**Journal of Mathematical Sciences: Advances and Applications**
Volume 55, 2019, Pages 21-35
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100122029

# HYERS-ULAM STABILITY OF A NONLINEAR MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATION ON BOUNDED TIME INTERVAL

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## Abstract

In this paper, we study the Hyers-Ulam stability of a nonlinear multi-term fractional differential equation with Caputo derivative on bounded time interval. First we discuss the existence and uniqueness for the solutions of initial value problem of a nonlinear multi-term fractional differential equation and second we give a criteria for Hyers-Ulam stability which is our main purpose. The technique relies on the fixed point theorem in the Banach space and the equivalence between Chebyshev norm and Bielecki norm.

## 1. Introduction

Nowadays, it became widely known that fractional differential equations serve a strong tool in modelling many phenomena in various fields such as engineering, economics, chemistry, etc. For more details

2010 Mathematics Subject Classification: 34A08, 34G20.
Keywords and phrases: Hyers-Ulam stability, fractional differential equation, Chebyshev norm, Bielecki norm.
Received January 22, 2019

on fractional differential equations, one can see the monographs of Kilbas et al. [1], Podlubny [2], and Miller and Ross [3]. The research of the fractional differential equations is more difficult than that of integer-order differential equations, because they are nonlocal and have weakly singular kernels.

Recently, one of the most active research topics of fractional differential equations is the Ulam stability theory. Ulam stability of differential equations is very important, since it guarantees the existence of solution and provides the theoretical basis of approximate solution. Obloza [4] initiated the study of the Ulam stability of differential equations and after that several results of the Ulam stability of classical differential equations were given by many researchers [5-12].

In recent years, the study of the Ulam stability of fractional differential equations is proceeding widely and many remarkable results on that have come out. Wang et al. [13] discussed the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the fractional differential equation

$$ {}^{c}D^{\alpha}x(t) = f(t, x(t)), \quad t \in [a, b], $$

where $\alpha \in (0, 1)$ and ${}^{c}D^{\alpha}$ is Caputo fractional derivative of order $\alpha$. Wang et al. [14] obtained the criteria for the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of fractional differential equation

$$ {}^{c}D^{\alpha}_{a+}y(x) = F(x, y(x)), \quad t \in I, $$

where $I$ is a given interval and $\alpha \in (0, 1)$.

Wang and Xu [15] proved the Hyers-Ulam stability of two types of fractional linear differential equations

$$ ({}^{c}D^{\alpha}_{0+}y)(x) - \lambda y(x) = f(x), $$

$$ ({}^{c}D^{\alpha}_{0+}y)(x) - \lambda({}^{c}D^{\beta}_{0+}y)(x) = g(x), $$

where $x > 0$, $\lambda \in (-\infty, +\infty)$, $0 < \beta < \alpha$, $f(x)$ and $g(x)$ are real functions defined on $\mathbf{R}_+$. Mohamed [16] studied the Ulam stability of fractional impulsive differential equations by using singular Gronwall inequality. Besides, in many papers, the Ulam stability of classical differential equations has been extended to the several types of fractional differential equations. For more details, one can see [17-22].

Although there are many articles and papers on the Ulam stability of fractional differential equations, to our best knowledge, most results on that are restricted to monomial fractional differential equations. So we discuss the Hyers-Ulam stability of the following nonlinear multi-term fractional differential equation on bounded time interval.

$$ {}^{c}D_{0+}^{\alpha}u(t) = f(t, u(t), {}^{c}D_{0+}^{\beta}u(t)), \tag{1}$$

where ${}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$, $0 < \beta < \alpha$ and $0 < t < b < +\infty$.

The rest of this paper is consisted as follows: In Section 2, we introduce the basic definitions and lemmas which are used in this paper. In Section 3, we give the criteria for the Hyers-Ulam stability of the Equation (1).

## 2. Preliminaries

**Definition 2.1.** The fractional integral of order $\alpha > 0$ with the lower limit $a$ for function $f \in L_1[a, b]$ is defined as

$$I_{a+}^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau, \quad a \le t \le b,$$

where $\Gamma(.)$ is the Gamma function.

**Definition 2.2.** The Caputo fractional derivative of order $\alpha > 0$ for function $f$ is defined as

$$ {}^{c}D_{a+}^{\alpha} f(t) := I_{a+}^{n-\alpha}[D^{n} f(t)] = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, $$

where $f^{(n)} \in L_1[a, b]$, $n = [\alpha] + 1$ and $[\alpha]$ is integer part of $\alpha$.

**Lemma 2.1.** *For a function* $f(x) = (x-a)^{p}$, *let* $p > -1$, $\alpha > 0$. *Then*

$$ I_{a+}^{\alpha} f(t) = \frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} (t-a)^{\alpha+p}. $$

**Lemma 2.2.** *If* $\alpha \in \boldsymbol{R}_{+}$, $n-1 < \alpha \le n$, $n \in \boldsymbol{N}$, $f \in AC^{n}[a, b]$, *then*

$$ I_{a+}^{\alpha}\, {}^{c}D_{a+}^{\alpha} f(t) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (t-a)^{j}. $$

**Definition 2.3.** A function $u(t)$ such that ${}^{c}D_{0+}^{\alpha} u \in C[0, b]$ is called solution of Equation (1) if it satisfies Equation (1).

**Definition 2.4.** Equation (1) is Hyers-Ulam stable if for a function $u(t)$ satisfying the following inequality:

$$ \| {}^{c}D_{0+}^{\alpha} u(t) - f(t, u(t), {}^{c}D_{0+}^{\beta} u(t)) \|_{C[0,b]} \le \varepsilon, \tag{2} $$

and for some $\varepsilon > 0$, there exist a solution $u_0(t)$ of the Equation (1) such that

$$ \| u - u_0 \|_{C[0,b]} \le K(\varepsilon), \tag{3} $$

where $K(\varepsilon)$ is only dependent on $\varepsilon$ and $\| \cdot \|_{C[0,b]}$ is the Chebyshev norm.

**Definition 2.5.** $u(t)$ such that ${}^{c}D_{0+}^{\alpha} u \in C[0, b]$ is called solution of inequality (2) if it satisfies Equation (2).

## 3. Hyers-Ulam Stability of Nonlinear Multi-term Fractional Differential Equation

Let $u(t)$ be the solution of inequality (2). If we define $h(t)$ as below,

$$h(t) := {}^{c}D_{0+}^{\alpha}u(t) - f(t, u(t), {}^{c}D_{0+}^{\beta}u(t)), \ t \in [0, b], \tag{4}$$

then $\|h\|_{C[0,b]} \leq \varepsilon$.

**Lemma 3.1.** *Let us suppose* $u(t)$ *is a solution of inequality* (2). *Then* $\nu(t)$ *such that* ${}^{c}D_{0+}^{\alpha}u(t) = \nu(t)$ *satisfies the following fractional integral equation*:

$$\nu(t) - f(t, u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t), {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}\nu(t)) = h(t), \ t \in [0, b]. \tag{5}$$

*Conversely*, $u(t)$ *such that*

$$u(t) = u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t) \tag{6}$$

*is a solution of inequality* (2), *where* $\nu(t)$ *satisfies Equation* (5) *in* $C[0, b]$.

**Proof.** First, we prove the first part of the lemma.

From the hypothesis of the first part, $u(t)$ is a solution of inequality (2) and

$${}^{c}D_{0+}^{\alpha}u(t) = \nu(t). \tag{7}$$

${}^{c}D_{0+}^{\alpha}u(t) \in C[0, b]$, so we can apply fractional integral operator $I_{0+}^{\alpha}$ to the both sides of the equality (7). Then we have the following equalities:

$$I_{0+}^{\alpha}\,{}^{c}D_{0+}^{\alpha}u(t) = I_{0+}^{\alpha}\nu(t),$$

$$u(t) - u(0) - u'(0)t = I_{0+}^{\alpha}\nu(t),$$

$$u(t) = u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t),$$

$${}^{c}D_{0+}^{\beta}u(t) = {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}\nu(t).$$

By substituting above equalities into (4), Equation (5) is obtained.

Now, we prove the second part of the lemma.

$\nu(t)$ is a solution of Equation (5) and $u(t) = u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t)$. By applying the fractional differential operators ${}^{c}D_{0+}^{\beta}$, ${}^{c}D_{0+}^{\alpha}$ to the both sides of (6), the following equalities are obtained:

$${}^{c}D_{0+}^{\beta}u(t) = {}^{c}D_{0+}^{\beta}(u(0) + u'(0)t) + {}^{c}D_{0+}^{\beta}I_{0+}^{\alpha}\nu(t) = {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}\nu(t), \quad (8)$$

$${}^{c}D_{0+}^{\alpha}u(t) = {}^{c}D_{0+}^{\alpha}(u(0) + u'(0)t) + {}^{c}D_{0+}^{\alpha}I_{0+}^{\alpha}\nu(t) = \nu(t). \quad (9)$$

From the Equation (5) and equalities (7)-(9), we have equality (4). □

**Lemma 3.2.** *Under the assumption*:

$(H_0)$ *There exist* $l_1 > 0$, $l_2 > 0$ *such that* $|f(t, x_1, y_1) - f(t, x_2, y_2)| \le l_1|x_1 - x_2| + l_2|y_1 - y_2|$ *for any* $(x_1, y_1), (x_2, y_2) \in R^2$.

*Then the following nonlinear fractional integral equation has the unique solution in* $C[0, b]$.

$$\nu(t) - f(t, u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t), {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}\nu(t)) = 0, \ t \in [0, b]. \quad (10)$$

**Proof.** Let us consider the following iterative scheme:

$$\nu_{n+1}(t) = f(t, u(0) + u'(0)t + I_{0+}^{\alpha}\nu_n(t), g(t) + I_{0+}^{\alpha-\beta}\nu_n(t)),$$

where $g(t) = {}^{c}D_{0+}^{\beta}(u'(0)t)$.

Indeed, $\{\nu_n(t)\}$ is Cauchy sequence in $C[0, b]$. By considering the assumption $(H_0)$,

$$|\nu_{n+1}(t) - \nu_n(t)|$$

$$= |f(t, u(0) + u'(0)t + I_{0+}^{\alpha}\nu_n(t), g(t) + I_{0+}^{\alpha-\beta}\nu_n(t)) - f(t, u(0) + u'(0)t + I_{0+}^{\alpha}\nu_{n-1}(t), g(t) + I_{0+}^{\alpha-\beta}\nu_{n-1}(t))|$$

$$\leq l_1|I_{0+}^{\alpha}\nu_n(t) - I_{0+}^{\alpha}\nu_{n-1}(t)| + l_2|I_{0+}^{\alpha-\beta}\nu_n(t) - I_{0+}^{\alpha-\beta}\nu_{n-1}(t)|.$$

Now we use the following Bielecki norm which is equivalent to Chebyshev norm.

$$\|x\|_k := \max\{e^{-kt}|x(t)|\},\ x \in C[0, b],\ k > 0.$$

Then the following inequalities hold:

$$|I_{0+}^{\alpha}\nu_n(t) - I_{0+}^{\alpha}\nu_{n-1}(t)| \leq I_{0+}^{\alpha}|\nu_n(t) - \nu_{n-1}(t)| = I_{0+}^{\alpha}e^{-kt}|\nu_n(t) - \nu_{n-1}(t)|e^{kt}$$

$$\leq \|\nu_n - \nu_{n-1}\|_k I_{0+}^{\alpha}e^{kt} \leq \|\nu_n - \nu_{n-1}\|_k \frac{e^{kt}}{k^{\alpha}}.$$

In the same way, we obtain

$$|I_{0+}^{\alpha-\beta}\nu_n(t) - I_{0+}^{\alpha-\beta}\nu_{n-1}(t)| \leq \|\nu_n - \nu_{n-1}\|_k \frac{e^{kt}}{k^{\alpha-\beta}}.$$

Therefore, we have

$$|\nu_{n+1}(t) - \nu_n(t)| \leq \|\nu_n - \nu_{n-1}\|_k e^{kt}\left(\frac{l_1}{k^{\alpha}} + \frac{l_2}{k^{\alpha-\beta}}\right),$$

and by multiplying $e^{-kt}$ to the both sides,

$$\|\nu_{n+1} - \nu_n\|_k \leq \|\nu_n - \nu_{n-1}\|_k\left(\frac{l_1}{k^{\alpha}} + \frac{l_2}{k^{\alpha-\beta}}\right)$$

is obtained.

Now we choose $k_* := k$ such that $p =: \left(\frac{l_1}{k^{\alpha}} + \frac{l_2}{k^{\alpha-\beta}}\right) < 1$, then the following relation holds:

$$\|\nu_{n+1} - \nu_n\|_{k_*} \leq p\|\nu_n - \nu_{n-1}\|_{k_*}.$$

So $\{\nu_n(t)\}$ converges as $n \to \infty$ and the limit $\nu_*(t)$ is the unique solution of Equation (10). □

**Lemma 3.3.** *If $w(t)$ is the unique solution of Equation* (10)*, then the function*

$$u_0(t) := u(0) + u'(0)t + I_{0+}^{\alpha} w(t) \tag{11}$$

*is a solution of Equation* (1)*, where $w(t)$ is the unique solution of Equation* (10).

**Proof.** From the assumption of the lemma, the following equality holds:

$$w(t) - f(t,\, u(0) + u'(0)t + I_{0+}^{\alpha} w(t),\, {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta} w(t)) = 0,\ t \in [0,\, b]. \tag{12}$$

By applying the fractional differential operator ${}^{c}D_{0+}^{\alpha}$, ${}^{c}D_{0+}^{\beta}$ to the both sides of (11):

$${}^{c}D_{0+}^{\beta} u_0(t) = {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta} w(t), \tag{13}$$

$${}^{c}D_{0+}^{\alpha} u_0(t) = w(t). \tag{14}$$

By substituting (11), (13), and (14) into (12), the following equality is obtained:

$${}^{c}D_{0+}^{\alpha} u_0(t) = f(t,\, u_0(t),\, {}^{c}D_{0+}^{\beta} u_0(t)),\ t \in [0,\, b].$$ □

**Theorem 3.4.** *Under the assumption* $(H_0)$, *the Equation* (1) *is Hyers-Ulam stable.*

**Proof.** For any $\varepsilon > 0$, let $u(t)$ be the solution of inequality (2). From the definition of fractional differential inequality, ${}^{c}D_{0+}^{\alpha}u(t)$ exists and so $u(0)$ and $u'(0)$ also exist. Therefore, from the Lemma 3.1, $u(t)$ is denoted as below.

$$u(t) = u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t),$$

where $\nu(t)$ is the solution of Equation (5) and

$$h(t) := {}^{c}D_{0+}^{\alpha}u(t) - f(t, u(t), {}^{c}D_{0+}^{\beta}u(t)), \ t \in [0, b].$$

On one side, from Lemma 3.2, fractional integral equation

$$w(t) - f(t, u(0) + u'(0)t + I_{0+}^{\alpha}w(t), {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}w(t)) = 0, \ t \in [0, b] \tag{15}$$

has the unique solution. So from Lemma 3.3, a function $u_0(t)$ such that

$$u_0(t) := u(0) + u'(0)t + I_{0+}^{\alpha}w(t) \tag{16}$$

is the solution of Equation (1).

Now, we estimate $\|u - u_0\|_{C[0,b]}$. From (6) and (16), we have

$$|u(t) - u_0(t)| = |I_{0+}^{\alpha}\nu(t) - I_{0+}^{\alpha}w(t)|. \tag{17}$$

By using Bielecki norm as in the proof of Lemma 3.2,

$$|I_{a+}^{\alpha}\nu(t) - I_{a+}^{\alpha}w(t)| \le I_{a+}^{\alpha}|\nu(t) - w(t)| = I_{a+}^{\alpha}e^{kt}e^{-kt}|\nu(t) - w(t)| \le \|\nu(t) - w(t)\|_k I_{a+}^{\alpha}e^{kt}.$$

From the inequality, the following inequality holds:

$$|I_{0+}^{\alpha}\nu(t) - I_{0+}^{\alpha}w(t)| \le \|\nu - w\|_k \frac{e^{kt}}{k^{\alpha}}. \tag{18}$$

By multiplying $e^{-kt}$ to the both sides of (18), we obtain

$$\|I_{0+}^{\alpha}\nu - I_{0+}^{\alpha}w\|_k \leq \frac{1}{k^{\alpha}}\|\nu - w\|_k. \tag{19}$$

From (17) and (19),

$$\|u - u_0\|_k = \|I_{0+}^{\alpha}\nu - I_{0+}^{\alpha}w\|_k \leq \frac{1}{k^{\alpha}}\|\nu - w\|_k. \tag{20}$$

From (5) and (15), $|\nu(t) - w(t)|$ is estimated as the following:

$$\begin{aligned}
|\nu(t) - w(t)| \leq\ & |f(t,\ u(0) + u'(0)t + I_{0+}^{\alpha}\nu(t),\ {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}\nu(t)) \\
& - f(t,\ u(0) + u'(0)t + I_{0+}^{\alpha}w(t),\ {}^{c}D_{0+}^{\beta}(u'(0)t) + I_{0+}^{\alpha-\beta}w(t))| + |h(t)| \\
\leq\ & l_1|I_{0+}^{\alpha}\nu(t) - I_{0+}^{\alpha}w(t)| + l_2|I_{0+}^{\alpha-\beta}\nu(t) - I_{0+}^{\alpha-\beta}w(t)| + |h(t)| \\
\leq\ & l_1\|\nu - w\|_k \frac{e^{kt}}{k^{\alpha}} + l_2\|\nu - w\|_k \frac{e^{kt}}{k^{\alpha-\beta}} + |h(t)|,\ t \in [0,\ b].
\end{aligned}$$

Therefore,

$$e^{-kt}|\nu(t) - w(t)| \leq l_1\|\nu - w\|_k \frac{1}{k^{\alpha}} + l_2\|\nu - w\|_k \frac{1}{k^{\alpha-\beta}} + e^{-kt}|h(t)|,$$

$$\|\nu - w\|_k \leq \left(\frac{l_1}{k^{\alpha}} + \frac{l_2}{k^{\alpha-\beta}}\right)\|\nu - w\|_k + \|h\|_k \leq \frac{l_1 + l_2}{k^{\alpha-\beta}}\|\nu - w\|_k + \|h\|_k.$$

Now we can easily investigate there exists $k_* > 0$ such that $q := \dfrac{l_1 + l_2}{k_*^{\alpha-\beta}} > 0$,

$1 - q > 0$, $\dfrac{1}{k_*^{\alpha}}\dfrac{1}{1-q} \leq 1$. Then

$$\|\nu - w\|_{k_*} \leq \frac{1}{1-q}\|h\|_{k_*}, \tag{21}$$

and from (20) and (21), the following inequalities hold:

$$\|u - u_0\|_{k_*} \leq \left(\frac{1}{{k_*}^{\alpha}} \frac{1}{1-q}\right) \|h\|_{k_*} \leq \|h\|_{k_*}. \tag{22}$$

The following inequalities show us the equivalence of Chebyshev norm and Bielecki norm.

$$e^{-kb}\|x\|_{C[0,b]} \leq \|x\|_k \leq \|x\|_{C[0,b]}.$$

Then

$$e^{-k_* b}\|u - u_0\|_{C[0,b]} \leq \|u - u_0\|_{k_*} \leq \|h\|_{k_*} \leq \|h\|_{C[0,b]},$$

and therefore,

$$\|u - u_0\|_{C[0,b]} \leq e^{k_* b}\|h\|_{C[0,b]}.$$

From $\|h\|_{C[0,b]} \leq \varepsilon$, we obtain the following result:

$$\|u - u_0\|_{C[0,b]} \leq K\varepsilon,\ K = e^{k_* b}. \tag{23}$$

(13) shows us the Equation (1) is Hyers-Ulam stable. □

**Theorem 3.5.** *Let* $D_1 := \{k | k > (l_1 + l_2)^{1/(\alpha-\beta)}\}$ *and* $D_2 := \{k | b < \ln (k^{\alpha} - k^{\beta}(l_1 + l_2))^{1/k}\}$. *If* $D_1 \cap D_2 \neq \phi$, *then* (23) *is estimated as the following*:

$$\|u - u_0\|_{C[0,b]} \leq \varepsilon. \tag{24}$$

**Proof.** $D_1 \cap D_2 \neq \phi$, so if we select $k_*$ in $D_1 \cap D_2$, then $1 - q > 0$ from the definition of $D_1$, where

$$q = \frac{l_1 + l_2}{{k_*}^{\alpha-\beta}}.$$

On one side, the following relations hold:

$$
\begin{aligned}
& b < \ln({k_*}^{\alpha} - {k_*}^{\beta}(l_1 + l_2))^{1/k_*} \\
& \Leftrightarrow k_* b < \ln({k_*}^{\alpha} - {k_*}^{\beta}(l_1 + l_2)) \\
& \Leftrightarrow 0 < -k_* b + \ln({k_*}^{\alpha} - {k_*}^{\beta}(l_1 + l_2)) \\
& \Leftrightarrow 1 < e^{-k_* b}({k_*}^{\alpha} - {k_*}^{\beta}(l_1 + l_2)) \\
& \Leftrightarrow 1 < e^{-k_* b}{k_*}^{\alpha}(1 - q) \\
& \Leftrightarrow \frac{1}{{k_*}^{\alpha}} \frac{1}{1 - q} < e^{-k_* b}.
\end{aligned}
$$

From (20) and (21),

$$
\|u - u_0\|_{k_*} \leq \left( \frac{1}{{k_*}^{\alpha}} \frac{1}{1 - q} \right) \|h\|_{k_*} \leq e^{-k_* b} \|h\|_{k_*},
$$

and due to the equivalence of Chebyshev norm and Bielecki norm,

$$
e^{-k_* b} \|u - u_0\|_{C[0, b]} \leq \|u - u_0\|_{k_*} \leq e^{-k_* b} \|h\|_{k_*} \leq e^{-k_* b} \|h\|_{C[0, b]}.
$$

Therefore, we have

$$
\|u - u_0\|_{C[0, b]} \leq e^{-k_* b} e^{k_* b} \|h\|_{C[0, b]} = \|h\|_{C[0, b]},
$$

and by considering $\|h\|_{C[0, b]} \leq \varepsilon,$ inequality (24) holds. □

**Remark 3.6.** In (23), $K$ is a kind of characteristic of Hyers-Ulam stability. Theorem 3.5 shows us $K = 1$ in some strict condition.

## 4. Conclusion

In this paper, we give a criteria for the Hyers-Ulam stability of a nonlinear multi-term fractional differential equation on bounded interval by using fixed point theorem in Banach space, Chebyshev norm

and Bielecki norm. We think this method can be extended to the various differential equations. In the similar way, we can also expect the affirmative results on the Hyers-Ulam stability of fractional impulsive differential equations and fractional differential equations with $p$-Laplacian.

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