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APPROXIMATION IN WEIGHTED LORENTZ SPACES

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Abstract

We define a convolution type transform for the fractional derivatives of the functions in weighted Lorentz spaces and estimate this transform with the best trigonometric approximation in this space.

1. Introduction and Main Result

The convolution type transforms are very convenient in trigonometric approximation theory to build the approximating polynomials. So, we

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need to investigate the relations among these transforms and the sequences of the best approximations in the weighted Lorentz spaces.

A measurable function $\omega : [-\pi, \pi] \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero. Let $\mathbb{T} := [-\pi, \pi]$ and ω be a weight function. Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x)dx. \tag{1}$$

We define the decreasing rearrangement $f_\omega^*(t)$ of $f : \mathbb{T} \to \mathbb{R}$ with respect to the Borel measure (1)

$$f_\omega^*(t) = \inf\{\tau \geq 0 : \omega(x \in \mathbb{T} : |f(x)| > \tau) \leq t\}.$$

The weighted Lorentz space $L_\omega^{pq}(\mathbb{T})$ is defined ([4], p. 20), as

$$L_\omega^{pq}(\mathbb{T}) = \left\{ f \in \mathbf{M}(\mathbb{T}) : \|f\|_{pq,\omega} = \left(\int_{\mathbb{T}} (f^{**}(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{1/q} < \infty, \quad 1 < p, q < \infty \right\},$$

where $\mathbf{M}(\mathbb{T})$ is the set of 2π periodic integrable functions on $\mathbb{T}$ and

$$f^{**}(t) = \frac{1}{t} \int_0^t f_\omega^*(u)du.$$

The weighted Lorentz space $L_\omega^{pq}(\mathbb{T})$ is a Banach space with this norm. If $p = q$, $L_\omega^{pq}(\mathbb{T})$ turns into the weighted Lebesgue space $L_\omega^p(\mathbb{T})$ ([4], p. 20).

A weight function $\omega : \mathbb{T} \to [0, \infty]$ belongs to the Muckenhoupt class A_p [7], $1 < p < \infty$, if

$$\sup \frac{1}{|I|} \int_I \omega(x) dx \left(\frac{1}{|I|} \int_I \omega^{1-p'}(x) dx \right)^{p-1} = C_{A_p} < \infty, \quad p' := \frac{p}{p-1},$$

with a finite constant C_{A_p} independent of I, where the supremum is taken with respect to all intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I. The constant C_{A_p} is called the Muckenhoupt constant of ω.

For $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p, q < \infty$, $\omega \in A_p$, the operator σ_h is defined as

$$(\sigma_h f)(x, u) := \frac{1}{2h} \int_{-h}^{h} f(x + tu) dt, \quad 0 < h < \pi, \; x \in \mathbb{T}, \quad -\infty < u < \infty. \quad (2)$$

Whenever $\omega \in A_p$, $1 < p, q < \infty$, the Hardy-Littlewood maximal function of $f \in L_\omega^{pq}(\mathbb{T})$ belongs to $L_\omega^{pq}(\mathbb{T})$ ([2], Theorem 3). Therefore the operator $\sigma_h f$ belongs to $L_\omega^{pq}(\mathbb{T})$.

Since $L_\omega^{pq}(\mathbb{T}) \subset L^1(\mathbb{T})$ when $\omega \in A_p$, $1 < p, q < \infty$ (see [5], the proof of Proposition 3.3), we can define the Fourier series of $f \in L_\omega^{pq}(\mathbb{T})$. By not loosing of generalization suppose that Fourier series of f is

$$\sum_{r=1}^{\infty} c_r(f) e^{irx} =: \sum_{r=1}^{\infty} A_r(f, x), \quad (3)$$

where $c_r(f)$ is the Fourier coefficients of the function f.

Let $S_n f(x) (n = 0, 1, 2, \ldots)$ be the n-th partial sum of the series (3) at the point x, that is,

$$S_n f(x) := \sum_{r=1}^{\infty} A_r(f, x).$$

If $\alpha > 0$, then α-th *integral* of $f \in L^1(\mathbb{T})$ is defined as

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k(f)(ik)^{-\alpha} e^{ikx},$$

where $(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \, sign\, k}$ and $\mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}$ ([9], p. 347).

For $\alpha \in (0, 1)$, the α-th *fractional derivative* of f is defined by

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x, f),$$

$$f^{(\alpha+r)}(x) := (f^{(\alpha)}(x))^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x, f),$$

if the right hand sides exist, where $r \in \mathbb{Z}^+ := \{1, 2, 3, \ldots\}$ ([9], p. 347).

Let $W^\alpha_{pq, \omega}$, $\alpha > 0$, be the class of functions $f \in L^{pq}_\omega(\mathbb{T})$ such that $f^{(\alpha)} \in L^{pq}_\omega(\mathbb{T})$. $W^\alpha_{pq, \omega}$, $1 < p, q < \infty$, $\alpha > 0$, becomes a Banach space with the norm

$$\|f\|_{W^\alpha_{pq, \omega}} := \|f\|_{L^{pq}_\omega} + \|f^{(\alpha)}\|_{L^{pq}_\omega}.$$

By $E_n(f)_{pq, \omega}$ we denote the best approximation of $f \in L^{pq}_\omega(\mathbb{T})$ by polynomials in $\mathcal{T}_n$, i.e.,

$$E_n(f)_{pq, \omega} = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{pq, \omega},$$

where $\mathcal{T}_n$ is the set of trigonometric polynomials of degree $\le n$.

Since the weighted Lorentz spaces are non-invariant with respect to the usual shift $f(x - hu)$, we define the convolution type transforms by using the mean value function $(\sigma_h f)(x, u)$.

For $f \in L_\omega^{pq}$, we denote the norm of the convolution type transform by $D(f, \mu, h, pq)$:

$$D(f, \mu, h, pq) := \left\| \int_{-\infty}^{\infty} (\sigma_h f)(x, u) d\mu(u) \right\|_{pq, \omega}, \tag{4}$$

where $\mu(u)$ is a real function of bounded variation on the real axis (for the definition of such a function, see ([6], p. 328)).

Throughout this paper, the constant c denotes a generic constant, i.e., a constant whose values can change even between different occurrences in a chain of inequalities. In this paper, we will use the following notation:

$$A(x) \preceq B(x) \Leftrightarrow \exists c > 0 : A(x) \leq cB(x).$$

Theorem 1. *Let* $1 < p, q < \infty, \omega \in A_p, f \in W_{pq, \omega}^{\alpha}$ *and* $\alpha \geq 0.$ *Then for every natural number* m

$$D(f^{(\alpha)}, \mu, h, pq) \preceq \sum_{r=0}^{m} (E_{2^r}(f^{(\alpha)})_{pq, \omega} \cdot \delta_{2^r, h}) + E_{2^{m+1}}(f^{(\alpha)})_{pq, \omega},$$

where

$$\delta_{2^r, h} := \sum_{l=2^r}^{2^{r+1}} |\hat{\mu}(lh) - \hat{\mu}((l+1)h)| + |\hat{\mu}(2^r h)|,$$

$$\hat{\mu}(x) := \int_{-\infty}^{\infty} \frac{\sin ux}{ux} d\mu(u), \quad 0 < h \leq \pi.$$

In Orlicz, weighted Orlicz and weighted Lorentz spaces the similar theorems were proved for $\alpha = 0$ in [3, 8, 10, 11]. In this paper, we obtain this theorem in weighted Lorentz spaces for the fractional derivatives of functions.

Proof of Theorem 1. Let $h \leq 2^{-m-1}$, $f \in W^{\alpha}_{pq,\omega}$ and

$$S_{2^{m+1}} f^{(\alpha)} = \sum_{r=1}^{2^{m+1}} c_r(f^{(\alpha)}) e^{irx}, \tag{5}$$

be the partial sum of Fourier series of the function $f^{(\alpha)} \in L^{pq}_{\omega}(\mathbb{T})$. By (4) and the properties of the norm, we have

$$D(f^{(\alpha)}, \mu, h, pq) = \left\| \int_{-\infty}^{\infty} (\sigma_h f^{(\alpha)})(x, u) d\mu(u) \right\|_{pq,\omega}$$

$$\leq \left\| \int_{-\infty}^{\infty} [(\sigma_h f^{(\alpha)})(x, u) - (\sigma_h S_{2^{m+1}} f^{(\alpha)})(x, u)] d\mu(u) \right\|_{pq,\omega}$$

$$+ \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f^{(\alpha)})(x, u) d\mu(u) \right\|_{pq,\omega}.$$

From ([12], Theorem 1.3), ([5], Proposition 3.2) and the boundedness of the operator σ_h, we get

$$D(f^{(\alpha)}, \mu, h, pq) \preceq \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f^{(\alpha)})(x, u) d\mu(u) \right\|_{pq,\omega} + E_{2^{m+1}}(f^{(\alpha)})_{pq,\omega}.$$

By (2) and (5), we obtain

$$\int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f^{(\alpha)})(x, u) d\mu(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2h} \int_{-h}^{h} S_{2^{m+1}} f^{(\alpha)} (x + tu) dt \right) d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2h} \int_{-h}^{h} \sum_{r=1}^{2^{m+1}} c_r (f^{(\alpha)}) e^{ir(x+tu)} dt \right) d\mu(u)$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2h} \sum_{r=1}^{2^{m+1}} c_r (f^{(\alpha)}) e^{irx} \int_{-h}^{h} e^{irtu} dt \right) d\mu(u)$$

$$= \sum_{r=1}^{2^{m+1}} A_r (f^{(\alpha)}, x) \int_{-\infty}^{\infty} \frac{e^{irhu} - e^{-irhu}}{2irhu} d\mu(u)$$

$$= \sum_{r=1}^{2^{m+1}} A_r (f^{(\alpha)}, x) \hat{\mu}(rh).$$

Therefore, we have

$$D(f^{(\alpha)}, \mu, h, pq) \preceq \left\| \sum_{r=1}^{2^{m+1}} A_r (f^{(\alpha)}, x) \hat{\mu}(rh) \right\|_{pq, \omega} + E_{2^{m+1}} (f^{(\alpha)})_{pq, \omega}. \quad (6)$$

If we put

$$\Delta_{r, \mu} := \sum_{l=2^r}^{2^{r+1}} A_l (f^{(\alpha)}, x) \hat{\mu}(lh),$$

as a consequence of the properties of the norm of the weighted Lorentz space $L_{\omega}^{pq}(\mathbb{T})$, we get

$$\left\| \sum_{r=1}^{2^{m+1}} A_r (f^{(\alpha)}, x) \hat{\mu}(rh) \right\|_{pq, \omega} \preceq \left\| \sum_{r=0}^{m} \Delta_{r, \mu} \right\|_{pq, \omega} \leq \sum_{r=0}^{m} \| \Delta_{r, \mu} \|_{pq, \omega}.$$

If we apply the Abel transform to $\Delta_{r,\mu}$

$$\Delta_{r,\mu} = \sum_{l=2^r}^{2^{r+1}} [S_l f^{(\alpha)}(x) - S_{2^{r+1}} f^{(\alpha)}(x)][\hat{\mu}(lh) - \hat{\mu}((l+1)h)]$$

$$+ [S_{2^{r+1}} f^{(\alpha)}(x) - S_{2^r} f^{(\alpha)}(x)]\hat{\mu}(2^r h).$$

From ([12], Theorem 1.3)

$$\|\Delta_{r,\mu}\|_{pq,\omega} \leq \sum_{l=2^r}^{2^{r+1}} \|S_l f^{(\alpha)}(x) - S_{2^{r+1}} f^{(\alpha)}(x)\|_{pq,\omega} |\hat{\mu}(lh) - \hat{\mu}((l+1)h)|$$

$$+ \|S_{2^{r+1}} f^{(\alpha)}(x) - S_{2^r} f^{(\alpha)}(x)\|_{pq,\omega} |\hat{\mu}(2^r h)|$$

$$\leq E_{2^r}(f^{(\alpha)})_{pq,\omega} S_{2^r,h}.$$

Then

$$\left\| \sum_{r=1}^{2^{m+1}} A_r(f^{(\alpha)}, x)\hat{\mu}(rh) \right\|_{pq,\omega} \preceq \sum_{r=0}^{m} E_{2^r}(f^{(\alpha)})_{pq,\omega} S_{2^r,h}.$$

Now, we have the desired estimate

$$D(f^{(\alpha)}, \mu, h, pq) \preceq \sum_{r=0}^{m} (E_{2^r}(f^{(\alpha)})_{pq,\omega} \cdot \delta_{2^r,h}) + E_{2^{m+1}}(f^{(\alpha)})_{pq,\omega}.$$

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