Journal of Mathematical Sciences: Advances and Applications
Volume 44, 2017, Pages 51-71
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121759

DIRECT AND INVERSE TASK FOR THE OPERATOR STURM-LIOUVILLE TYPE WITH LINEAR DELAY

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Abstract

This work is dedicated to solving the direct and inverse spectral task Sturm-Liouville type with linear delay and dispensed boundary conditions. Under the direct spectral assignment, we mean the construction of solutions $D^2y = z^2y$.

The aim of this paper is to solve the inverse task for the operator D^2, which to

2010 Mathematics Subject Classification: 35Qxx.
Keywords and phrases: characteristic function, asymptotic zero, linear delay, potential, Fourier coefficients.
Received January 17, 2017; Revised February 8, 2017

our knowledge is not known in the literature. We propose a new method. This method is based on the transformation of F in such a form that establishes a connection between the Fourier coefficients of a specific set of functions and values of their own operator D^2, second. In this article, we will assume that $h = 0$.

1. Introduction

This paper is dedicated to solving the direct and inverse spectral assignment type Sturm-Liouville with linear delay

$$\tau(x) = \alpha x + \beta,\ (\alpha,\ \beta) \in (0,\ 1) \times \left[\frac{(1-\alpha)^3}{\alpha^2 - 3\alpha - 3}\beta,\ \frac{(1-\alpha)^2}{2-\alpha}\pi \right),$$

and partitioned boundary conditions. Let is

$q \in L_2[0,\ \pi],\ \gamma(x) = (1-\alpha)x - \beta,\ \xi_1 = \dfrac{\beta}{1-\alpha}$. Therefore $\gamma(\xi_1) = 0$.

Studying border assignments $D^2 y = \lambda y$ given by

$$-y''(x) + q(x)y(\gamma(x)) = \lambda y(x);\ \lambda = z^2, \tag{1}$$

$$y(\gamma(x),\ z) \equiv 0, \quad x \in [0,\ \xi_1), \tag{2}$$

$$y'(0,\ z) - hy(0,\ z) = 0, \tag{3}$$

$$y'(\pi,\ z) + Hy(\pi,\ z) = 0. \tag{4}$$

The potential q and numbers h, H, α, β are parameters operator. In particular, if

$\alpha = 0$, holds $\gamma(x) = x - \beta$ we have tasks with constant delay expires (see [4], [5], [6], [7]). If $\beta = 0$ and $a = 1 - \alpha$, we have tasks with homogeneously delay (see [8] and [9]). When $\alpha = \beta = 0$ assignment detail thrown in the works ([1], [2], [3], and [10]).

2. Direct Spectral Assignments

In this paper, under the direct spectral assignment mean construction solutions $D^2y = z^2y$ and the construction of the characteristic functions F and its asymptotic zero.

2.1. Solution task (1, 2, 3)

Lemma 1. *Task* (1, 2, 3) *is equivalent to the integral equation type Voltaire*

$$y(x, z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\int_{\xi_1}^{x} q(t_1)\sin z(x - t_1)y(\gamma(t_1), z)dt_1. \qquad (5)$$

Indeed, the method of variation of constants in the Equation (1) using the initial condition (2) and the boundary condition (3) the relation (5) follows directly. We construct a series of functions

$$\gamma^k(x) = \gamma\left(\gamma^{k-1}(x)\right) \text{ and series of numbers } \xi_k : \gamma^k(\xi_k) = 0.$$

Lemma 2. *There* $k_0 \in N$ *are such that it is valid* $\xi_{k_0} < \pi < \xi_{k_0+1}$.

Proof.

$$\gamma^1(x) = \gamma(x) = (1 - \alpha)x - \beta = 0 \Rightarrow \xi_1 = \frac{\beta}{1 - \alpha}.$$

$$\gamma^2(x) = \gamma(\gamma(x)) = (1 - \alpha)^2 x - (1 - \alpha)\beta - \beta = 0 \Rightarrow$$

$$\Rightarrow \xi_2 = \frac{\beta}{(1 - \alpha)^2} + \frac{\beta}{1 - \alpha} = \frac{\beta}{1 - \alpha}\left[1 + \frac{1}{1 - \alpha}\right] = \frac{2 - \alpha}{(1 - \alpha)^2}\beta = \frac{2 - \alpha}{1 - \alpha}\xi_1.$$

Mathematical induction to show valid

$$\gamma^k(x) = (1 - \alpha)^k x - (1 - \alpha)^{k-1} x - \cdots - (1 - \alpha)\beta - \beta,$$

and it follows

$$\xi_k = \gamma^{-1}(0) = \frac{\beta}{1 - \alpha}\left[1 + \frac{1}{1 - \alpha} + \cdots + \frac{1}{(1 - \alpha)^{k-1}}\right] = \frac{\beta}{\alpha}\frac{1 - (1 - \alpha)^k}{(1 - \alpha)^k}.$$

How to order $\sum_{k=1}^{\infty} \frac{1}{(1-\alpha)^k}$ diverges, there $k_0 \in N$ are such that it is valid $\xi_{k_0} < \pi < \xi_{k_0+1}$.

Therefore,

$$(\xi_1, \pi] = \bigcup_{k=2}^{k_0} (\xi_{k-1}, \xi_k] \cup (\xi_{k_0}, \pi].$$

In the following, we take the functions

$$a_{sc}(x, z) = \int_{\xi_1}^{x} q(t_1) \sin(x - t_1) \cos z\gamma(t_1) dt_1,$$

$$a_{s^2}(x, z) = \int_{\xi_1}^{x} q(t_1) \sin(x - t_1) \sin z\gamma(t_1) dt_1,$$

and the recurrent relations

$$a_{s^k c}(x, z) = \int_{\xi_k}^{x} q(t_1) \sin z(x - t_1) a_{s^{k-1} c}(\gamma(t_1), z) dt_1, \quad k > 1,$$

$$a_{s^{k+1}}(x, z) = \int_{\xi_k}^{x} q(t_1) \sin z(x - t_1) a_{s^k}(\gamma(t_1), z) dt_1, \quad k > 1.$$

Theorem 1. *At a distance* $(\xi_l, \xi_{l+1}]$, $l = \overline{1, k_0}$ *solution of the Equation* (5) *has the form of*

$$y(x, z) = \cos xz + \frac{h}{z} \sin xz + \sum_{k=1}^{l} \frac{1}{z^k} \left[a_{s^k c}(x, z) + \frac{h}{z} a_{s^{k+1}}(x, z) \right]. \qquad (7)$$

Proof. For $x \in (0, \xi_1]$ valid $y(x, z) = \cos xz + \frac{h}{z} \sin xz$. For $x \in (\xi_1, \xi_2]$, we have

$$y(x, z) = \cos xz + \frac{h}{z} \sin xz + \frac{1}{z} a_{sc}(x, z) + \frac{h}{z^2} a_{s^2}(x, z).$$

For $x \in (\xi_2, \xi_3]$, we have

$$y(x, z) = \cos xz + \frac{h}{z} \sin xz + \frac{1}{z} a_{sc}(x, z) + \frac{h}{z^2} a_{s^2}(x, z) + \frac{1}{z^2} a_{s^2c}(x, z) + \frac{h}{z^3} a_{s^3}(x, z).$$

The notion that (7) holds for l and $l+1$. So at a distance $(\xi_{l+1}, \xi_{l+2}]$ intial solution is exactly the distance $(\xi_l, \xi_{l+1}]$ and it is determined summands

$$\frac{1}{z^{l+1}} \left[a_{s^{l+1}c}(x, z) + \frac{h}{z} a_{s^{l+2}}(x, z) \right],$$

which is exactly the (7) for $l+1$.

2.2. Construction characteristic function

Construction of characteristic function is based on the solution of the equation at a distance $(\xi_{k_0}, \pi]$ and condition (4). From (7) for $l = k_0$ get

$$\frac{dy}{dx}(x, z) = -z \sin xz + h \cos xz + \sum_{k=1}^{k_0} \frac{1}{z^{k-1}} \left[a_{cs^{k-1}c}(x, z) + \frac{h}{z} a_{cs^k}(x, z) \right], \quad (8)$$

where is it

$$a_{cs^{k-1}c}(x, z) = \int_{\xi_k}^{x} q(t_1) \cos z(x - t_1) a_{s^{k-1}c}(\gamma(t_1), z) dt_1,$$

$$a_{cs^k}(x, z) = \int_{\xi_k}^{x} q(t_1) \cos z(x - t_1) a_{s^k}(\gamma(t_1), z) dt_1. \quad (9)$$

From (7) and (8) on the basis of (4) come to the characteristic function $F(z)$:

$$F(z) = \left(-z + \frac{hH}{z}\right)\sin \pi z + (h + H)\cos \pi z + \sum_{k=1}^{k_0}\left\{\frac{1}{z^{k-1}}\, a_{cs^{k-1}c}(z) + \frac{1}{z^k}\left[h a_{cs^k}(z) + H a_{s^k c}(z)\right] + \frac{hH}{z^{k+1}}\, a_{s^{k+1}}(z)\right\}, \tag{10}$$

where we have the markings π.

The function F is the whole function of exponential type with transparent singularity included in the point $z = 0$ specified. Within the circle $|z - n| = r_n$, where is r_n sufficiently small number, there is exactly one zero z_n function *F*, which is well known from [3] or [13].

In this study limits may be the case $k_0 = 2$. Therefore

$$F(z) = \left(-z + \frac{hH}{z}\right)\sin \pi z + (h + H)\cos \pi z + a_{c^2}(z) + \frac{1}{z}\left[h a_{cs}(z) + H a_{sc}(z)\right] + \frac{hH}{z^2}\, a_{c^2}(z) + \frac{1}{z}\, a_{\mathrm{csc}}(z) + \frac{1}{z^2}\left[h a_{cs^2}(z) + H a_{s^2 c}(z)\right] + \frac{hH}{z^3}\, a_{s^3}(z). \tag{11}$$

In the right conditions $k_0 = 2$ limited space allowed for β and is $\beta \in \left[\frac{(1-\alpha)^3}{\alpha^2 - 3\alpha - 3}\beta, \frac{(1-\alpha)^2}{2-\alpha}\pi\right)$.

2.3. The transformation function *F*

The finish line of this work is to solve the inverse task for the operator D^2, which to our knowledge is not known in the literature. Since methods Borg, Geljfand, Marchenko, Crane, Levinson, and other Lejbenzona not apply to the operator with a deviation, we propose a new method. This method is based on the transformation function *F* in such a

form that establishes a connection between the Fourier coefficients of a specific set of functions and eigen values of the operator D^2, second. In this paper, we will assume that $h = 0$, which only simplifies technical procedure. First, let us introduce the following codes:

$$\nu_{-2}^{(1)} = \frac{1}{2}\xi_1;\ \nu_{-1}^{(1)} = \frac{1}{2}(\alpha\pi + \beta) = \frac{\tau(\pi)}{2};\ \nu_0^{(1)} = \frac{1}{2}\xi_2,\ \nu_1^{(1)} = \frac{\alpha}{2}\pi + l_1\xi_1,$$

$$\nu_2^{(1)} = l_1\tau(\pi),\ \nu_3^{(1)} = l_2\pi - l_1\beta,\ \nu_4^{(1)} = l_1(\pi - \xi_1),\ \nu_5^{(1)} = \pi - \frac{1}{2}\tau(\pi),$$

where is

$$l_1 = 1 - \frac{\alpha}{2}, \quad l_2 = 1 - \alpha + \frac{\alpha^2}{2}, \quad l_1^* = \frac{2}{\alpha} - 1.$$

Also put

$$\alpha_0 = \frac{3 - \sqrt{5}}{2},\ \underline{\beta}(\alpha) = \frac{(1-\alpha)^3}{\alpha^2 - 3\alpha - 3}\pi,\ \bar{\beta}(\alpha) = \frac{(1-\alpha)^2}{2 - \alpha}\pi,$$

$$\varphi(\alpha, \beta) = \xi_2(\alpha, \beta) - \tau(\pi) = \left[\frac{2 - \alpha}{(1-\alpha)^2} - 1\right]\beta - \alpha\pi.$$

In area $D = \left\{(\alpha, \beta) : \alpha \in (0, 1),\ \beta \in \left[\underline{\beta}(\alpha), \bar{\beta}(\alpha)\right)\right\}$ there is a fault $\beta = r(\alpha)$ with a starting point in the $\left(\alpha_0, \underline{\beta}(\alpha_0)\right)$ and end at point $(1, 0) \notin D$ such that it is valid

$\varphi(\alpha, r(\alpha)) = 0,\ \alpha \in [\alpha_0, 1)$. In area

$$D_+ =$$

$$\left\{(\alpha, \beta) \in D : \left(\alpha \in (0, \alpha_0) \wedge \beta \in \left[\underline{\beta}(\alpha), \bar{\beta}(\alpha)\right)\right) \vee \alpha \in (\alpha_0, 1) \wedge \beta \in \left(r(\alpha), \bar{\beta}(\alpha)\right)\right\}$$

$$\subsetneq D$$

is

$$0 < \nu_{-2}^{(1)} < \nu_{-1}^{(1)} < \nu_0^{(1)} < \nu_1^{(1)} < \nu_2^{(1)} < \nu_3^{(1)} < \nu_4^{(1)} < \nu_5^{(1)} < \pi.$$

Definition 1. Function

$$\begin{cases} 0, & \theta \in [0,\ \nu_{-2}^{(1)}) \cup (\nu_{5}^{(1)},\ \pi], \\ \dfrac{1}{\alpha} q\left(\dfrac{2\theta-\beta}{\alpha}\right) + \dfrac{1}{2-\alpha} q\left(\dfrac{2\theta+\beta}{2-\alpha}\right), & \theta \in [\nu_{-2}^{(1)},\ \nu_{-1}^{(1)}), \\ \dfrac{1}{2-\alpha} q\left(\dfrac{2\theta+\beta}{2-\alpha}\right), & \theta \in [\nu_{-1}^{(1)},\ \nu_{5}^{(1)}], \end{cases} \tag{11'}$$

called transition to function operator D^2.

We define the following auxiliary functions:

$$\varphi_1^{(1)}(\theta) = \frac{2}{\alpha} \int_{\frac{\theta}{\alpha 1}-\frac{\beta}{\alpha}}^{\frac{2}{\alpha}\theta - l_1^* \xi_1} q(t_1) q\left(\frac{2\theta - 2\beta - \alpha t_1}{\alpha}\right) dt_1,$$

$$\varphi_1^{(2)}(\theta) = \frac{2}{\alpha} \int_{\frac{\theta}{\alpha 1}-\frac{\beta}{\alpha}}^{\pi} q(t_1) q\left(\frac{2\theta - 2\beta - \alpha t_1}{\alpha}\right) dt_1,$$

$$\varphi_2^{(1)}(\theta) = \frac{2}{\alpha} \int_{\frac{\theta}{l_2}+\beta\frac{l_1}{l_2}}^{\frac{\theta}{l_1}+\xi_1} q(t_1) q\left(l_1^* t_1 - \frac{2\beta + 2\theta}{\alpha}\right) dt_1,$$

$$\varphi_2^{(2)}(\theta) = \frac{2}{\alpha} \int_{\frac{\theta}{l_2}+\beta\frac{l_1}{l_2}}^{\pi} q(t_1) q\left(l_1^* t_1 - \frac{2\beta + 2\theta}{\alpha}\right) dt_1,$$

$$\varphi_3^{(1)}(\theta) = \frac{1}{l_1} \int_{\frac{2}{\alpha}\theta - l_1 \xi_1}^{\frac{\theta}{l_1}+\beta\frac{l_1}{l_2}} q(t_1) q\left(\frac{\theta}{l_1} - \frac{t_1}{l_1^*}\right) dt_1,$$

$$\varphi_3^{(2)}(\theta) = \frac{1}{l_1} \int_{\frac{2}{\alpha}\theta - l_1 \xi_1}^{\pi} q(t_1) q\left(\frac{\theta}{l_1} - \frac{t_1}{l_1^*}\right) dt_1,$$

$$\varphi_4^{(1)}(\theta) = \frac{1}{l_1} \int\limits_{\frac{\theta}{\alpha l_1} - \frac{\beta}{\alpha}}^{\frac{\theta}{l_1} + \xi_1} q(t_1) q\left(t_1 - \frac{\theta}{l_1}\right) dt_1,$$

$$\varphi_4^{(2)}(\theta) = \frac{1}{l_1} \int\limits_{\frac{\theta}{\alpha l_1} - \frac{\beta}{\alpha}}^{\pi} q(t_1) q\left(t_1 - \frac{\theta}{l_1}\right) dt_1, \tag{12}$$

$$K_1(\theta,\, q(\theta)) = \begin{cases} 0, & \theta \in [0,\, \nu_0^{(1)}) \cup (\nu_4^{(1)},\, \pi], \\ \varphi_1^{(1)}(\theta) - \varphi_2^{(1)}(\theta) + \varphi_3^{(1)}(\theta) - \varphi_4^{(1)}(\theta), & \theta \in [\nu_0^{(1)},\, \nu_1^{(1)}), \\ \varphi_1^{(2)}(\theta) - \varphi_2^{(1)}(\theta) + \varphi_3^{(1)}(\theta) - \varphi_4^{(1)}(\theta), & \theta \in [\nu_1^{(1)},\, \nu_2^{(1)}), \\ \quad - \varphi_2^{(1)}(\theta) + \varphi_3^{(2)}(\theta) - \varphi_4^{(2)}(\theta), & \theta \in [\nu_2^{(1)},\, \nu_3^{(1)}), \\ \quad - \varphi_2^{(2)}(\theta) - \varphi_4^{(2)}(\theta), & \theta \in [\nu_3^{(1)},\, \nu_4^{(1)}). \end{cases} \tag{13}$$

Converting the product of trigonometric functions in the forest, by introducing new variables and changing the order of integration we obtain

$$a_{c^2}(z) = \int\limits_0^{\pi} \widetilde{q}^{+}(\theta) \cos z(\pi - 2\theta) d\theta = \widetilde{\alpha}_c{}^{+}(z),$$

$$\frac{a_{sc}(z)}{z} = \int\limits_0^{\pi} \widetilde{q}^{+}(\theta) \frac{\sin z(\pi - 2\theta)}{z} d\theta = \frac{1}{z} \widetilde{\alpha}_s{}^{+}(z),$$

$$\frac{a_{\csc}(z)}{z} = \frac{1}{4} \int\limits_0^{\pi} K_1(\theta,\, q(\theta)) \frac{\sin z(\pi - 2\theta)}{z} d\theta,$$

$$\frac{a_{s^2c}(z)}{z} = -\frac{1}{4} \int\limits_0^{\pi} K_1(\theta,\, q(\theta)) \frac{\cos z(\pi - 2\theta)}{z} d\theta. \tag{14}$$

Producing the partial integration in (14), we get

$$\frac{a_{\csc}(z)}{z} = \frac{1}{2}\int_0^{\pi}\int_0^{\theta} K_1(\theta_1,\, q(\theta_1))d\theta_1 \cos z(\pi - 2\theta)d\theta,$$

$$\frac{a_{s^2c}(z)}{z} = \frac{1}{2}\int_0^{\pi}\int_0^{\theta} K_1(\theta_1,\, q(\theta_1))d\theta_1 \sin z(\pi - 2\theta)d\theta. \tag{15}$$

By the relation (15), we have come with the use of the fact that

$$\lim_{z\to 0}\frac{a_{s^2c}(z)}{z} = 0,$$

and then must be

$$\int_0^{\pi} K_1(\theta,\, q(\theta))d\theta = 0.$$

So, when $h = 0,$ the function (11) is transformed into the form of

$$F(z) = -z\sin\pi z + H\cos\pi z + \widetilde{a}_c^{\,+}(z) + H\frac{\widetilde{a}_s^{\,+}(z)}{z}$$

$$+ \frac{1}{2}\int_0^{\pi}\int_0^{\theta} K_1(\theta_1,\, q(\theta_1))d\theta_1 \cos z(\pi - 2\theta)d\theta$$

$$+ \frac{H}{2z}\int_0^{\pi}\int_0^{\theta} K_1(\theta_1,\, q(\theta_1))d\theta_1 \sin z(\pi - 2\theta)d\theta. \tag{16}$$

2.4. Asymptotic zero function *F*

Lemma 3. *If it is* $q \in L_2[0,\, \pi]$ *then zero* $z_n,\ n \in N$ *have the following asymptotic*:

$$z_n = \pm\left[n + \left(\frac{H}{\pi} + \frac{1}{\pi}\int_0^{\pi}\widetilde{q}^{+}(\theta)\cos 2n\theta d\theta\right)\frac{1}{n} + O\left(\frac{\int_0^{\pi}\widetilde{q}^{+}(\theta)\cos 2n\theta d\theta}{n^2}\right)\right] (\text{as } n\to\infty),$$

where is $\widetilde{q}^{+}$ transition function. Indeed, the direct inclusion z_n in (16) and (13) follows the accuracy results.

From (16) follow

$$\lambda_n = n^2 + \frac{2}{\pi} H + \frac{2}{\pi} D^2 \int_0^{\pi} \widetilde{q}^{+}(\theta) \cos 2n\theta d\theta + O\left(\frac{1}{n} \int_0^{\pi} \widetilde{q}^{+}(\theta) \cos 2n\theta d\theta \right). \quad (17)$$

Therefore, eigenvalues operator D^2 have asymptotic (17).

Note 1. Oscillating member $\widetilde{\alpha}_{2n}^{+} = \frac{2}{\pi} \int_0^{\pi} \widetilde{q}^{+}(\theta) \cos 2n\theta d\theta$ in the (17) is cosine coefficient transition function $\widetilde{q}^{+}$ that with spaces $[0, \nu_1^{(0)})$ and $(\nu_4^{(1)}, \pi]$ annulled. Because

$\widetilde{q}^{+}(\nu_0^{(1)}) = \frac{2}{\alpha(2-\alpha)}$ and $\widetilde{q}^{+}(\nu_5^{(1)}) = \frac{1}{2-\alpha} q(\pi)$ we think it's worth $q(\xi_1)q(\pi) \neq 0$ which means that the transition function $\widetilde{q}^{+}$ in items $\nu_0^{(1)}$ and $\nu_5^{(1)}$ has interrupted.

3. Inverse Spectral Assignments

Determination of parameters q, α, β, H operator D^2 called solving the inverse spectral assignment. We proceed from the assumption that the two sets of data own values λ_{nj} operator D^2, with the potential $q : q(\xi_1)q(\pi) \neq 0$. From the necessary conditions that we know is true asymptotic

$$z_n = n + c_j + \widetilde{\alpha}_{2n}^{+} + O\left(\frac{\widetilde{\alpha}_{2n}^{+}}{2} \right), \text{ (as } n \to \infty), \; j = 1, 2, \quad (17')$$

where are c_j and $\widetilde{\alpha}_{2n}^{+}$ cosine Fourier coefficients of a function that is reversed at intervals $[0, \mu_1)$ and $(\mu_2, \pi]$ where are μ_1 and μ_2 known numbers and $\mu_1 < \mu_2$.

From $\lim_{n\to\infty}\left(\lambda_{nj} - n^2\right) = c_j,\ j = 1, 2,$ we obtain $H_j = \frac{\pi}{2} c_j$. Still, we are celebrating $\frac{\beta}{2(1-\alpha)} = \mu_1$ and $\pi - \frac{1}{2}(\alpha\pi + \beta) = \mu_2$ to we obtain the parameters α and β of the equation. Thus, the parameters $H_j,\ j = 1, 2.$ and α and β are obtained uniformly.

3.1. Basic identity

According Adamar's theorems obtain representation of F by their zero $\pm\sqrt{\lambda_{nj}},\ n \in N_0,\ j = 1, 2$ in which

$$F_j(z) = \pi\lambda_{0j} \prod_{n=1}^{\infty} \frac{\lambda_{nj}}{n^2}\left(1 - \frac{z^2}{\lambda_{0j}}\right)\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_{nj}}\right), \tag{18}$$

numbers α and β are as defined. In the future we will assume that the following applies:

$$\alpha \in \left(0,\ \frac{3-\sqrt{5}}{2}\right) \wedge \beta \in \left[\frac{(1-\alpha)^3}{\alpha^3 - 3\alpha + 3}\pi,\ \frac{(1-\alpha)^2}{2-\alpha}\pi\right), \text{ or}$$

$$\alpha \in \left(\frac{3-\sqrt{5}}{2},\ 1\right) \wedge \beta \in \left(r(\alpha),\ \frac{(1-\alpha)^2}{2-\alpha}\pi\right), \text{ where is}$$

$$\left[\frac{2-\alpha}{(1-\alpha)^2} - 1\right] r(\alpha) - \alpha\pi = 0, \quad \alpha \in \left(\frac{3-\sqrt{5}}{2},\ 1\right).$$

In this way, we can conclude that the other functions on the expression (18) must be of the form (15), i.e., $k_0 = 2$. First, we put

$$a_c^{(1)}(z) = \int_0^{\pi}\left(\int_0^{\theta} K_1(\theta_1,\ q(0,\ 1)d\theta)\cos z(\pi - 2\theta)d\theta,\right.$$

$$a_s^{(1)}(z) = \int_0^{\pi}\left(\int_0^{\theta} K_1(\theta_1,\ q(\theta,\ 1)d\theta_1)\sin z(\pi - 2\theta)\,d\theta.\right.$$

We establish basic identities

$$A(z) = \frac{H_2 F_1(z) - H_1 F_2(z)}{H_2 - H_1} + z \sin \pi z,$$

$$B(z) = z \frac{F_2(z) - F_1(z)}{H_2 - H_1} z - z \cos \pi z. \tag{19}$$

From (19) we come to the system

$$\widetilde{a}_c^+(z) + \frac{1}{2} a_c^{(1)}(z) = A(z),$$

$$\widetilde{a}_s^+(z) + \frac{1}{2} \widetilde{a}_s^{(1)}(z) = B(z), \quad z \in C. \tag{20}$$

System identity (20) is equivalent infinite system

$$\frac{2}{\pi} \widetilde{a}_c^+(m) + \frac{2}{\pi} \frac{1}{2} a_c^{(1)}(m) = \frac{2}{\pi} A(m), \quad m \in N \cup \{0\},$$

$$\frac{2}{\pi} \widetilde{a}_s^+(m) + \frac{2}{\pi} \frac{1}{2} a_s^{(1)}(m), \quad m \in N. \tag{21}$$

Because

$$\frac{2}{\pi} \widetilde{a}_c^+(m) = (-1)^m \frac{2}{\pi} \int_o^{\pi} \widetilde{q}^+(\theta) \cos 2m\theta d\theta = (-1)^m \widetilde{a}_{2m}^+,$$

$$\frac{2}{\pi} \widetilde{a}_s^+(m) = (-1)^{m+1} \frac{2}{\pi} \int_o^{\pi} \widetilde{q}^+(\theta) \sin 2m\theta d\theta = (-1)^{m+1} \widetilde{b}_{2m}^+,$$

$$\frac{2}{\pi} a_c^{(1)}(m) = (-1)^m \frac{2}{\pi} \int_0^{\pi} \left(\int_0^{\theta} K_1(\theta_1, q(\theta_1) d\theta_1) \cos 2m\theta d\theta = (-1)^m a_{2m}^{(1)}, \right.$$

$$\frac{2}{\pi} a_s^{(1)}(m) = (-1)^{m+1} \frac{2}{\pi} \int_0^{\pi} \left(\int_0^{\theta} K_1(\theta_1, q(\theta_1) d\theta_1) \sin 2m\theta d\theta = (-1)^{m+1} b_{2m}^{(1)}. \right.$$

And leading mark

$$\frac{2}{\pi} A(m)(-1)^m = A_{2m}, \frac{2}{\pi} B(m)(-1)^{m+1} = B_{2m} \text{ from (21) we get}$$

$$\tilde{a}_{2m}^{+} + \frac{1}{2} a_{2m}^{(1)} = A_{2m},\ m \in N_0,$$

$$\tilde{b}_{2m}^{+} + \frac{1}{2} b_{2m}^{(1)} = B_{2m},\ m \in N. \tag{22}$$

From (19) proves to be true $A_{2m} \to 0$, $B_{2m} \to 0$, $m \to 0$. So arrays $\{A_{2m}, B_{2m}\}$ represent the Fourier coefficients of a function $f \in L_2[0, \pi]$. Multiplying (22) with $\cos 2mx^* i \sin 2mx^*$, $x^* \in [0, \pi]$, after summarizing come to the equation

$$\tilde{q}^{+}(x^*) + \frac{1}{2} \int_0^{x^*} K_1(\theta, q(\theta)) d\theta = f(x^*). \tag{23}$$

Transformation of the transition function $\tilde{q}^{+}$ and function q is the tool that we used the method of determining the solutions reduce to solving integral equations of type Volterra. From (11') and (13), we conclude that the Equation (23) breaks up in different forms at intervals $\left[\nu_{-1}^{(1)}, \nu_0^{(1)}\right]$ and $\left[\nu_4^{(1)}, \nu_5^{(1)}\right]$ kernel K_1 is reversed and the Equation (23) becomes

$$\tilde{q}^{+}(x) = f(x). \tag{23$_1$}$$

For

$$x^* \in \left[\nu_{-1}^{(1)}, \nu_5^{(1)}\right] \text{ valid } \tilde{q}^{+}(x^*) = \frac{1}{2-\alpha} q\left(\frac{2x^* + \beta}{2-\alpha}\right).$$

Let's

$$x_1 = \frac{\alpha\pi + 2\beta}{2-\alpha}, \quad x_2 = \frac{\alpha^2 - 3\alpha + 3}{(2-\alpha)(1-\alpha)}\beta, \quad x_3 = \pi - \frac{\beta}{(2-\alpha)(1-\alpha)},$$

and

$$\widetilde{f}(x) = (2-\alpha)f\left(\frac{1}{2}[(2-\alpha)x-\beta]\right).$$

When out of (23_1) following $q(x) = \widetilde{f}(x)$, $x \in [x_1, x_2] \cup [x_3, \pi]$.

On the segment $\left[\nu_3^{(1)}, \nu_4^{(1)}\right]$ valid

$$K_1(\theta, q(\theta)) = -\left(\varphi_2^{(2)}(\theta) + \varphi_4^{(2)}(\theta)\right),$$

so we get

$$\frac{1}{2-\alpha}q\left(\frac{2x^*+\beta}{2-\alpha}\right) - \frac{1}{2}\int\limits_{\nu_3^{(1)}}^{x^*}\left(\varphi_2^{(2)}(\theta) + \varphi_4^{(2)}(\theta)\right)d\theta = f(x^*).$$

From (12), we have

$$\int\limits_{\nu_3^{(1)}}^{x^*}\varphi_2^{(2)}(\theta)d\theta = \frac{2}{\alpha}\int\limits_{\nu_3^{(1)}}^{x^*}\int\limits_{\frac{\theta}{l_2}+\beta\frac{l_1}{l_2}}^{\pi} q(t_1)q\left(\frac{(2-\alpha)-t_1 2\beta+2\theta}{\alpha}\right)dt_1 dt.$$

By changing the order of integration and the replacement of variables

$$\frac{(2-\alpha)t_1 - 2\beta - 2\theta}{\alpha} = t_2,$$

we have

$$\int\limits_{\nu_3^{(1)}}^{x^*}\varphi_2^{(2)}(\theta)d\theta = \int\limits_{\mu_1^{(1)}}^{\xi_1+\frac{1}{l_1}x^*} q(t_1)\left(\int\limits_{\alpha\xi_1}^{l_1^* t_1 - \frac{2l_2}{\alpha}\pi-\beta} q(t_2)dt_2\right)dt_1. \qquad (\Phi_1)$$

Similar to the previous valid

$$\int\limits_{\nu_3^{(1)}}^{x^*}\varphi_4^{(2)}(\theta)d\theta = \int\limits_{\mu_2^{(1)}}^{-\frac{\beta}{\alpha l_1^2}+\frac{x^*}{\alpha l_1}} q(t_1)\left(\int\limits_{(1-\alpha)t_1-\frac{\beta}{l_1^2}}^{\frac{l_2}{l_1}\pi-\beta} q(t_2)dt_2\right)dt_1, \qquad (\Phi_2)$$

where is

$$\mu_2^{(1)} = \frac{\pi}{\alpha} - \frac{\beta}{\alpha} - \frac{\beta}{\alpha l_1^2}, \quad \mu_1^{(1)} = \frac{l_2}{l_1}\pi + \alpha\xi_1.$$

Functions

$$\psi_1^{(1)}(t_1) = \int_{\alpha\xi_1}^{l_1^* t_1 - \mu_3^{(1)}} q(t_2)dt_2, \quad \mu_3^{(1)} = \frac{2l_2\pi}{\alpha} + \beta,$$

and

$$\psi_2^{(1)}(t_1) = \int_{(1-\alpha)t_1 - \frac{\beta}{l_1^2}}^{\frac{l_2}{l_1}\pi - \beta} q(t_2)dt_2,$$

are absolutely continuous on fraud

$$\left[\mu_1^{(1)},\ \xi_1 + \frac{1}{l_1}x^*\right], \left[\mu_2^{(1)},\ -\frac{\beta}{\alpha l_1^2} + \frac{x^*}{\alpha l_1}\right] \text{ and } x^* \in \left[\nu_3^{(1)},\ \nu_4^{(1)}\right].$$

So, these segments are limited and have performed almost everywhere

Let's

$$x = \frac{2x^* + \beta}{2 - \alpha},\ x_4 = \frac{l_2}{l_1}\pi + \frac{\alpha\beta}{2l_1},\ x_5 = \pi - \frac{\xi_1}{l_1^*},\ x_6 = \frac{\beta}{\alpha}\left(\frac{1}{2l_1} + \frac{1}{l_2^2}\right).$$

When (23) becomes

$$q(x) = \widetilde{f}(x) + \int_{\mu_1^{(1)}}^{x + \frac{\xi_1}{2l_1}} q(t_1)(2-\alpha)\psi_1^{(1)}(t_1)dt_1 + \int_{\mu_2^{(1)}}^{\frac{1}{\alpha}x - x_6} q(t_1)(2-\alpha)\psi_2^{(1)}(t_1)dt_1.$$

$$(\Phi_3)$$

Let's $\underline{t_1} = \min\left\{\mu_2^{(1)}, \mu_2^{(2)}\right\}$ and $s_1 x + s_2 = \max\left\{x + \frac{\xi_1}{2l_1}, \frac{1}{\alpha} x - x_6\right\}$,

$$\psi(t_1) = (2 - \alpha)\psi_1^{(1)}(t_1) + (2 - \alpha)\psi_2^{(1)}(t_1), \quad h(x) = q(x) - \widetilde{f}(x).$$

When (Φ_3) becomes

$$h(x) = \int_{t_1}^{s_1 x + s_2} \left(h(t_1) + \widetilde{f}(t_1)\right)\psi(t_1)dt_1. \tag{23$_2$}$$

Operator $A(h) = \int_{t_1}^{s_1 x + s_2} \left(h(t_1) + \widetilde{f}(t_1)\right)\psi(t_1)dt_1$ with some degree of contraction operator and a uniform fixed point. This provides the solution for q at a distance $[x_4, x_3]$, where is $x_4 = \frac{l_2}{l_1}\pi - \frac{1 - \alpha}{2 - \alpha}\beta$. With intervals $\left[\nu_0^{(1)}, \nu_1^{(1)}\right]$, $\left[\nu_1^{(1)}, \nu_2^{(1)}\right]$, and $\left[\nu_2^{(1)}, \nu_3^{(1)}\right]$, the solution was prepared by the same procedure as in interval $\left[\nu_3^{(1)}, \nu_4^{(1)}\right]$.

Theorem. *There is a single partition segments* $[\xi_1, \pi]$ *and* $\left[\nu_{-2}^{(1)}, \nu_5^{(1)}\right]$ *such that*

$$[\xi_1, \pi] = \bigcup_{k=0}^{\infty} [x_{k+1}, x_k], \; x_0 = \pi, \; x_\infty = \xi_1,$$

and

$$\left[\nu_{-2}^{(1)}, \nu_5^{(1)}\right] = \bigcup_{k=0}^{\infty} [x_{k+1}^*], \; x_0^* = \nu_5^{(1)}, \; x_\infty = \nu_{-2}^{(1)}, \; x_1^* = \nu_{-1}^{(1)}.$$

Proof. By using the function $x^* = \frac{\alpha x + \beta}{2}$ point π exceeding the point $\nu_{-1}^{(1)}$ and using the tool $x^* = \frac{(2-\alpha)x - \beta}{2}$ point $x_1 = \frac{\alpha\pi + 2\beta}{2-\alpha}$ exceeding the point $\nu_{-1}^{(1)}$ because $x_1 = \frac{2\left(\frac{\alpha\pi + \beta}{2}\right) + \beta}{2-\alpha} = \frac{\alpha\pi + 2\beta}{2-\alpha}$. Further,

$$\frac{\alpha x_1 + \beta}{2} = \frac{1}{2}\left(\frac{\alpha^2\pi}{2-\alpha} + \frac{2\alpha\beta}{2-\alpha} + \beta\right) = x_2^*,$$

and is

$$x_2 = \frac{\left(\frac{\alpha^2\pi}{2-\alpha} + \frac{2\alpha\beta}{2-\alpha} + \beta + \beta\right)}{2-\alpha} = \frac{\alpha^2\pi}{(2-\alpha)^2} + \frac{2\alpha\beta}{(2-\alpha)^2} + \frac{2\beta}{2-\alpha}.$$

Also,

$$\frac{\alpha x_2 + \beta}{2} = \frac{1}{2}\left[\frac{\alpha^3\pi}{(2-\alpha)^2} + \frac{2\alpha^2\pi}{(2-\alpha)^2} + \frac{2\beta}{2-\alpha} + \beta\right] = x_3^*,$$

and

$$x_l = \left(\frac{\alpha}{2-\alpha}\right)^l \pi + \frac{2\beta}{2-\alpha}\left[\left(\frac{\alpha}{2-\alpha}\right)^{l-1} + \left(\frac{\alpha}{2-\alpha}\right)^{l-2} + \cdots + \frac{\alpha}{2-\alpha} + 1\right]. \quad (23_2)$$

Directly checking that the formula (23_2) is true when using in (23_2) instead of l, $l+1$ put and the formula are valid for every $l \in N$.

Moreover valid

$$\lim_{l\to\infty} x_l = \lim_{l\to\infty}\left(\frac{\alpha}{2-\alpha}\right)^l \pi + \frac{2\beta}{2-\alpha}\lim_{l\to\infty}\left[1 + \frac{\alpha}{2-\alpha} + \cdots + \left(\frac{\alpha}{2-\alpha}\right)^{l-1}\right]$$

$$= \frac{2\beta}{2-\alpha}\lim_{l\to\infty}\frac{1 - \left(\frac{\alpha}{2-\alpha}\right)^l}{1 - \frac{\alpha}{2-\alpha}} = \frac{\beta}{1-\alpha} = \xi_1 = x_\infty.$$

Like checking that applies

$$\lim_{l\to\infty} x_l^* = \nu_{-2}^{(1)} = \frac{1}{2}\xi_1.$$

This theorem evidence.

Effect 1: While variable x lies in the segment $[x_{k+1},\ x_k],\ k = 0,\ 1,\ 2,\ \ldots$ when valid

$$\frac{(2-\alpha)x-\beta}{2} \in [x_{k+1}^*,\ x_k^*], \text{ and } \frac{\alpha x+\beta}{2} \in [x_{k+2}^*,\ x_{k+1}^*].$$

Effect 2: Since the solution of the Equation (23) defined on $[x_1,\ \pi]$ stops finding potential to $[\xi_1,\ x_1)$. Equation (23) at a distance $\left[\nu_{-2}^{(1)},\ \nu_{-1}^{(1)}\right]$ has the form

$$\frac{1}{\alpha}q\left(\frac{2x^*-\beta}{\alpha}\right) \pm \frac{1}{2-\alpha}q\left(\frac{2x^*+\beta}{2-\alpha}\right) = f(x^*)$$

$$x = \frac{2x^*+\beta}{2-\alpha} \in [x_{k+2},\ x_{k+1}] \Rightarrow \frac{2x^*-\beta}{\alpha} = \frac{2-\alpha}{\alpha}x - \frac{2\beta}{\alpha} \in [x_{k+1},\ x_k],$$

$$k = 0,\ 1,\ 2,\ \ldots.$$

Therefore,

$$q(x) = -\frac{2-\alpha}{\alpha}q\left(\frac{2-\alpha}{\alpha}x - \frac{2\beta}{\alpha}\right) + f\left(\frac{(2-\alpha)x-\beta}{2}\right). \qquad (23_3)$$

If it is $x \in [x_{k+1},\ x_k],\ k = 1,\ 2,\ \ldots$ when it is $\frac{2-\alpha}{\alpha}x - \frac{2\beta}{\alpha} \in [x_k,\ x_{k-1}]$.

For $k = 0$, we determine the potential q in the previous treatments to the right side (23_3) is known. Relation (23_3) yields the potential q on the segment $[x_2,\ x_1]$. Successive potential is determined on each segment $[x_{k+1},\ x_k],\ k = 1,\ 2,\ \ldots.$ Finally, this procedure gives potential on $[\xi_1,\ x_1]$. Thus we have proved the main result:

Theorem 3. *Some were given two sets of eigenvalues* λ_{nj}, $j = 1, 2$. $n \in N_0$ *operator* $D^2(q, \alpha, \beta, 0, H)$ *with potential* $q : \exists \mu_1 \in (0, \pi) : q(\xi_1)$ $q(\pi) \neq 0$;

$$(\alpha, \beta) \in (0, 1) \times \left(\underline{\beta}(\alpha), \bar{\beta}(\alpha)\right); \underline{\beta}(\alpha) = \frac{-(1-\alpha)^3}{\alpha^2 - 3\alpha - 3}\pi, \bar{\beta}(\alpha) = (1-\alpha)\pi,$$

and if λ_{nj} *him have asymptotic* (17) *whereby* $\tilde{\alpha}_{2n}{}^{+}$ *series of cosine Fourier coefficients function* $\tilde{q}^{+} \in L_2[0, \pi] : \tilde{q}^{+}(x)\, x \in [0, \mu_1) \cup (\mu_2, \pi], \mu_1 < \mu_2$, *a numbers* μ_1, μ_2 *are known. Then the operator* D^2 *is known.*

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