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EXTENSION OF A BETA FUNCTION INEQUALITY

PÉTER IVÁDY

Dózsa u. 35, 9934
Hegyhátszentjakab
Hungary
e-mail: ivadyp314@t-online.hu

Abstract

In this paper, we present a beta function inequality on $[1, \infty) \times [1, \infty)$ which extends an inequality of the author. The inequality improves some beta function inequalities of Alzer and Cerone. Moreover, some elementary inequalities of two real variables are proved.

1. Introduction

For $x > 0$, the classical gamma function Γ and the psi function or digamma function Ψ are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{and} \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The derivatives Ψ', Ψ'', Ψ''', … are known as polygamma functions.

Closely related to the gamma function is the beta function which is the real function of two variables defined by

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$$B(x,\ y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x > 0,\ y > 0.$$

A well-known equation connecting the beta and the gamma functions

$$B(x,\ y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1}$$

For a proof of (1.1) we refer the reader to [4], where a good reference for these functions is also given. Further literature is given in [2, 3, 6, 7, 9-14] where more references could be find.

In [12], it was shown that for all $x,\ y \geq 1$ the following inequality holds:

$$B(x,\ y) \leq \frac{\alpha}{xy}\,\frac{x+y}{(x+1)(y+1)}, \tag{1.2}$$

with the best possible constant $\alpha = 2$. Equality occurs in (1.2) if and only if $x = y = 1$. This result is in fact an extension of the right side of the inequality of Alzer ([3], p. 738, Theorem 3.1), which reads as follows:

$$\frac{1}{xy}\left(1 - \alpha\,\frac{1-x}{1+x}\,\frac{1-y}{1+y}\right) \leq B(x,\ y) \leq \frac{1}{xy}\left(1 - \beta\,\frac{1-x}{1+x}\,\frac{1-y}{1+y}\right), \quad \text{for all } x,\ y \in (0,\ 1], \tag{1.3}$$

with the best possible constants $\alpha = 2 / 3\pi^2 - 4 = 2.57973\ldots$ and $\beta = 1$, respectively. In [10, 11], it was established that for all $x,\ y \in (0,\ 1]$

$$B(x,\ y) \leq \frac{1}{xy}\,\frac{x+y}{1+xy}, \tag{1.4}$$

holds. In view of (1.3), (1.4), and (1.2), it is natural to ask whether (1.4) could be true also for all $x,\ y \geq 1$. The main purpose of this paper is to answer this question. We show that for all $x,\ y \geq 1$ the inequality (1.4) remains true.

Let us mention two related results in this direction. In ([3], p. 741, Theorem 3.4), the following result is proved. For all real numbers $x,\ y \geq 1,$ we have

$$\frac{1}{\min(x,\ y)}\left(\frac{1}{\max(x,\ y)}\right)^{\min(x,\ y)} \leq B(x,\ y) \leq \frac{1}{\min(x,\ y)}\left[1-\left(1-\frac{1}{\max(x,\ y)}\right)^{\min(x,\ y)}\right]. \quad (1.5)$$

Recently, Cerone ([6], p.79, Corollary 1) established for all $x,\ y \geq 1-$ among others – the estimations

$$\max\left\{\frac{1}{xy^x},\ \frac{1}{yx^y}\right\} \leq B(x,\ y) \leq \min\left\{\frac{1}{x}\left[1-\left(1-\frac{1}{y}\right)^x\right],\ \frac{1}{y}\left[1-\left(1-\frac{1}{x}\right)^y\right]\right\}. \quad (1.6)$$

We will point out that (1.4) improves for all $x,\ y \geq 1$ (1.2), the right side of (1.5) and also the right side of (1.6).

We recall the *Hermite-Hadamard* inequality for a real valued convex function *f* as follows ([15], p. 50). Let $f : [a,\ b] \to \mathbb{R}$ be a convex function. Then we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}, \quad (1.7)$$

with $a < b$.

2. Lemmas

For proving our main result, we need the following lemmas. The first one can be found in ([1], Chapter 6).

Lemma 2.1. *For all x, we have*

$$\Gamma(x+1) = x\Gamma(x). \tag{2.1}$$

For all $x \neq i (i = 1, 2, \ldots)$, *we have*

$$\Psi(x) = -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty}\frac{1}{k(x+k)}, \qquad (x \neq 0, -1, -2, \ldots), \tag{2.2}$$

$$\Psi^{(m)}(x) = (-1)^{m+1} m! \sum_{i=0}^{\infty}\frac{1}{(x+i)^{m+1}},$$

$$(x \neq 0, -1, -2, \ldots \textit{and } m = 1, 2, \ldots), \tag{2.3}$$

where $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) = 0.57721\ldots$ *is the Euler-Mascheroni constant*.

In Gordon ([8], p. 860, Theorem 4) – among other estimations – the following inequalities are given.

Lemma 2.2. *For all* $t > 0$, *we have*

$$\frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6\left(t + \frac{1}{14}\right)^3} < \Psi'(t) < \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3}. \tag{2.4}$$

Lemma 2.3. *Let* $x, y \geq 1$, *then we have*

$$p(x, y) := 2xy^3 - y^3 + 6x^2y^2 - y^2 + 4x^3y - 2xy + y - 1 > 0.$$

Proof. It is clear that $p(1, 1) = 8$. Differentiation gives

$$\frac{\partial p(x, y)}{\partial x} = 2y^3 + 12xy^2 + 12x^2y - 2y > 0,$$

therefore $p(x, y)$ is strictly increasing in x and we have

$$p(x, y) \geq p(1, y) = y^3 + 5y^2 + 3y - 1 > 0. \qquad \square$$

3. Main Result

Now we give the main result of this paper.

Theorem. *For all real numbers* $x \geq 1$ *and* $y \geq 1$, *we have*

$$B(x,\ y) \leq \frac{1}{xy}\frac{x+y}{1+xy}, \tag{3.1}$$

and equality occurs in (3.1) *if and only if* $x = 1$ *or* $y = 1$. *For* $x \geq 1$ *and* $0 < y \leq 1$ *or* $y \geq 1$ *and* $0 < x \leq 1$, (3.1) *is reversed.*

Proof. To establish (3.1), let

$$F(x,\ y) := \log\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y)} - \log\frac{x+y}{1+xy}.$$

Without loss of any generality, we can assume that $1 \leq x < y < \infty$. Building the derivative, we get

$$\frac{\partial F(x,\ y)}{\partial x} = \frac{(y+1)(y-1)}{(x+y)(1+xy)} - \Psi(x+y) + \Psi(x+1). \tag{3.2}$$

In fact we have $\partial F(x,\ y)/\partial x < 0$. Since $\Psi'(t)$ is convex [cf. (2.3)] and applying the *Hermite-Hadamard* inequality (1.7), we find

$$(y-1)\Psi'\left(x+\frac{y+1}{2}\right) \leq \int_{x+1}^{x+y}\Psi'(t)dt = \Psi(x+y) - \Psi(x+1). \tag{3.3}$$

In view of (3.2), (3.3) and in conjunction with the left side of (2.4), if we have

$$\frac{(y+1)(y-1)}{(x+y)(1+xy)} \leq (y-1)\left[\frac{1}{x+\frac{y+1}{2}} + \frac{1}{2\left(x+\frac{y+1}{2}\right)^2}\right] \leq (y-1)\Psi'\left(x+\frac{y+1}{2}\right)$$

$$\leq \Psi(x+y) - \Psi(x+1),$$

then we imply $\partial F(x,\ y)/\partial x < 0$. Now we obtain after some computations

$$\Delta(x,\ y) := \frac{(y+1)}{(x+y)(1+xy)} - \left[\frac{1}{x+\frac{y+1}{2}} + \frac{1}{2\left(x+\frac{y+1}{2}\right)^2}\right]$$

$$= -\frac{P(x,\ y)}{(x+y)(1+xy)(1+2x+y)^2},$$

where

$$P(x,\ y) := 2xy^3 - y^3 + 6x^2y^2 - y^2 + 4x^3y - 2xy + y - 1.$$

According to Lemma 2.3, we claim that $P(x,\ y) > 0$ for all $x,\ y \geq 1$. Therefore we get $\Delta(x,\ y) < 0$, and $\partial F(x,\ y)/\partial x < 0$ which means $F(x,\ y)$ is strictly decreasing in x, i.e., $F(x,\ y) \leq F(1,\ y) = 0$ and the Theorem is proved.

The reversed inequality could be prove by a similar argument, using the *Hermite-Hadamard inequality* (1.7) again, so we omit the details. □

4. Concluding Remarks

Now we show that (3.1) improves (1.2), (1.5), and (1.6) as announced in the Introduction. It is clear that the following relations hold:

Corollary 4.1. *Let* $x,\ y \geq 1$, *then we have*

$$\frac{x+y}{1+xy} \leq 1 - \frac{x-1}{\sqrt{x^2+1}}\frac{y-1}{\sqrt{y^2+1}} \leq \frac{2(x+y)}{(x+1)(y+1)}. \tag{4.1}$$

Equality in the first inequality of (4.1) *holds if and only if* $x = 1$ *or* $y = 1$ *or* $x = y$. *Whilst equality for the second inequality in* (4.1) *holds if and only if* $x = 1$ *or* $y = 1$.

The next corollary shows that (3.1) improves (1.5).

Corollary 4.2. *For all real numbers* $x,\ y \geq 1,$ *we have*

$$\frac{1}{xy}\frac{x+y}{1+xy} \leq \frac{1}{\min(x,\ y)}\left[1-\left(1-\frac{1}{\max(x,\ y)}\right)^{\min(x,\ y)}\right],$$

with equality if and only if $x = y = 1.$

Proof. Suppose that $1 < x < y < \infty.$ Let

$$H(x,\ y) := x\log\left(1-\frac{1}{y}\right) - \log\left(1-\frac{x+y}{y(1+xy)}\right).$$

Differentiation gives

$$\frac{\partial H(x,\ y)}{\partial x} = \log\left(1-\frac{1}{y}\right) - \frac{1}{x(1+xy)} < 0,$$

i.e., $H(x,\ y)$ is strictly decreasing in x thus we have $H(x,\ y) \leq H(1,\ y) = 0$ which we had to prove. □

Finally, we show that (3.1) also improves (1.6).

Corollary 4.3. *For all real numbers* $x,\ y \geq 1,$ *we have*

$$\frac{1}{xy}\frac{x+y}{1+xy} \leq \min\left\{\frac{1}{x}\left[1-\left(1-\frac{1}{y}\right)^{x}\right],\ \frac{1}{y}\left[1-\left(1-\frac{1}{x}\right)^{y}\right]\right\},$$

with equality if and only if $x = y = 1.$

Proof. We prove only

$$\frac{1}{xy}\frac{x+y}{1+xy} \leq \frac{1}{x}\left[1-\left(1-\frac{1}{y}\right)^{x}\right],$$

because the other one could be proved similarly. We define

$$H_1(x,\ y) := x\log\left(1-\frac{1}{y}\right) - \log\left(\frac{x(y+1)(y-1)}{y(1+xy)}\right).$$

Then we get

$$\frac{\partial H_1(x,\ y)}{\partial x} = \log\left(1 - \frac{1}{y}\right) - \frac{1}{x(1 + xy)} < 0,$$

therefore $H_1(x,\ y)$ is strictly decreasing in x so we conclude $H_1(x,\ y) \leq H_1(1,\ y) = 0$. □

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