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THE LIGHT EDGES IN TOROIDAL GRAPHS WITHOUT ADJACENT 3- AND 4-CYCLES

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Abstract

A subgraph of a graph is light if the sum of the degrees of the vertices of the subgraph in the graph is small. In this paper, we show that every toroidal graph with minimum vertex degree 3 and without adjacent 3- and 4-cycles has a light edge with the weight not greater than 9.

Keywords: toroidal graph, light edges, adjacent triangles.

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1. Introduction

All graphs considered are finite and simple. Let $G = (V, E, F)$ be a graph with V, E, and F being the set of vertices, edges, and faces of G, respectively. The weight $w(e)$ of an edge $e = uv$ is defined by $d_G(u) + d_G(v)$.

The research on graph theory particularly deals with the structural properties of graphs. A subgraph of a graph is light if the sum of the degrees of the vertices of the subgraph in the graph is small. The study and research of the lightness of graphs is a beneficial stage when considers graph coloring and other problems. The light subgraphs are often reducible if considered in a minimal counterexample. See [1, 2, 3] for a survey.

A torus is a closed surface (compact, connected 2-manifold without boundary) that is a sphere with a unique handle, and a toroidal graph is a graph embeddable in the torus. For a toroidal graph G, we still use G to denote an embedding of G in the torus.

Let $\mathcal{G}$ denote the set of toroidal graphs without adjacent 3- and 4-cycles. In this article, we focus on the light edges in $\mathcal{G}$ and give the following Theorem 1.

Theorem 1. *Let G be a connected graph with* $\delta(G) \geq 3$ *in* $\mathcal{G}$. *Then there is an edge* $e \in E(G)$ *with* $w(e) \leq 9$.

We use $N_G(v)$ and $d_G(v)$ to denote the set and number of vertices adjacent to a vertex v, respectively, and use $\delta(G)\,(\Delta(G))$ to denote the minimum (maximum) degree of G. A face f is incident with all vertices and edges on $b(f)$. The *degree* of a face f of G, denoted also by $d_G(f)$, is the length of its boundary walk, where cut edges are counted twice. A vertex (face) of degree k is called a k-vertex (k-face). If $r \leq k$ or $1 \leq k \leq r$, then a k-vertex (k-face) is called an r^+- or r^--vertex (r^+- or r^--face), respectively. A k-cycle is a cycle with k edges. For a vertex $v \in V(G)$, let $n_i(v)$ denote the number of i-vertices adjacent to v for $i \geq 1$, and $m_j(v)$ the number of j-faces incident with v for $j \geq 1$. For a face $f \in F(G)$, let $n_i(f)$ denote the number of i-vertices incident with f for $i \geq 2$.

2. Proof of the Theorems

Proof of Theorem 1. Assume to the contrary that the theorem is false. Let G be a connected graph with $\delta(G) \geq 3$ in $\mathcal{G}$. Then G contains no adjacent triangles and no adjacent 3- and 4-cycles, and $w(e) \geq 10$ for any edge in $E(G)$. The Euler's formula $|V| + |F| - |E| = 0$ can be rewritten in the following form:

$$\sum_{v \in V(G) \cup F(G)} \{d_G(v) - 4\} = 0. \tag{1}$$

Let ω be a weight on $V(G) \cup F(G)$ by defining $\omega(x) = d_G(x) - 4$ if $x \in V(G) \cup F(G)$. To prove Theorem 1, we will introduce some rules to transfers weights between the elements of $V(G) \cup F(G)$ so that the total sum of the weights is kept constant while the transferring is in progress. If we obtain a new nonnegative weight $\omega^*(x)$ for all $x \in V \cup F$ and some positive weight for some $x \in V \cup F$ by transferring weights from one element to another, then we have $0 = \sum_{x \in V \cup F} \omega(x) = \sum_{x \in V \cup F} \omega^*(x) > 0$. This contradiction will complete the proof.

Our transferring rules are as follows:

(R_1) Each 7^+-vertex gives charge $\frac{1}{3}$ to each incident 3-vertex.

(R_2) Each 7^+-vertex gives charge $\frac{1}{2}$ to each incident 3-face.

(R_3) Each 5 or 6-vertex gives charge $\frac{1}{2}$ to each incident 3-face.

For G contains no adjacent triangles and the weight of any edge is greater than 10, we have:

(O_1) $m_3(v) \le \left\lfloor \frac{d(v)}{2} \right\rfloor$;

(O_2) for any edge with two endvertices u and v, if $d(u) = 3$, then $d(v) \ge 7$;

(O_3) $n_3(v) + m_3(v) \le d(v)$ for any $v \in V(G)$.

Let v be a k-vertex of G.

While $k = 4$, then $\omega^*(v) = \omega(v) = 0$.

While $k = 3$. By (O_2), $n_{7^+}(v) \ge 3$. Then $\omega^*(v) = \omega(v) - 1 + 3 \cdot \frac{1}{3} = 0$ by (R_1).

While $k = 5$, then $m_3(v) \le 2$ by (O_1), so $\omega^*(v) \ge \omega(v) - 2 \cdot \frac{1}{2} = 0$ by (R_3).

While $k = 6$, then $m_3(v) \le 3$ by (O_1), so

$$\omega^*(v) \ge \omega(v) - 3 \cdot \frac{1}{2} = \frac{1}{2} > 0, \tag{2}$$

by (R_3).

While $k \ge 7$, then $n_3(v) + m_3(v) \le d(v)$ for any $v \in V(G)$ by (O_3), we have

$$\omega^*(v) \ge \omega(v) - n_3(v) \cdot \frac{1}{3} - m_3(v) \cdot \frac{1}{2} \ge d(v) - 4 - \frac{1}{3} d(v) + \frac{1}{3} m_3(v) - \frac{1}{2} m_3(v)$$

$$\ge \frac{7d(v) - 48}{12} > 0 \tag{3}$$

by (R_1) and (R_2).

Now let f be a face with $d(f) = h$.

If $h \geq 4$, then $\omega^*(f) \geq 0$.

If $h = 3$, then $n_3(f) \leq 1$ by $w(e) \geq 10$ for every edge $e \in E(G)$. If $n_3(f) = 0$, that is, f must be incident with at least two 5^+-vertices, then $\omega^*(f) \geq \omega(f) + 2 \cdot \frac{1}{2} = 0$ by (R_3). If $n_3(f) = 1$, then the degree of the other two vertices incident with f is at least 7, so $\omega^*(f) = \omega(f) + 2 \cdot \frac{1}{2} = 0$ by (R_2).

Now, we get that $\omega^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. If $\sum_{x \in V(G) \cup F(G)} \omega^*(x) > 0$, we are done. Assume that $\sum_{x \in V(G) \cup F(G)} \omega^*(x) = 0$. Then, by Equations (2) and (3), G contains neither 7^+-vertices nor 6-vertices, and no 4-vertices consequently, G also contains no 3-vertices because $w(e) \geq 10$ for any $e \in E(G)$. What's more, G contains no 5^+-face. But now we only have a 5-vertex with all its incident faces are 3- or 4-faces, this contradict completes the proof. □

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