence $\tilde{F}^{*'}$ is a subautomaton of $\tilde{F}^*$. □

Theorem 2.10. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton and* λ *be a fuzzy subset of* Q. *For* $x \in \sum^*$ *and* $q \in Q$, *define the fuzzy subset* λx *of* Q *by*

$$(\lambda x)(q) = \bigvee_{p \in Q} \{\lambda(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q)\}.$$

Then λ *is a fuzzy subsystem of* $\tilde{F}^*$ *if and only if* $\lambda x \subseteq \lambda$, $\forall x \in \sum^*$.

Proof. Let λ be a fuzzy subsystem of $\tilde{F}^*$ and $x \in \sum^*$, $q \in Q$. Then

$$(\lambda x)(q) = \bigvee_{p \in Q} \{\lambda(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q)\} \leq \lambda(q).$$

Hence, $\lambda x \subseteq \lambda$.

Conversely, let $x \in \sum^*$ and $q \in Q$. Then for all $p \in Q$, we have

$$\lambda(q) \geq (\lambda x)(q) = \bigvee_{p \in Q} \{\lambda(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q)\} \geq \lambda(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q).$$

Hence, λ is a fuzzy subsystem of $\tilde{F}^*$. □

Theorem 2.11. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* $\alpha \in [0, 1]$ *and* $p, q \in Q$. *Define the fuzzy subset* $q_\alpha \sum^*$ *of* Q *by*

$$(q_\alpha \sum^*)(p) = \bigvee_{x \in \sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}, \forall q \in Q.$$

Then $q_\alpha \sum^*$ *is a fuzzy subsystem of* $\tilde{F}^*$.

Proof. Let $x' \in \sum^*$, then we have

$$(q_\alpha \sum^*)(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x', r)$$

$$= \bigvee_{x \in \sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\} \wedge \tilde{\delta}^*((p, \mu^t(p)), x', r)$$

$$= \bigvee_{x \in \sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x', r)\}$$

$$\leq \bigvee_{x \in \sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), xx', r)\}$$

$$\leq \bigvee_{u \in \sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), u, r)\}$$

$$= (q_\alpha \sum^*)(r).$$

Hence, $q_\alpha \sum^*$ is a fuzzy subsystem of $\tilde{F}^*$. □

Theorem 2.12. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* λ *is a fuzzy subset of* Q, $\alpha \in [0, 1]$, $q \in Q$ *and* $\alpha \leq \lambda(q)$. *Then* λ *is a fuzzy subsystem of* $\tilde{F}^*$ *if and only if* $q_\alpha \sum^* \subseteq \lambda$.

Proof. Let λ be a fuzzy subsystem of $\tilde{F}^*$. Then for $p \in Q$, we have

$$(q_\alpha \sum^*)(p) = \vee_{x \in \sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}$$

$$\leq \vee_{x \in \sum^*} \{\lambda(q) \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}$$

$$\leq \vee_{x \in \sum^*} \lambda(p) = \lambda(p).$$

Hence, $q_\alpha \sum^* \subseteq \lambda$.

Conversely, let $q_\alpha \sum^* \subseteq \lambda$, $p \in Q$ and $u \in \sum^*$. If $\lambda(p) = 0$ or $\tilde{\delta}^*((p, \mu^t(p)), x, q) = 0$, then $\lambda(q) \geq \lambda(p) \wedge \tilde{\delta}^*((p, \mu^t(p)), x, q) = 0$.

Suppose $\lambda(p) \neq 0$ and $\tilde{\delta}^*((p, \mu^t(p)), x, q) \neq 0$. Let $\alpha = \lambda(q)$ and $u \in \sum^*$. Thus, by the hypothesis, $q_\alpha \sum^* \subseteq \lambda$. Then we have

$$\lambda(p) \geq (q_\alpha \sum^*)(p) = \vee_{x \in \Sigma^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}$$

$$\geq \alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), u\ p)$$

$$= \lambda(q) \wedge \tilde{\delta}^*((q, \mu^t(q)), u\ p).$$

Hence, λ is a fuzzy subsystem of $\tilde{F}^*$. □

Theorem 2.13. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* $\alpha \in [0, 1]$ *and* $q \in Q$. *Then supp* $(q_\alpha \sum^*) = S(q)$.

Proof. We have

$$p \in S(q) \Leftrightarrow \exists x \in \sum^* : \tilde{\delta}^*((q, \mu^t(q)), x\ p) > 0$$

$$\Leftrightarrow \bigvee_{x \in \Sigma^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\} > 0$$

$$\Leftrightarrow (q_\alpha \sum^*)(p) > 0$$

$$\Leftrightarrow p \in supp(q_\alpha \sum^*).$$

Thus $supp(q_\alpha \sum^*) = S(q)$. □

Definition 2.14. Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and λ be a fuzzy subsystem of $\widetilde{F}^*$. Then λ is called cyclic if there exist $\alpha \in [0, 1]$ and $q \in Q$ with $\alpha \leq \lambda(q)$ such that $\lambda = q_\alpha \sum^*$.

In this case, we call q_α a generator of λ.

Theorem 2.15. *Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, $\alpha \in [0, 1]$, $q \in Q$, $\alpha \leq \lambda(q)$ and λ be a fuzzy subsystem of $\widetilde{F}^*$. If λ be cyclic with generator q_α, then $\lambda(q) = \alpha$.*

Proof. Since $\lambda = q_\alpha \sum^*$, then we have

$$\begin{aligned}\lambda(q) &= (q_\alpha \sum^*)(q) \\ &= \bigvee_{x \in \sum^*} \{\alpha \wedge \widetilde{\delta}^*((q, \mu^t(q)), x, q)\} \\ &= \alpha \wedge \bigvee_{x \in \sum^*} \{\widetilde{\delta}^*((q, \mu^t(q)), x, q)\} = \alpha \wedge 1 = \alpha.\end{aligned}$$

□

Theorem 2.16. *Let $\widetilde{F}^* = (Q, \sum, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, $\alpha \in [0, 1]$, $q \in Q$, $\alpha \leq \lambda(q)$ and λ be a fuzzy subsystem of $\widetilde{F}^*$. If λ be cyclic with generator q_α, then $\lambda(q) \geq \lambda(p)$, $\forall p \in Q$.*

Proof. Let $p \in Q$. Since $\lambda = q_\alpha \sum^*$, then we have

$$\begin{aligned}\lambda(p) &= (q_\alpha \sum^*)(p)\\ &= \bigvee_{x\in\sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}\\ &\le \bigvee_{x\in\sum^*} \{\lambda(q) \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}\\ &= \lambda(q) \wedge \bigvee_{x\in\sum^*} \{\tilde{\delta}^*((q, \mu^t(q)), x, p)\} \le \lambda(q).\end{aligned}$$

□

Theorem 2.17. *Let* $\tilde{F}^* = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* $\alpha \in [0, 1]$, $q \in Q$, $\alpha \le \lambda(q)$ *and* λ *be a fuzzy subsystem of* $\tilde{F}^*$. *If* λ *be cyclic with generator* q_α, *then for any fuzzy subsystem* λ' *of* $\tilde{F}^*$ *such that* $\lambda' \subseteq \lambda$ *and* $\lambda'(q) \ge \lambda'(r)$, $\forall r \in Q$, *we have* $\lambda' = q_{\lambda'(q)} \sum^*$.

Proof. Let $p \in Q$. Since $\lambda' \subseteq \lambda$, we have $\lambda'(p) \le \lambda(p)$. Then $\lambda'(p) = \lambda'(p) \wedge \lambda(p)$. Also, since $\lambda = q_\alpha \sum^*$, then we have

$$\begin{aligned}\lambda(p) &= (q_\alpha \sum^*)(p)\\ &= \bigvee_{x\in\sum^*} \{\alpha \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}\\ &= \bigvee_{x\in\sum^*} \{\lambda(q) \wedge \tilde{\delta}^*((q, \mu^t(q)), x, p)\}\\ &= \lambda(q) \wedge \bigvee_{x\in\sum^*} \{\tilde{\delta}^*((q, \mu^t(q)), x, p)\}.\end{aligned}$$

Hence

$$\lambda'(p) = \lambda'(p) \wedge \lambda(p)$$

$$= (\lambda'(p) \wedge \lambda(q)) \wedge \bigvee_{x \in \sum^*} \{\widetilde{\delta}^*((q, \mu^t(q)), x, p)\}.$$

Since $\lambda'(q) \geq \lambda'(p)$, then we have

$$\lambda'(p) \leq (\lambda'(q) \wedge \lambda(q)) \wedge \bigvee_{x \in \sum^*} \{\widetilde{\delta}^*((q, \mu^t(q)), x, p)\}$$

$$= \lambda'(q) \wedge \bigvee_{x \in \sum^*} \{\widetilde{\delta}^*((q, \mu^t(q)), x, p)\}$$

$$= \bigvee_{x \in \sum^*} \{\lambda'(q) \wedge \widetilde{\delta}^*((q, \mu^t(q)), x, p)\} = (q_{\lambda'(q)} \sum^*)(p).$$

Therefore $\lambda' \subseteq q_{\lambda'(q)} \sum^*$. By Theorem 2.12, we have $\lambda' = q_{\lambda'(q)} \sum^*$. □

3. Conclusion

In this paper, we have defined the notions of a successor of the state q and subautomaton of a max-min general fuzzy automaton. Then have defined a fuzzy subsystem of a max-min general fuzzy automaton. Finally, we have obtained the relationships between them.

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