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HERMITE-HADAMARD TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR GENERALIZED $(1, m, \varphi)$-PREINVEX FUNCTIONS

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Abstract

In the present paper, by using new identity for fractional integrals, some new estimates on generalizations of Hermite-Hadamard type inequalities for the class of generalized $(1, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integral are established. At the end, some applications to special means are given.

1. Introduction

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the

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interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K, $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following definition will be used in the sequel.

Definition 1.2. The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

for $c > b > 0$ and $|z| < 1$, where $\beta(x, y)$ is the Euler beta function for all $x, y > 0$.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [11], [12], [13]). For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions (see [9]) and the references cited therein, also (see [8]) and the references cited therein.

Fractional calculus (see [9]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [9]-[16]) and the references cited therein.

Now, let us recall some definitions of various convex functions.

Definition 1.4 (see [3])**.** A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}_{\circ}$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \quad t \in [0, 1].$$

Definition 1.5 (see [6])**.** A function $f : \mathbb{R}_{\circ} \to \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y), \tag{1.2}$$

for all $x, y \in \mathbb{R}_{\circ}$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_{\circ}$ as usual. The s-convex functions in the second sense have been investigated in (see [6]).

Definition 1.6 (see [4])**.** A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details, please see ([4], [5]) and the references therein.

Definition 1.7 (see [7])**.** The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

Motivated by these results, the aim of this paper is to establish some generalizations of Hermite-Hadamard type inequalities using new identity given in Section 2 for generalized $(1, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integral. In Section 3, some applications to special means are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

2. Main Results

Definition 2.1 (see [2])**.** A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \to \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$.

Definition 2.3 (see [1])**.** Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0, 1] \to \mathbb{R}^n$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. For $f : K \to \mathbb{R}$ and any fixed $s, m \in (0, 1]$, if

$$f(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)) \leq m(1-t)^s f(\varphi(x)) + t^s f(\varphi(y)), \tag{2.1}$$

is valid for all $x, y \in K$, $t \in [0, 1]$, then we say that $f(x)$ is a generalized (s, m, φ)-preinvex function with respect to η.

Remark 2.4. For $s = 1$ in Definition 2.3, we say that $f(x)$ is a generalized $(1, m, \varphi)$-preinvex function with respect to η.

Remark 2.5. In Definition 2.3, it is worthwhile to note that the class of generalized (s, m, φ)-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.5.

Throughout this paper, we denote

$$\begin{aligned} &A_\alpha(x; \eta, \varphi, \lambda, \mu, m, a, b) \\ &= \frac{\eta(\varphi(a), \varphi(x), m)^\alpha}{\eta(\varphi(b), \varphi(a), m)} \{\lambda f(m\varphi(x) + \eta(\varphi(a), \varphi(x), m)) + (1-\lambda) f(m\varphi(x))\} \\ &\quad + \frac{\eta(\varphi(b), \varphi(x), m)^\alpha}{\eta(\varphi(b), \varphi(a), m)} \{\mu f(m\varphi(x)) + (1-\mu) f(m\varphi(x) + \eta(\varphi(b), \varphi(x), m))\} \\ &\quad - \frac{\Gamma(\alpha+1)}{\eta(\varphi(b), \varphi(a), m)} \\ &\quad \times \left[J^\alpha_{(m\varphi(x)+\eta(\varphi(a), \varphi(x), m))-} f(m\varphi(x)) + J^\alpha_{(m\varphi(x)+\eta(\varphi(b), \varphi(x), m))-} f(m\varphi(x)) \right], \end{aligned}$$

where $x = rb + (1-r)a$ for $r \in [0, 1]$.

In this section, in order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized $(1, m, \varphi)$-preinvex function via fractional integrals, we need the following lemmas:

Lemma 2.6. *Let* $\varphi : I \to \mathbb{R}$ *be a continuous increasing function. Suppose* $K \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : K \times K \times (0, 1] \to \mathbb{R}$ *for any fixed* $m \in (0, 1]$ *and let* $\varphi(a), \varphi(b) \in K$, $a < b$ *with* $\eta(\varphi(b), \varphi(a), m) \neq 0$. *Assume that* $f : K \to \mathbb{R}$ *is a differentiable function on* K° *and* $f' \in L_1[\varphi(a), \varphi(b)]$. *Then for any* $\lambda, \mu \in [0, 1]$ *and* $\alpha > 0$, *the following identity holds*:

$$A_\alpha(x; \eta, \varphi, \lambda, \mu, m, a, b)$$

$$= \frac{\eta(\varphi(b), \varphi(x), m)^{\alpha+1}}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 (t^\alpha - \mu) f'(m\varphi(x) + t\eta(\varphi(b), \varphi(x), m))dt$$

$$- \frac{\eta(\varphi(a), \varphi(x), m)^{\alpha+1}}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 (1 - \lambda - t^\alpha) f'(m\varphi(x) + t\eta(\varphi(a), \varphi(x), m))dt, \tag{2.2}$$

where $x = rb + (1 - r)a$ *for* $r \in [0, 1]$.

Proof. A simple proof of the equality (2.2) can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

□

For the simplicities of notations, let

$$\delta_1(\alpha, \xi) = \int_0^1 |t^\alpha - \xi| dt, \quad \delta_2(\alpha, \xi) = \int_0^1 t|t^\alpha - \xi| dt,$$

$$\varrho_1(\alpha, \xi, p) = \int_0^1 |t^\alpha - \xi|^p dt. \tag{2.3}$$

Lemma 2.7. *For* $0 \le \xi \le 1$, *we have*

(a)

$$\delta_1(\alpha, \xi) = \begin{cases} \dfrac{1}{\alpha+1}, & \xi = 0; \\ \dfrac{2\alpha\xi^{1+\frac{1}{\alpha}}+1}{\alpha+1} - \xi, & 0 < \xi < 1; \\ \dfrac{\alpha}{\alpha+1}, & \xi = 1. \end{cases}$$

(b)

$$\delta_2(\alpha, \xi) = \begin{cases} \dfrac{1}{\alpha+2}, & \xi = 0; \\ \dfrac{\alpha\xi^{1+\frac{2}{\alpha}}+1}{\alpha+2} - \dfrac{\xi}{2}, & 0 < \xi < 1; \\ \dfrac{\alpha}{2(\alpha+2)}, & \xi = 1. \end{cases}$$

(c)

$$\varrho_1(\alpha, \xi, p) = \begin{cases} \dfrac{1}{p\alpha+1}, & \xi = 0; \\ \dfrac{\xi^{p+\frac{1}{\alpha}}}{\alpha}\beta\left(\dfrac{1}{\alpha}, p+1\right) + \dfrac{(1-\xi)^{p+1}}{\alpha(p+1)} & \\ \times {}_2F_1\left(1-\dfrac{1}{\alpha}, 1; p+2; 1-\xi\right), & 0 < \xi < 1; \\ \dfrac{1}{\alpha}\beta\left(p+1, \dfrac{1}{\alpha}\right), & \xi = 1. \end{cases}$$

Proof. These equalities follows from a straightfoward computation of definite integrals. □

Now we turn our attention to establish new integral inequalities of Hermite-Hadamard type for generalized $(1, m, \varphi)$-preinvex functions via fractional integrals. Using Lemmas 2.6 and 2.7, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative:

Theorem 2.8. *Let* $\varphi : I \to \mathbb{R}$ *be a continuous increasing function. Suppose* $A \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : A \times A \times (0, 1] \to \mathbb{R}$ *for any fixed* $m \in (0, 1]$ *and let* $\varphi(a), \varphi(b) \in A$, $a < b$ *with* $\eta(\varphi(b), \varphi(a), m) \neq 0$. *Assume that* $f : A \to \mathbb{R}$ *is a differentiable function on* A°. *If* $|f'|^q$ *is a generalized* $(1, m, \varphi)$*-preinvex function on* $[\varphi(a), \varphi(b)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, *then, for any* $\lambda, \mu \in [0, 1]$ *and* $\alpha > 0$, *the following inequality for fractional integrals holds*:

$$|A_\alpha(x; \eta, \varphi, \lambda, \mu, m, a, b)|$$

$$\leq \frac{|\eta(\varphi(a), \varphi(x), m)|^{\alpha+1}}{|\eta(\varphi(b), \varphi(a), m)|} \varrho_1^{\frac{1}{p}}(\alpha, 1-\lambda, p) \left(\frac{|f'(\varphi(a))|^q + m|f'(\varphi(x))|^q}{2} \right)^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(b), \varphi(x), m)|^{\alpha+1}}{|\eta(\varphi(b), \varphi(a), m)|} \varrho_1^{\frac{1}{p}}(\alpha, \mu, p) \left(\frac{|f'(\varphi(b))|^q + m|f'(\varphi(x))|^q}{2} \right)^{\frac{1}{q}}. \tag{2.4}$$

Proof. Suppose that $q > 1$. Using Lemmas 2.6 and 2.7, the fact that $|f'|^q$ is a generalized $(1, m, \varphi)$-preinvex function, property of the modulus and Hölder's inequality, we have

$$|A_\alpha(x; \eta, \varphi, \lambda, \mu, m, a, b)|$$

$$\leq \frac{|\eta(\varphi(a), \varphi(x), m)|^{\alpha+1}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 |1 - \lambda - t^\alpha| \, |f'(m\varphi(x) + t\eta(\varphi(a), \varphi(x), m))| \, dt$$

$$+ \frac{|\eta(\varphi(b), \varphi(x), m)|^{\alpha+1}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 |t^\alpha - \mu| \, |f'(m\varphi(x) + t\eta(\varphi(b), \varphi(x), m))| \, dt$$

$$\leq \frac{|\eta(\varphi(a),\ \varphi(x),\ m)|^{\alpha+1}}{|\eta(\varphi(b),\ \varphi(a),\ m)|}\left(\int_0^1 |1-\lambda-t^{\alpha}|^{p}\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f'(m\varphi(x)+t\eta(\varphi(a),\ \varphi(x),\ m))|^{q}\,dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b),\ \varphi(x),\ m)|^{\alpha+1}}{|\eta(\varphi(b),\ \varphi(a),\ m)|}\left(\int_0^1 |t^{\alpha}-\mu|^{p}\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f'(m\varphi(x)+t\eta(\varphi(b),\ \varphi(x),\ m))|^{q}\,dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(a),\ \varphi(x),\ m)|^{\alpha+1}}{|\eta(\varphi(b),\ \varphi(a),\ m)|}\,\varrho_1^{\frac{1}{p}}(\alpha,\ 1-\lambda,\ p)$$

$$\times\left(\int_0^1 (m(1-t)|f'(\varphi(x))|^{q}+t|f'(\varphi(a))|^{q}\,dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b),\ \varphi(x),\ m)|^{\alpha+1}}{|\eta(\varphi(b),\ \varphi(a),\ m)|}\,\varrho_1^{\frac{1}{p}}(\alpha,\ \mu,\ p)$$

$$\times\left(\int_0^1 (m(1-t)|f'(\varphi(x))|^{q}+t|f'(\varphi(b))|^{q}\,dt\right)^{\frac{1}{q}}$$

$$=\frac{|\eta(\varphi(a),\ \varphi(x),\ m)|^{\alpha+1}}{|\eta(\varphi(b),\ \varphi(a),\ m)|}\,\varrho_1^{\frac{1}{p}}(\alpha,\ 1-\lambda,\ p)\left(\frac{|f'(\varphi(a))|^{q}+m|f'(\varphi(x))|^{q}}{2}\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b),\ \varphi(x),\ m)|^{\alpha+1}}{|\eta(\varphi(b),\ \varphi(a),\ m)|}\,\varrho_1^{\frac{1}{p}}(\alpha,\ \mu,\ p)\left(\frac{|f'(\varphi(b))|^{q}+m|f'(\varphi(x))|^{q}}{2}\right)^{\frac{1}{q}}.$$

The proof of Theorem 2.8 is completed. □

Corollary 2.9. *Under the conditions of Theorem* 2.8, *if we choose* $\lambda = \mu = \frac{1}{2}$, $x = \frac{a+b}{2}$, $m = 1$ *and* $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, *we get the following generalized Hermite-Hadamard type inequality for fractional integrals*:

$$\left| \frac{(\varphi(a) - \varphi(\frac{a+b}{2}))^{\alpha}}{\varphi(b) - \varphi(a)} \left\{ \frac{f(\varphi(a)) + f(\varphi(\frac{a+b}{2}))}{2} \right\} \right.$$

$$+ \frac{(\varphi(b) - \varphi(\frac{a+b}{2}))^{\alpha}}{\varphi(b) - \varphi(a)} \left\{ \frac{f(\varphi(b)) + f(\varphi(\frac{a+b}{2}))}{2} \right\}$$

$$\left. - \frac{\Gamma(\alpha+1)}{\varphi(b) - \varphi(a)} \left[J^{\alpha}_{\varphi(a)-} f\left(\varphi\left(\frac{a+b}{2}\right)\right) + J^{\alpha}_{\varphi(b)-} f\left(\varphi\left(\frac{a+b}{2}\right)\right) \right] \right|$$

$$\leq \frac{(\varphi(\frac{a+b}{2}) - \varphi(a))^{\alpha+1}}{\varphi(b) - \varphi(a)} \varrho_1^{\frac{1}{p}}\left(\alpha, \frac{1}{2}, p\right) \left(\frac{\left|f'(\varphi(\frac{a+b}{2}))\right|^q + |f'(\varphi(a))|^q}{2} \right)^{\frac{1}{q}}$$

$$+ \frac{(\varphi(b) - \varphi(\frac{a+b}{2}))^{\alpha+1}}{\varphi(b) - \varphi(a)} \varrho_1^{\frac{1}{p}}\left(\alpha, \frac{1}{2}, p\right) \left(\frac{\left|f'(\varphi(\frac{a+b}{2}))\right|^q + |f'(\varphi(b))|^q}{2} \right)^{\frac{1}{q}}. \quad (2.5)$$

Theorem 2.10. *Let* $\varphi : I \to \mathbb{R}$ *be a continuous increasing function. Suppose* $A \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : A \times A \times (0, 1] \to \mathbb{R}$ *for any fixed* $m \in (0, 1]$ *and let* $\varphi(a), \varphi(b) \in A$, $a < b$ *with* $\eta(\varphi(b), \varphi(a), m) \neq 0$. *Assume that* $f : A \to \mathbb{R}$ *is a differentiable function on* A°. *If* $|f'|^q$ *is a generalized* $(1, m, \varphi)$*-preinvex function on* $[\varphi(a), \varphi(b)]$, $q \geq 1$, *then, for any* $\lambda, \mu \in [0, 1]$ *and* $\alpha > 0$, *the following inequality for fractional integrals holds*:

$$
\begin{aligned}
&|A_\alpha(x;\,\eta,\,\varphi,\,\lambda,\,\mu,\,m,\,a,\,b)| \\
&\leq \frac{|\eta(\varphi(a),\,\varphi(x),\,m)|^{\alpha+1}}{|\eta(\varphi(b),\,\varphi(a),\,m)|}\,\delta_1^{1-\frac{1}{q}}(\alpha,\,1-\lambda) \\
&\quad\times\Big[\delta_2(\alpha,\,1-\lambda)|f'(\varphi(a))|^q + m(\delta_1(\alpha,\,1-\lambda)-\delta_2(\alpha,\,1-\lambda))|f'(\varphi(x))|^q\Big]^{\frac{1}{q}} \\
&\quad+\frac{|\eta(\varphi(b),\,\varphi(x),\,m)|^{\alpha+1}}{|\eta(\varphi(b),\,\varphi(a),\,m)|}\,\delta_1^{1-\frac{1}{q}}(\alpha,\,\mu) \\
&\quad\times\Big[\delta_2(\alpha,\,\mu)|f'(\varphi(b))|^q + m(\delta_1(\alpha,\,\mu)-\delta_2(\alpha,\,\mu))|f'(\varphi(x))|^q\Big]^{\frac{1}{q}}. \qquad (2.6)
\end{aligned}
$$

Proof. Suppose that $q \geq 1$. Using Lemmas 2.6 and 2.7, the fact that $|f'|^q$ is a generalized $(1,\,m,\,\varphi)$-preinvex function, property of the modulus and the well-known power mean inequality, we have

$$
\begin{aligned}
&|A_\alpha(x;\,\eta,\,\varphi,\,\lambda,\,\mu,\,m,\,a,\,b)| \\
&\leq \frac{|\eta(\varphi(a),\,\varphi(x),\,m)|^{\alpha+1}}{|\eta(\varphi(b),\,\varphi(a),\,m)|}\int_0^1|1-\lambda-t^\alpha|\,|f'(m\varphi(x)+t\eta(\varphi(a),\,\varphi(x),\,m))|\,dt \\
&\quad+\frac{|\eta(\varphi(b),\,\varphi(x),\,m)|^{\alpha+1}}{|\eta(\varphi(b),\,\varphi(a),\,m)|}\int_0^1|t^\alpha-\mu|\,|f'(m\varphi(x)+t\eta(\varphi(b),\,\varphi(x),\,m))|\,dt \\
&\leq \frac{|\eta(\varphi(a),\,\varphi(x),\,m)|^{\alpha+1}}{|\eta(\varphi(b),\,\varphi(a),\,m)|}\left(\int_0^1|1-\lambda-t^\alpha|\,dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1|1-\lambda-t^\alpha|\,|f'(m\varphi(x)+t\eta(\varphi(a),\,\varphi(x),\,m))|^q\,dt\right)^{\frac{1}{q}} \\
&\quad+\frac{|\eta(\varphi(b),\,\varphi(x),\,m)|^{\alpha+1}}{|\eta(\varphi(b),\,\varphi(a),\,m)|}\left(\int_0^1|t^\alpha-\mu|\,dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1|t^\alpha-\mu|\,|f'(m\varphi(x)+t\eta(\varphi(b),\,\varphi(x),\,m))|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$\leq \frac{|\eta(\varphi(a),\, \varphi(x),\, m)|^{\alpha+1}}{|\eta(\varphi(b),\, \varphi(a),\, m)|}\, \delta_1^{1-\frac{1}{q}}(\alpha,\, 1-\lambda)$$

$$\times \left[\delta_2(\alpha,\, 1-\lambda)|f'(\varphi(a))|^q + m(\delta_1(\alpha,\, 1-\lambda) - \delta_2(\alpha,\, 1-\lambda))\,|f'(\varphi(x))|^q\right]^{\frac{1}{q}}$$

$$+ \frac{|\eta(\varphi(b),\, \varphi(x),\, m)|^{\alpha+1}}{|\eta(\varphi(b),\, \varphi(a),\, m)|}\, \delta_1^{1-\frac{1}{q}}(\alpha,\, \mu)$$

$$\times \left[\delta_2(\alpha,\, \mu)|f'(\varphi(b))|^q + m(\delta_1(\alpha,\, \mu) - \delta_2(\alpha,\, \mu))\,|f'(\varphi(x))|^q\right]^{\frac{1}{q}}.$$

The proof of Theorem 2.10 is completed. □

Corollary 2.11. *Under the conditions of Theorem* 2.10, *if we choose* $\lambda = \mu = \frac{1}{2}$, $x = \frac{a+b}{2}$, $m = 1$ *and* $\eta(\varphi(y),\, \varphi(x),\, 1) = \varphi(y) - \varphi(x)$, *we get the following generalized Hermite-Hadamard type inequality for fractional integrals*:

$$\left|\frac{(\varphi(a) - \varphi(\frac{a+b}{2}))^{\alpha}}{\varphi(b) - \varphi(a)}\left\{\frac{f(\varphi(a)) + f(\varphi(\frac{a+b}{2}))}{2}\right\}\right.$$

$$+ \frac{(\varphi(b) - \varphi(\frac{a+b}{2}))^{\alpha}}{\varphi(b) - \varphi(a)}\left\{\frac{f(\varphi(b)) + f(\varphi(\frac{a+b}{2}))}{2}\right\}$$

$$\left. - \frac{\Gamma(\alpha+1)}{\varphi(b) - \varphi(a)}\left[J^{\alpha}_{\varphi(a)-}f\left(\varphi\left(\frac{a+b}{2}\right)\right) + J^{\alpha}_{\varphi(b)-}f\left(\varphi\left(\frac{a+b}{2}\right)\right)\right]\right|$$

$$\leq \frac{(\varphi(\frac{a+b}{2}) - \varphi(a))^{\alpha+1}}{\varphi(b) - \varphi(a)}\, \delta_1^{1-\frac{1}{q}}\left(\alpha,\, \frac{1}{2}\right)$$

$$\times \left[\delta_2\left(\alpha,\, \frac{1}{2}\right)|f'(\varphi(a))|^q + \left(\delta_1\left(\alpha,\, \frac{1}{2}\right) - \delta_2\left(\alpha,\, \frac{1}{2}\right)\right)\left|f'\left(\varphi\left(\frac{a+b}{2}\right)\right)\right|^q\right]^{\frac{1}{q}}$$

$$+ \frac{(\varphi(b) - \varphi(\frac{a+b}{2}))^{\alpha+1}}{\varphi(b) - \varphi(a)}\, \delta_1^{1-\frac{1}{q}}\left(\alpha,\, \frac{1}{2}\right)$$

$$\times \left[\delta_2\left(\alpha,\, \frac{1}{2}\right)|f'(\varphi(b))|^q + \left(\delta_1\left(\alpha,\, \frac{1}{2}\right) - \delta_2\left(\alpha,\, \frac{1}{2}\right)\right)\left|f'\left(\varphi\left(\frac{a+b}{2}\right)\right)\right|^q\right]^{\frac{1}{q}}. \quad (2.7)$$

3. Applications to Special Means

In the following, we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1 (see [17]). A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, is called a mean function if it has the following properties:

(1) Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$.

(2) Symmetry: $M(x, y) = M(y, x)$.

(3) Reflexivity: $M(x, x) = x$.

(4) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$.

(5) Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β $(\alpha \neq \beta)$.

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \; r \geq 1.$$

(5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}; \ |\alpha| \neq |\beta|, \ \alpha\beta \neq 0.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \ p \in \mathbb{R} \setminus \{-1, 0\}, \ \alpha \neq \beta.$$

(8) The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}},$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0 (i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well-known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \to \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \to \mathbb{R}$ be a continuous increasing function, therefore one can obtain various inequalities using the results of Section 2 for these means as follows:

Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting $\eta(\varphi(a), \varphi(b)) = M(\varphi(a), \varphi(b))$, $\eta(\varphi(a), \varphi(x)) = M(\varphi(a), \varphi(x))$, $\eta(\varphi(b), \varphi(x)) = M(\varphi(b), \varphi(x))$, $\forall x \in A$, for value $m = 1$ in (2.4) and (2.6), one can obtain the following interesting inequalities involving means:

$$\left|\frac{M(\varphi(a),\,\varphi(x))^{\alpha}}{M(\varphi(a),\,\varphi(b))}\{\lambda f(\varphi(x)+M(\varphi(a),\,\varphi(x)))+(1-\lambda)f(\varphi(x))\}\right.$$

$$+\frac{M(\varphi(b),\,\varphi(x))^{\alpha}}{M(\varphi(a),\,\varphi(b))}\{\mu f(\varphi(x))+(1-\mu)f(\varphi(x)+M(\varphi(b),\,\varphi(x)))\}$$

$$-\frac{\Gamma(\alpha+1)}{M(\varphi(a),\,\varphi(b))}$$

$$\left.\times\left[J^{\alpha}_{(\varphi(x)+M(\varphi(a),\,\varphi(x)))-}f(\varphi(x))+J^{\alpha}_{(\varphi(x)+M(\varphi(b),\,\varphi(x)))-}f(\varphi(x))\right]\right|$$

$$\leq\frac{M(\varphi(a),\,\varphi(x))^{\alpha+1}}{M(\varphi(a),\,\varphi(b))}\varrho_1^{\frac{1}{p}}(\alpha,\,1-\lambda,\,p)\left(\frac{|f'(\varphi(a))|^q+|f'(\varphi(x))|^q}{2}\right)^{\frac{1}{q}}$$

$$+\frac{M(\varphi(b),\,\varphi(x))^{\alpha+1}}{M(\varphi(a),\,\varphi(b))}\varrho_1^{\frac{1}{p}}(\alpha,\,\mu,\,p)\left(\frac{|f'(\varphi(b))|^q+|f'(\varphi(x))|^q}{2}\right)^{\frac{1}{q}},\tag{3.1}$$

$$\left|\frac{M(\varphi(a),\,\varphi(x))^{\alpha}}{M(\varphi(a),\,\varphi(b))}\{\lambda f(\varphi(x)+M(\varphi(a),\,\varphi(x)))+(1-\lambda)f(\varphi(x))\}\right.$$

$$+\frac{M(\varphi(b),\,\varphi(x))^{\alpha}}{M(\varphi(a),\,\varphi(b))}\{\mu f(\varphi(x))+(1-\mu)f(\varphi(x)+M(\varphi(b),\,\varphi(x)))\}$$

$$-\frac{\Gamma(\alpha+1)}{M(\varphi(a),\,\varphi(b))}$$

$$\left.\times\left[J^{\alpha}_{(\varphi(x)+M(\varphi(a),\,\varphi(x)))-}f(\varphi(x))+J^{\alpha}_{(\varphi(x)+M(\varphi(b),\,\varphi(x)))-}f(\varphi(x))\right]\right|$$

$$\leq\frac{M(\varphi(a),\,\varphi(x))^{\alpha+1}}{M(\varphi(a),\,\varphi(b))}\delta_1^{1-\frac{1}{q}}(\alpha,\,1-\lambda)$$

$$\times\left[\delta_2(\alpha,\,1-\lambda)|f'(\varphi(a))|^q+(\delta_1(\alpha,\,1-\lambda)-\delta_2(\alpha,\,1-\lambda))|f'(\varphi(x))|^q\right]^{\frac{1}{q}}$$

$$+\frac{M(\varphi(b),\,\varphi(x))^{\alpha+1}}{M(\varphi(a),\,\varphi(b))}\delta_1^{1-\frac{1}{q}}(\alpha,\,\mu)$$

$$\times\left[\delta_2(\alpha,\,\mu)|f'(\varphi(b))|^q+(\delta_1(\alpha,\,\mu)-\delta_2(\alpha,\,\mu))|f'(\varphi(x))|^q\right]^{\frac{1}{q}}.\tag{3.2}$$

Letting $M(\varphi(x), \varphi(y)) = A, G, H, P_r, I, L, L_p, M_p, \forall x, y \in A,$ in (3.1) and (3.2), we get the inequalities involving means for a particular choices of a differentiable generalized $(1, 1, \varphi)$-preinvex function f. The details are left to the interested reader.

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