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SUBHARMONIC SOLUTIONS OF SUBQUADRATIC HAMILTONIAN SYSTEMS AND THE MASLOV-TYPE INDEX THEORY

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Abstract

In this paper, we mainly use the Galerkin approximation method and the iteration inequalities of the Maslov type index theory to study the properties of subharmonic solutions for the Hamiltonian systems $\dot{z}(t) = J\nabla H(t, z(t))$, where $H(t, z) = \frac{1}{2}(\hat{B}(t)z, z) + \hat{H}(t, z)$, $\hat{B}(t)$ is a semipositive symmetric continuous matrix and $\hat{H}$ is unbounded. We prove that these Hamiltonian systems exist subharmonic solutions, and these solutions are pairwise geometrically distinct under certain conditions.

Keywords: subharmonic solution, Maslov-type index, Hamiltonian systems.

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1. Introduction

In this section, we consider the subharmonic solutions of the Hamiltonian systems

$$\dot{z}(t) = J\nabla H(t, z(t)), \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the symplectic matrix, I_n is the unit matrix of order n, $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and $\nabla H(t, z)$ is the gradient of $H(t, z)$ respect to z. Recall that a solution $z(t)$ of (1.1) is called subharmonic if $z(t)$ is kT-periodic for some positive integer k. Given an integer j and a kT-periodic solution z of (1.1), the phase shift $j * z$ of z is defined by

$$(j * z)(t) = z(t + jT).$$

We shall say two solutions z_1 and z_2 are geometrically distinct if

$$j * z_1 \neq l * z_2, \quad \forall j, l \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers.

The generic form of Hamiltonian function H is given by

$$H(t, z) = \frac{1}{2}(\hat{B}(t)z, z) + \hat{H}(t, z),$$

where $\hat{B}(t)$ is a T-periodic symmetric continuous matrix and $\hat{H}$ has subquadratic behaviour, and $\hat{H}$ and $\hat{B}$ satisfy the following conditions:

(h1) There is a constant $M > 0$ such that $|\nabla\hat{H}(t, z)| \leq M$ for all $t \in [0, T]$ and $z \in \mathbb{R}^{2n}$;

(h2) $\hat{H}(t, z) \to +\infty$ as $|z| \to +\infty$ uniformly for $t \in [0, T]$;

(h3) $\hat{B}(t)$ is a T-periodic symmetric continuous matrix, $|\hat{B}|_{C^0} \leq \delta$ for some $\delta \geq 0$, and $\hat{B}(t)$ is semipositive for all $t \in [0, T]$.

We state the main result:

Theorem 1.1. *Suppose that* (h1)-(h3) *hold, then for each positive* k, (1.1) *has a nonconstant* kT*-periodic solution* z_k *such that* z_j *and* z_{pj} *are geometrically distinct for* $p > (i_T(\hat{B}) + \nu_T(\hat{B}) + n + 1)/(i_T(\hat{B}) + \nu_T(\hat{B}) - n + 1)$. *If all* z_k *are nondegenerate, then* z_j *and* z_{pj} *are geometrically distinct for* $p > 1$.

The first result on subharmonic solutions was obtained by Rabinowitz in his pioneer work [17]. Since then, many new contributions have appeared. See, for example, [5, 6, 8, 9, 10, 11, 13] and the references therein. In [6], Ekeland and Hofer proved that under a strict convex condition and a superquadratic condition, the Hamiltonian system (1.1) possesses subharmonic solution z_k for each integer $k \geq 1$ and all of these solutions are pairwise geometrically distinct. In [11], Liu generalized the result of [6] to the nonconvex case conditionally. For the subquadratic Hamiltonian systems, many papers, for example, [17] and the references therein, proved the existence of subharmonic solutions.

2. Preliminaries

In this section, we briefly recall the Maslov-type index theory for symplectic matrix paths. For the systematic statements of this index theory can be found in reference [13].

As usual, the $2n$-dimensional symplectic group $Sp(2n)$ is defined by

$$Sp(2n) = \{M \in \mathscr{L}(\mathbb{R}^{2n}) | M^T JM = J\},$$

where $\mathscr{L}(\mathbb{R}^{2n})$ is the set of all real $2n \times 2n$ matrices, M^T is the transpose of matrix M. Denote by $\mathscr{L}_s(\mathbb{R}^{2n})$ the subset of $\mathscr{L}(\mathbb{R}^{2n})$ consisting of symmetric matrices.

In the case of linear Hamiltonian systems

$$\dot{y} = JB(t)y, \quad \forall y \in \mathbb{R}^{2n},$$

where $B \in C(\mathbb{R}, \mathscr{L}_s(\mathbb{R}^{2n}))$ is T-periodic. Its fundamental solution $\gamma = \gamma_B$ is a symplectic path starting from identity matrix I_{2n}, i.e., $\gamma = \gamma_B \in \mathcal{P}(2n)$ with $\mathcal{P}(2n) = \{\gamma \in C([0, T], Sp(2n)) | \gamma(0) = I_{2n}\}$.

In the study of periodic solutions of Hamiltonian systems, an index theory for such symplectic path was introduced by Conley and Zehnder in [4] for nondegenerate elements in $\mathcal{P}(2n)$ with $n \geq 2$, the $n = 1$ case of the nondegenerate elements in $\mathcal{P}(2n)$ was introduced by Long and Zehnder in [15], the general case for any symplectic path in $\mathcal{P}(2n)$ was introduced by Long in [14] and Viterbo in [18]. We call this index theory the Maslov-type index theory and denote it by

$$(i_1, \nu_1) = (i_1(\gamma), \nu_1(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\},$$

where $i_1 = i_1(\gamma)$ is the index part or rotation number of γ and $\nu_1 = \nu_1(\gamma)$ is the nullity.

The Maslov-type index of $\gamma = \gamma_B$ on the interval $[0, T]$ is also called the Maslov-type index of the matrix function B. In this case the nullity is defined by

$$\nu_T = \dim \ker(\gamma(T) - I_{2n}).$$

We know that B is also a kT-periodic matrix function, thus its fundamental solution can be extended to the interval $[0, kT]$. In fact, we define a symplectic path $\gamma^k : [0, kT] \to Sp(2n)$ by

$$\gamma^k(t) = \gamma(t - jT)\gamma(T)^j \text{ for } jT \leq t \leq (j+1)T, \quad 0 \leq j \leq k-1.$$

We denote the corresponding Maslov-type index of γ^k on the interval $[0, kT]$ by

$$(i_{kT}, \nu_{kT}) = (i_{kT}(\gamma), \nu_{kT}(\gamma)) \equiv (i_{kT}(\gamma^k), \nu_{kT}(\gamma^k)).$$

3. Proof of Theorem 1.1

Consider the Hamiltonian systems

$$\begin{cases} \dot{z}(t) = J\nabla H(t, z(t)), \\ z(0) = z(T). \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that* (h1)-(h3) *hold, then* (3.1) *has a nonconstant T-periodic solution z whose Maslov-type pair satisfies*

$$i_T(z) \le i_T(\hat{B}) + \nu_T(\hat{B}) + 1 \le i_T(z) + \nu_T(z).$$

In order to prove Theorem 3.1, we need the following arguments. Let $S_T = \mathbb{R} / (TZ)$. Denote $E = W^{1/2, 2}(S_T, \mathbb{R}^{2n})$ the Sobolev space of all $z(t)$ in $L^2(S_T, \mathbb{R}^{2n})$ whose Fourier series

$$z(t) = \sum_{k=-\infty}^{k=+\infty} \exp(2k\pi tJ / T)z_k, \; z_k \in \mathbb{R}^{2n},$$

satisfies

$$\|z\|_E^2 \equiv T|z_0|^2 + T\sum_{k=-\infty}^{k=+\infty} |k| \cdot |z_k|^2 < +\infty.$$

The inner product of E is defined by

$$(z_1, z_2)_E = T(z_0^1, z_0^2) + T\sum_{k=-\infty}^{k=+\infty} |k|(z_k^1, z_k^2),$$

where $z(t) = \sum_{k=-\infty}^{k=+\infty} \exp(2k\pi tJ / T)z_k^i, \; i = 1, 2.$

Denote the linear operator A and $\hat{B}$ on E by extending the bilinear form

$$(Au, v)_E = \int_0^T (-J\dot{u}, v)dt, \; (\hat{B}u, v)_E = \int_0^T (\hat{B}u, v)dt.$$

Then A is a bounded self-adjoint linear operator and $\hat{B}$ is compact self-adjoint linear operator. Denote the eigenvalues of self-adjoint operator $A - \hat{B}$ by

$$\cdots \le \lambda_2' \le \lambda_1' < 0 < \lambda_1 \le \lambda_2 \le \cdots.$$

Let $\{e_j'\}$ and $\{e_j\}$ be eigenvectors of $A - \hat{B}$ corresponding to λ_j' and λ_j, respectively. For $m > 0$, set

$$E_m^+ = span\{e_1, \cdots, e_m\}, \; E_m^- = span\{e_1', \cdots, e_m'\},$$

$$E^0 = \ker(A - \hat{B}), \; E_m = E_m^+ \oplus E_m^- \oplus E^0,$$

let $P_m : E \to E_m$ be the orthogonal projection. Then $\Gamma = \{P_m, m = 0, 1, \cdots\}$ is a Galerkin approximation frame with respect to $A - \hat{B}$.

For $d > 0$, we denote by $M_d^*(\cdot), * = +, 0, -$ the eigenspaces corresponding to the eigenvalues λ belong to $[d, +\infty), (-d, d)$ and $(-\infty, -d]$, respectively. And denote by $M^*(\cdot), * = +, 0, -$ the eigenspaces corresponding to the eigenvalues λ belong to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$, respectively. For any adjoint operator Q, we denote $Q^\sharp = (Q|_{\operatorname{Im} Q})^{-1}$, and we also denote $P^m Q P^m = (P^m Q P^m)|_{W_L^m}$ (see [11]).

Theorem 3.2. *For any continuous T-periodic symmetric matrix function* $B(t)$ *with the Maslov-type index pair* $(i_T(B), \nu_T(B))$, *there exists an* $m_0 > 0$ *such that for* $m > m_0$,

$$\dim M_d^+(P_m(A-B)P_m) = m - i_T(B) - \nu_T(B) + i_T(\hat{B}) + \nu_T(\hat{B}),$$

$$\dim M_d^-(P_m(A-B)P_m) = m - i_T(\hat{B}) + i_T(B),$$

$$\dim M_d^0(P_m(A-B)P_m) = \nu_T(B).$$

We also need the following two iteration inequalities (see [11-13]).

Theorem 3.3. *For any* $k \in \mathbb{N}$, *and any continuous T-periodic symmetric matrix function* $B(t)$ *with the Maslov-type index pair* $(i_T(B), \nu_T(B))$, *there hold*

$$k(i_T(B) + \nu_T(B) - n) + n - \nu_T(B) \leq i_{kT}(B) \leq k(i_T(B) + n) - n - \nu_{kT}(B) + \nu_T(B),$$

$$k(i_T(B) + \nu_T(B) - n) - n \leq i_{kT}(B) \leq k(i_T(B) + n) + n - \nu_{kT}(B).$$

Define a function $\hat{\varphi} \in C^2(E, \mathbb{R})$ as above, i.e.,

$$\hat{\varphi}(z) = \frac{1}{2}(Az, z)_E - \int_0^T H(t, z(t))dt$$

$$= \frac{1}{2}((A - \hat{B})z, z)_{\mathrm{E}} - \int_0^T \hat{H}(t, z(t))dt, \ \forall z \in E.$$

Let $\hat{\varphi}_m = \hat{\varphi}|_{E_m}$ be the restriction of $\hat{\varphi}$ on E_m. Similar to [8], we have the following two lemmas.

Lemma 3.1. *For every* $m > 1$, $\hat{\varphi}_m$ *satisfies* $(PS)_c$ *condition for every* $c \in \mathbb{R}$, *i.e., if* $\{z_m\} \subset X^m$ *satisfies* $\hat{\varphi}'_m(z_m) \to 0$ *and* $\hat{\varphi}_m(z_m) \to c$, *then* $\{z_m\}$ *has a convergent subsequence.*

Lemma 3.2. *$\hat{\varphi}$ satisfies $(PS)_c^*$ condition for $c \in \mathbb{R}$. i.e., each sequence $\{z_m\}$ such that $z_m \in X^m$, $\hat{\varphi}'_m(z_m) \to 0$ and $\hat{\varphi}_m(z_m) \to c$ possesses a convergent subsequence in X.*

In order to prove Theorem 3.1, we need the following definition and the saddle-point theorem (see [7]).

Definition 3.1. Let E be a C^2-Riemannian manifold and D be a closed subset of E. A family of subset of E, $\phi(\alpha)$ is said to be a homological family of dimensional q with boundary D if for some nontrivial class $\alpha \in H_q(E, D)$. The family $\phi(\alpha)$ is defined by

$$\phi(\alpha) = \{G \subseteq E : \alpha \text{ is in the image of } i_* : H_q(G, D) \to H_q(E, D)\},$$

where i_* is the homomorphism induced by the immersion $i : G \to E$.

Theorem 3.4. *For above E, D and α, let $\phi(\alpha)$ be a homological family of dimension q with boundary D, suppose that $f \in C^2(E, \mathbb{R})$ satisfies (PS) condition. Define*

$$c = \inf_{G \in \phi(\alpha)} \sup_{x \in G} f(x).$$

Suppose $\sup_{x \in D} f(x) < c$ and f' is Fredholm on

$$\mathscr{K}_c(f) \equiv \{x \in E : f'(x) = 0,\ f(x) = c\}.$$

Then there exists an $x \in \mathscr{K}_c(f)$ such that the Morse indices $m^-(x)$ and $m^0(x)$ of the functional f at x satisfy

$$m^-(x) \leq q \leq m^-(x) + m^0(x).$$

It is clear that a critical point of $\hat{\varphi}$ is a solution of (3.1). For a critical point $z = z(t)$, we define the linearized system at z by $\dot{y}(t) = JH''(t, z(t))y(t)$. Let $B(t) = H''(t, z(t))$, then the Maslov-type index pair of z is defined by $(i_T(z), \nu_T(z)) = (i_T(B), \nu_T(B))$.

Proof of Theorem 3.1. We also prove this result in 3 steps.

Step 1. The critical points of $\hat{\varphi}_m$.

Set $U_m = E_m^- \oplus E^0$, $V_m = E_m^+$. Then $\dim U_m = m + \dim E^0 = m + \nu_T(\hat{B})$, $\dim V_m = m$. In the following, we prove that $\hat{\varphi}_m(z)$ satisfies:

(1) $\hat{\varphi}_m(z) \geq \hat{\beta}_1 > 0$, $\forall z \in \partial B_\rho(0) \cap V_m$;

(2) $\hat{\varphi}_m(z) < \hat{\xi}_1 < \hat{\beta}_1$, $\forall z \in \partial Q_m$, where $Q_m = \{re | r \in [0, r_1]\} \oplus (B_{r_2}(0) \cap U_m)$, $e \in V_m \cap \partial B_1(0)$, $r_1 > \rho$, $r_2 > 0$.

First we prove (1). Take $z \in V_m$, we have that $|\hat{H}(t, z)| \leq a + b|z|$ for some constants a, b and all $z \in \mathbb{R}^{2n}$, then

$$\hat{\varphi}_m(z) = \frac{1}{2}((A - \hat{B})z, z)_E - \int_0^T \hat{H}(t, z(t))dt$$

$$\geq \frac{\lambda_1}{2}\|z\|_E^2 - \int_0^T \hat{H}(t, z(t))dt$$

$$\geq \frac{\lambda_1}{2}\|z\|_E^2 - \int_0^T (a + b|z|)dt$$

$$\geq \frac{\lambda_1}{2}\|z\|_E^2 - \hat{b}_1\|z\|_E - aT,$$

where $\hat{b}_1 = \hat{a}_1 b$. Choose $\rho > 0$ large enough such that

$$\frac{\lambda_1}{2}\rho^2 - \hat{b}_1\rho - aT \geq \hat{\beta}_1 > 0,$$

which are independent of m.

Now we prove (2). Let $z = z^0 + z^- \in U_m$, denote $M_0 = \|A - \hat{B}\|_E$, then

$$\hat{\varphi}_m(z + re) = \frac{1}{2}((A - \hat{B})(z + re), z + re)_E - \int_0^T \hat{H}(t, z + re)dt$$

$$= \frac{1}{2}((A - \hat{B})z^-, z^-)_E + \frac{r^2}{2}((A - \hat{B})e, e)_E - \int_0^T \hat{H}(t, z + re)dt$$

$$\leq \frac{\lambda_1'}{2}\|z^-\|_E^2 + \frac{M_0}{2}r^2 - \int_0^T \hat{H}(t, z^0)dt - \int_0^T [\hat{H}(t, z + re) - \hat{H}(t, z^0)]dt,$$

while

$$\left|\int_0^T \hat{H}(t, z + re)dt - \int_0^T \hat{H}(t, z^0)dt\right| \leq \int_0^T |\hat{H}(t, z + re) - \hat{H}(t, z^0)|dt$$

$$\leq \int_0^T \int_0^1 |\nabla \hat{H}(t, z^0 + sW)| \cdot |W| ds dt$$

$$\leq M\int_0^T |W|dt \quad \leq \quad \hat{M}_1\|W\|_E,$$

where $\hat{M}_1 = \hat{a}_1 M$, $W = z^- + re$. Then

$$\hat{\varphi}_m(z + re) \leq \frac{\lambda_1'}{2}\|z^-\|_E^2 + \frac{M_0}{2}r^2 - \int_0^T \hat{H}(t, z^0)dt + \hat{M}_1\|W\|_E$$

$$= \frac{\lambda_1'}{2}\|z^-\|_E^2 + \frac{M_0}{2}r^2 - \int_0^T \hat{H}(t, z^0)dt + \hat{M}_1\|z^-\|_E + \hat{M}_1 r.$$

We can choose large enough $r_1, r_2 > \rho$ independent of m such that

$$\hat{\varphi}_m < \hat{\xi}_1 < \hat{\beta}_1, \ \forall z \in \partial Q_m.$$

Step 2. Since Q_m is deformation retract of E_m, there holds $H_q(Q_m, \partial Q_m) \cong H_q(E_m, \partial Q_m)$, where $q = \dim U_m + 1 = m + \nu_T(\hat{B}) + 1 = \dim Q_m$, and ∂Q_m is the boundary of Q_m in $U_m \oplus \{\mathbb{R}e\}$. But $H_q(Q_m, \partial Q_m) \cong H_{q-1}(S^{q-1}) \cong \mathbb{R}$. Denote by $i : Q_m \to P_m E$ the inclusion map. Let $\alpha = [Q_m] \in H_q(Q_m, D)$ be a generator. Then $i_* \alpha$ is nontrivial in $H_q(P_m E, \partial Q_m)$, and $\phi(i_* \alpha)$ defined by Definition 3.1 is a homological family of dimension q with boundary $D := \partial Q_m$ and $Q_m \in \phi(i_* \alpha)$. ∂Q_m and $B_\rho(0) \cap V_m$ are homologically link. By Lemma 3.1, $\hat{\varphi}_m$ satisfies $(PS)_c$ condition for $c \in \mathbb{R}$. Define $\hat{c}_m = \inf_{G \in \phi(i_* \alpha)} \sup_{z \in G} \hat{\varphi}_m(z)$. We have

$$\sup_{z \in \partial Q_m} \hat{\varphi}_m(z) < \hat{\xi}_1 < \hat{\beta}_1 \le \hat{c}_m \le \sup_{z \in Q_m} \hat{\varphi}_m(z) \le \frac{M_0}{2} r_1^2.$$

Since E_m is finite dimensional, $\hat{\varphi}'_m$ is Fredholm. By Theorem 3.4, $\hat{\varphi}_m$ has a critical point $\hat{z}_m$ with critical value $\hat{c}_m$, and Morse indices $m^-(\hat{z}_m)$ and $m^0(\hat{z}_m)$ of $\hat{z}_m$ satisfy

$$m^-(\hat{z}_m) \le m + \nu_T(\hat{B}) + 1 \le m^-(\hat{z}_m) + m^0(\hat{z}_m).$$

Since $\{\hat{c}_m\}$ is bounded, passing to a subsequence, suppose $\hat{c}_m \to \hat{c} \in [\hat{\beta}_1, \frac{M_0}{2} r_1^2]$. By $(PS)_c^*$ condition, passing to a subsequence, there exists an $\hat{z} \in E$ such that

$$\hat{z}_m \to \hat{z}, \ \hat{\varphi}(\hat{z}) = \hat{c}, \ \hat{\varphi}'(\hat{z}) = 0.$$

Then $\hat{z}$ is a solution of (3.1). Since $\hat{c} > 0$, $\hat{z}$ is nonconstant.

Step 3. Let $B(t) = H''(t, \hat{z}(t))$, $d = \frac{1}{4}\|(A-B)^{\sharp}\|_E$. Since

$$\|\hat{\varphi}''(x) - (A-B)\|_E \to 0 \text{ as } \|x - \hat{z}\|_E \to 0,$$

there exists a $r_3 > 0$ such that

$$\|\hat{\varphi}''(x) - (A-B)\|_E < \frac{1}{4}d, \ \forall x \in V_{r_3}(\hat{z}) = \{x \in E | \ \|x - \hat{z}\|_E \le r_3\}.$$

Then for m large enough, there holds

$$\|\hat{\varphi}''_m(x) - P_m(A-B)P_m\|_E < \frac{1}{2}d, \ \forall x \in V_{r_3}(\hat{z}) \cap E_m.$$

For $x \in V_{r_3}(\hat{z}) \cap E_m$, $\forall u \in M_d^-(P_m(A-B)P_m) \setminus \{0\}$, from above we have

$$\begin{aligned}(\hat{\varphi}''_m(x)u, u)_E &\le (P_m(A-B)P_m u, u)_E + \|\hat{\varphi}''_m(x) - P_m(A-B)P_m\|_E \cdot \|u\|_E^2 \\ &\le -\frac{1}{2}d\|u\|_E^2 < 0.\end{aligned}$$

Thus

$$\dim M^-(\hat{\varphi}''_m(x)) \ge \dim M_d^-(P_m(A-B)P_m), \ \forall x \in V_{r_3}(\hat{z}) \cap E_m.$$

Similarly, we have

$$\dim M^+(\hat{\varphi}''_m(x)) \ge \dim M_d^+(P_m(A-B)P_m), \ \forall x \in V_{r_3}(\hat{z}) \cap E_m.$$

By Theorem 3.2 and above inequalities, for large m, we have

$$\begin{aligned} m + \nu_T(\hat{B}) + 1 &\ge m^-(\hat{z}_m) \\ &\ge \dim M_d^-(P_m(A-B)P_m) \\ &= m - i_T(\hat{B}) + i_T(B). \end{aligned}$$

So we have $i_T(\hat{z}) \le i_T(\hat{B}) + \nu_T(\hat{B}) + 1,$ and we also have

$$m + \nu_T(\hat{B}) + 1 \le m^-(\hat{z}_m) + m^0(\hat{z}_m)$$

$$\le \dim M_d^-(P_m(A-B)P_m) \oplus \dim M_d^0(P_m(A-B)P_m)$$

$$= m + i_T(B) + \nu_T(B) - i_T(\hat{B}).$$

So we have $i_T(\hat{z}) + \nu_T(\hat{z}) \ge i_T(\hat{B}) + \nu_T(\hat{B}) + 1.$ Combining the above two inequalities, we have

$$i_T(\hat{z}) \le i_T(\hat{B}) + \nu_T(\hat{B}) + 1 \le i_T(\hat{z}) + \nu_T(\hat{z}).$$

□

We also use the following lemma:

Lemma 3.3. *Suppose* $\hat{B}(t)$ *is a T-periodic semipositive symmetric continuous matrix, then* $i_T(\hat{B}) + \nu_T(\hat{B}) \ge n$.

Proof of Theorem 1.1. For every positive integer k, $H(t, z)$ is a kT-periodic in t. By Theorem 3.1, (3.1) has a kT-periodic solution z_k whose Maslov-type index pair satisfies

$$i_T(z_k) \le i_T(\hat{B}) + \nu_T(\hat{B}) + 1 \le i_T(z_k) + \nu_T(z_k). \tag{3.2}$$

If z_j and z_{pj} are the same geometrically, there exists integer l and m such that

$$l * z_j = m * z_{pj},$$

where $(l * z_j)(t) = z_j(t + lT)$. By Lemma 4.1 of [11],

$$i_{jT}(l * z_j) = i_{jT}(z_j),\ \nu_{jT}(l * z_j) = \nu_{jT}(z_j),$$

$$i_{pjT}(m * z_{pj}) = i_{pjT}(z_{pj}),\ \nu_{pjT}(m * z_{pj}) = \nu_{pjT}(z_{pj}).$$

Take $k = j,\ pj$ in (3.2), respectively, we have

$$i_{pjT}(z_{pj}) \le i_T(\hat{B}) + \nu_T(\hat{B}) + 1 \le i_{jT}(z_j) + \nu_T(z_j).$$

Thus by above equations,

$$\begin{aligned} i_T(\hat{B}) + \nu_T(\hat{B}) + 1 \ge i_{pjT}(z_{pj}) &= i_{pjT}(m * z_{pj}) \\ &\ge p(i_{jT}(l * z_j) + \nu_T(l * z_j) - n) - n \\ &= p(i_{jT}(z_j) + \nu_{jT}(z_j) - n) + n \\ &\ge p(i_T(\hat{B}) + \nu_T(\hat{B}) + 1 - n) + n. \end{aligned}$$

Since $\hat{B}(t)$ is semipositive for all $t \in [0, T]$, by Lemma 3.3,

$$i_T(\hat{B}) + \nu_T(\hat{B}) + 1 - n > 0.$$

Hence $p \le (i_T(\hat{B}) + \nu_T(\hat{B}) + n + 1) / (i_T(\hat{B}) + \nu_T(\hat{B}) - n + 1)$.

In the case of all z_k are nondegenerate, if z_j and z_{pj} are the same geometrically, then

$$\begin{aligned} i_T(\hat{B}) + \nu_T(\hat{B}) + 1 &\ge i_{pjT}(z_{pj}) \\ &\ge p(i_{jT}(z_j) + \nu_{jT}(z_j) - n) + n - \nu_{jT}(z_j) \\ &\ge p(i_T(\hat{B}) + \nu_T(\hat{B}) + 1 - n) + n. \end{aligned}$$

Then $p \le 1$. □

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