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CONVERGENCE OF A SPECIAL SET OF TRIANGLES

Dedicated to Sherman Kopald Stein (11 August 1926-…) for his pleasant sense of humor and mathematical wisdom that the third author enjoyed as a colleague during the late 1980s at the University of California, Davis

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Abstract

An explicit expression for the convergence point of a given set of triangles is introduced leading to an open interval locus of convergence that contains the centroid of the triangles.

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1. Introduction

Using coordinate systems such as rectangular, polar, spherical and cylindrical require an origin but at times we need to locate points relative to existing points or local coordinates rather than the origin [1-4].

Mobius assigned coordinates (barycentric coordinates) to the point at which the center of gravity, or barycentre, was located, reflecting the ratio of weights attached to the ends of a given rod. He then extended this idea to a system of three weighted points, forming a triangle and pointing out that the center of gravity will remain the same if the weights are increased or decreased by a common factor [5, 6].

Central to barycentric coordinates are the concept of concurrency of lines in a triangle and Ceva's theorem that "Given any triangle, the segments from each vertex to the opposite sides of the triangle are concurrent precisely when the product of the ratios of the pairs of segments formed on each side of the triangle is equal to 1" [5, 7].

Ceva's theorem is used in the proofs of many well-known theorems including the theorem of Napolean's point, the theorem of Fermat's point, and the theorem of the Nagel point [8]. Barycentric coordinates are a fundamental tool for a wide variety of applications employing triangular meshes, underlining the definition of higher-order basis functions, the Bézier triangle in computer aided-design, and many interpolation and shading techniques in computer graphics [9, 10].

In the present paper, we use the convergence of a special set of triangles leading to the discovery of an explicit expression for the convergence point and the understanding that this point can be constructed using a compass and a straightedge ruler, even if the convergence process is infinite. These results are then generalized to Pappus triangles and generalized Pappus triangles [11-13]. Finally, we obtain an interesting open interval locus of convergence that contains the centroid of the triangle.

2. Terminology and Notation

The interior point D of the triangle ΔABC (Figure 1)

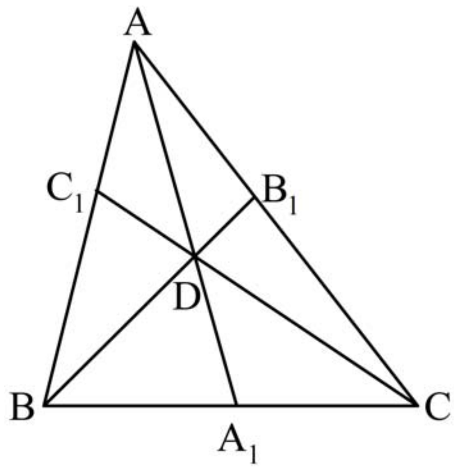

Figure 1. Representation of the point D.

shall be represented by two methods.

(a) Using the barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$, we obtain

$$x_D = \lambda_1 x_A + \lambda_2 x_B + \lambda_3 x_C,$$

$$y_D = \lambda_1 y_A + \lambda_2 y_B + \lambda_3 y_C, \text{ where}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \text{ and}$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0.$$

(b) Using the Ceva coordinates (α, β, γ), we obtain

AA_1, BB_1, and CC_1 which are three cevians that pass through D so that $\frac{BA_1}{A_1C} = \alpha$, $\frac{CB_1}{B_1A} = \beta$, and $\frac{AC_1}{C_1B} = \gamma$, where

$$\alpha\beta\gamma = 1, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0.$$

Known are the relations between the barycentric coordinates and the Ceva coordinates

$$\left.\begin{aligned} \lambda_1 &= \frac{\alpha\beta}{\alpha\beta + \alpha + 1} \\ \lambda_2 &= \frac{1}{\alpha\beta + \alpha + 1} \\ \lambda_3 &= \frac{\alpha}{\alpha\beta + \alpha + 1} \end{aligned}\right\}, \tag{1}$$

$$\left.\begin{aligned} \alpha &= \frac{\lambda_3}{\lambda_2} \\ \beta &= \frac{\lambda_1}{\lambda_3} \\ \gamma &= \frac{\lambda_2}{\lambda_1} \end{aligned}\right\}. \tag{2}$$

3. The Convergence Process

The convergence process used here are presented in two phases.

Case I: The Pappus configuration

Select the points $C_1, B_1, A_1,$ respectively, on the sides *CA*, *BC*, and *AB* so that $\frac{AA_1}{A_1B} = \frac{BB_1}{B_1C} = \frac{CC_1}{C_1A} \equiv \alpha.$ Similarly, select the points $C_2, B_2, A_2,$ respectively, on the sides C_1A_1, B_1C_1 and A_1B_1 so that $\frac{A_1A_2}{A_2B_1} = \frac{B_1B_2}{B_2C_1} = \frac{C_1C_2}{C_2A_1} \equiv \alpha.$

Repeating this process $(N+1)$-times (N is a whole number), we arrive at the points $C_{N+1}, B_{N+1}, A_{N+1},$ which are, respectively, on the sides $C_NA_N, B_NC_N,$ and $A_NB_N,$ so that they divide the sides of the triangle $A_NB_NC_N$ at a ratio that equals $\alpha.$

Note that the triangle $A_0B_0C_0$ is the original triangle *ABC* and the sides $A_0B_0, B_0C_0,$ and C_0A_0 are the sides of the original triangle *ABC*.

The famous Pappus' theorem claims that the triangles ΔABC and $\Delta A_1B_1C_1$ have a common centroid [14]. Therefore, it is clear that the triangles $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ also have a common centroid, and so on and so forth, up to the triangles $A_NB_NC_N$ and $A_{N+1}B_{N+1}C_{N+1}$. A continuous repetition of this process leads to the convergence of the triangles $A_{N+1}B_{N+1}C_{N+1}$ to a single centroid point as N approaches infinity and therefore we have the following:

Claim 1. $\bigcap \Delta A_{N+1}B_{N+1}C_{N+1}$ is the centroid point of ΔABC, as N tends to infinity.

Case II: Generalization

The points C_1, B_1, A_1 lie, respectively, on the sides CA, BC, and AB of the triangle ΔABC so that

$$\frac{BB_1}{B_1C} \equiv \alpha, \quad \frac{CC_1}{C_1A} \equiv \beta, \quad \frac{AA_1}{A_1B} \equiv \gamma \text{ (Figure 2).}$$

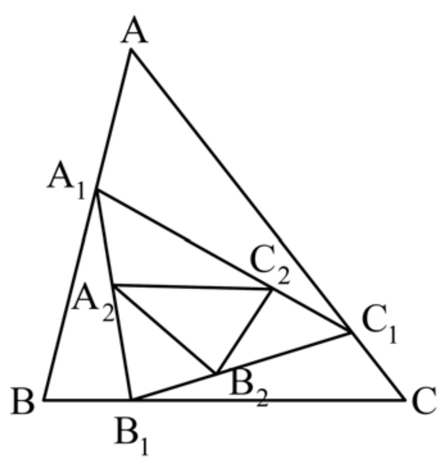

Figure 2. Convergence of Pappus triangles.

The points C_2, B_2, A_2 lie, respectively, on the sides C_1A_1, B_1C_1, and A_1B_1 of the triangle $\Delta A_1B_1C_1$ so that

$$\frac{B_1B_2}{B_2C_1} \equiv \alpha, \quad \frac{C_1C_2}{C_2A_1} \equiv \beta, \quad \frac{A_1A_2}{A_2B_1} \equiv \gamma.$$

Repeating this process $(N+1)$-times, we conclude that the points C_{N+1}, B_{N+1}, A_{N+1}, respectively, lie on the sides C_NA_N, B_NC_N, A_NB_N so that

$$\frac{B_NB_{N+1}}{B_{N+1}C_N} \equiv \alpha, \quad \frac{C_NC_{N+1}}{C_{N+1}A_N} \equiv \beta, \quad \frac{A_NA_{N+1}}{A_{N+1}B_N} \equiv \gamma.$$

Next we state and prove.

Claim 2. $\bigcap \Delta A_{N+1}B_{N+1}C_{N+1} = S$; a point of ΔABC, with the barycentric coordinates

$$(\lambda_1^S, \lambda_2^S, \lambda_3^S) = \left(\frac{\alpha\beta(\gamma+1)}{\alpha\beta(\gamma+1)+\beta\gamma(\alpha+1)+\alpha\gamma(\beta+1)},\right.$$

$$\left.\frac{\beta\gamma(\alpha+1)}{\alpha\beta(\gamma+1)+\beta\gamma(\alpha+1)+\alpha\gamma(\beta+1)}, \frac{\alpha\gamma(\beta+1)}{\alpha\beta(\gamma+1)+\beta\gamma(\alpha+1)+\alpha\gamma(\beta+1)}\right), \quad (3)$$

or with the Ceva coordinates

$$(\lambda_1^S, \lambda_2^S, \lambda_3^S) = \left(\frac{\alpha(\beta+1)}{\beta(\alpha+1)}, \frac{\beta(\gamma+1)}{\gamma(\beta+1)}, \frac{\gamma(\alpha+1)}{\alpha(\gamma+1)}\right). \quad (4)$$

Proof. We transform the triangle ABC by an affine transformation to a triangle in which A, B, and C have the coordinates $A(0, 1)$, $B(0, 0)$, and $C(1, 0)$.

Next, we transform the points A, C, B by an affine transformation to respective points C_1, B_1, A_1 so that $\frac{BB_1}{B_1C} = \alpha$, $\frac{CC_1}{C_1A} = \beta$, and $\frac{AA_1}{A_1B} = \gamma$.

Repeating this process $(N+1)$-times, we note that these affine transformations preserve the simple ratio where the triangle $\Delta A_NB_NC_N$

shall be transformed into the triangle $\Delta A_{N+1}B_{N+1}C_{N+1}$ so that the division ratios of the corresponding sides are α, β, and γ, as required.

The affine transformation f that transforms A, C, B to C_1, B_1, A_1, is denoted by $f(\Delta ABC) \equiv \Delta A_1B_1C_1$ (Figure 3). This is a continuous function from the convex domain ΔABC to the convex domain $\Delta A_1B_1C_1$ that is contained in ΔABC. Therefore f has a fixed point S. This point S shall be the only common point of the triangles $\bigcap \Delta A_{N+1}B_{N+1}C_{N+1}$.

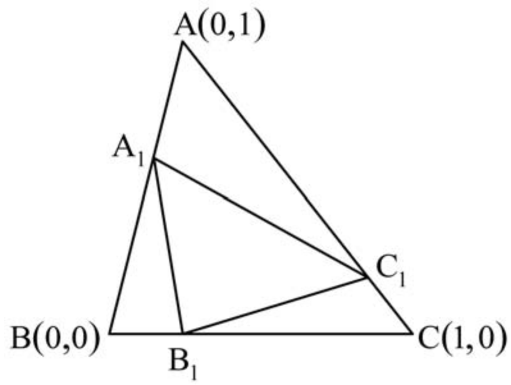

Figure 3. Affine transformation $f(\Delta ABC) = \Delta A_1B_1C_1$.

To derive the coordinates of the point S, observe that the coordinates of A_1, B_1, and C_1 are

$$A_1 = \left(0, \frac{1}{\gamma+1}\right),\ B_1 = \left(\frac{\alpha}{\alpha+1}, 0\right),\ C_1 = \left(\frac{1}{\beta+1}, \frac{\beta}{\beta+1}\right).$$

The fixed point for f has the same barycentric coordinates in ΔABC and $\Delta A_1B_1C_1$. Therefore it is a solution of the following system of equation:

$$x_S = \underbrace{\lambda_3 \cdot 1 + \lambda_1 \cdot 0 + \lambda_2 \cdot 0}_{\Delta ABC} = \underbrace{\lambda_1 \cdot 0 + \lambda_2 \cdot \frac{\alpha}{\alpha+1} + \lambda_3 \cdot \frac{1}{\beta+1}}_{\Delta A_1B_1C_1}.$$

In the other words,

$$\lambda_3 = \lambda_2 \cdot \frac{\alpha}{\alpha + 1} + \frac{\lambda_3}{\beta + 1}, \tag{5}$$

$$\lambda_1 = \frac{\lambda_1}{\gamma + 1} + \frac{\lambda_3 \beta}{\beta + 1}. \tag{6}$$

From (5), it follows that

$$\alpha_S = \frac{\lambda_3}{\lambda_2} = \frac{\alpha(\beta + 1)}{\beta(\alpha + 1)}. \tag{5*}$$

From (6), it follows that

$$\beta_S = \frac{\lambda_1}{\lambda_3} = \frac{\beta(\gamma + 1)}{\gamma(\beta + 1)}. \tag{6*}$$

And clearly, we obtain

$$\gamma_S = \frac{\lambda_2}{\lambda_1} = \frac{\gamma(\alpha + 1)}{\alpha(\gamma + 1)}.$$

A simple calculation, using Equation (1) gives $\lambda_1^S = \dfrac{\alpha\beta(\gamma + 1)}{\alpha\beta(\gamma + 1) + \beta\gamma(\alpha + 1) + \alpha\gamma(\beta + 1)}$. The process for λ_2^S and λ_3^S are similar and will be omitted.

Note. For given α, β, and γ, the point S can be constructed using a compass and a straightedge ruler (though the convergence process is infinite). An accurate description of segment construction, by using compass and straightedge, based on expressions whose representation is by segments, can be found in references [1, 2].

Next we deal with three particular cases (Figure 4).

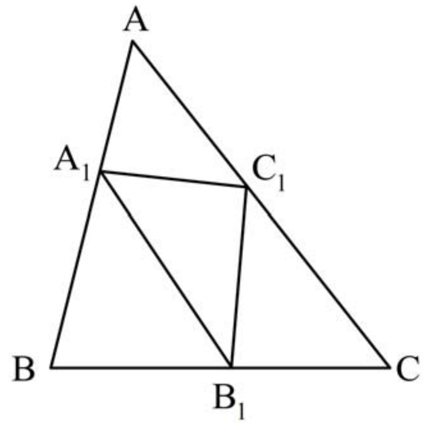

Figure 4. The case of the middle of the segments.

First particular case

If $\alpha = \beta = \gamma = 1$, then the convergence is at the centroid of ΔABC.

Second particular case (Pappus)

If $\alpha = \beta = \gamma = k$, where $k(0 < k < \infty)$ is fixed, then S is the centroid of ΔABC because $\alpha_S = \beta_S = \gamma_S = 1$.

Third particular case (Generalized Pappus)

If $\frac{BB_1}{B_1C} = k\alpha, \frac{CC_1}{C_1A} = k\beta$ and $\frac{AA_1}{A_1B} = k\gamma$, where $k(0 < k < \infty)$ is fixed, then

$$\left.\begin{aligned} \alpha_S &= \frac{\alpha(k\beta + 1)}{\beta(k\alpha + 1)} \\ \beta_S &= \frac{\beta(k\gamma + 1)}{\gamma(k\beta + 1)} \\ \gamma_S &= \frac{\gamma(k\alpha + 1)}{\alpha(k\gamma + 1)} \end{aligned}\right\}. \tag{7}$$

In the other words, aside from the Pappus' case, the point S depends on k and is therefore denoted by $S_{(k)}$.

Finally, we state and prove our

Claim 3. The locus of $S_{(k)}$ where $k\,(0 < k < \infty)$ is fixed, is the open interval straight-line segment bounded by two points M_1 and M_2 where M_1 is the centroid of the triangle and M_2 has the barycentric coordinates

$$\lambda_1{}^{M_2} = \frac{\alpha\beta}{\alpha\beta + \beta\gamma + \gamma\alpha},\ \lambda_2{}^{M_2} = \frac{\beta\gamma}{\alpha\beta + \beta\gamma + \gamma\alpha},\ \lambda_3{}^{M_2} = \frac{\alpha\gamma}{\alpha\beta + \beta\gamma + \gamma\alpha}.$$

Proof. When k tends to zero in (7), we obtain the Ceva coordinates $M_2 = \left(\frac{\alpha}{\beta}, \frac{\beta}{\gamma}, \frac{\gamma}{\alpha}\right)$, which means the barycentric coordinates

$$\left(\lambda_1{}^{M_2}, \lambda_2{}^{M_2}, \lambda_3{}^{M_2}\right) = \left(\frac{\alpha\beta}{\alpha\beta + \beta\gamma + \gamma\alpha}, \frac{\beta\gamma}{\alpha\beta + \beta\gamma + \gamma\alpha}, \frac{\alpha\gamma}{\alpha\beta + \beta\gamma + \gamma\alpha}\right).$$

When k tends to infinity, we obtain $(1, 1, 1) = \left(\alpha_{M_1}, \beta_{M_1}, \gamma_{M_1}\right)$, or barycentric coordinates $\left(\lambda_1{}^{M_2}, \lambda_2{}^{M_2}, \lambda_3{}^{M_2}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. We now prove that $S_{(k)}$ is located on the segment connecting M_1 and M_2. It is enough to prove that

$$\begin{vmatrix} \lambda_1{}^{M1} & \lambda_2{}^{M_1} & \lambda_3{}^{M_1} \\ \lambda_1{}^{M_2} & \lambda_2{}^{M_2} & \lambda_3{}^{M_2} \\ \lambda_1{}^{S_{(k)}} & \lambda_2{}^{S_{(k)}} & \lambda_3{}^{S_{(k)}} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha\beta & \beta\gamma & \alpha\gamma \\ \alpha\beta(k\gamma + 1) & \beta\gamma(k\alpha + 1) & \alpha\gamma(k\beta + 1) \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha\beta & \beta\gamma & \alpha\gamma \\ k\alpha\beta\gamma & k\alpha\beta\gamma & k\alpha\beta\gamma \end{vmatrix} = 0.$$

We will show that $S_{(k)}$ is located between M_1 and M_2. We connect A to M_1 and M_2, and obtain the respective points of intersection D_1 and D_2, (Figure 5).

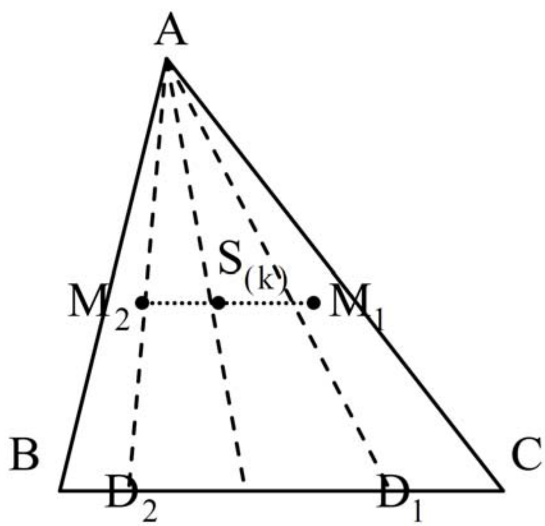

Figure 5. The location of the point $S_{(k)}$.

Without loss of generality, we assume that M_2 and D_2 are more to the left of M_1 and D_1, in the other words $\frac{\alpha}{\beta} < 1$.

In this case $\frac{\alpha}{\beta} < \alpha_{S_{(k)}} = \frac{\alpha(\beta k + 1)}{\beta(\alpha k + 1)} < 1$ and therefore $S_{(k)}$ is located between M_1 and M_2.

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