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TOTAL DOMINATION POLYNOMIALS OF SOME GRAPHS

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Abstract

A hypergraph is an ordered pair $H = (V, E)$, where V is a finite nonempty set called vertices and E is a collection of subsets of V, called hyper edges or simply edges. A subset T of vertices in a hypergraph H is called a vertex cover if T has a nonempty intersection with every edge of H. The vertex covering number $\tau(H)$ of H is the minimum size of a vertex cover in H. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of H with cardinality i and let $c(H, i)$ be the cardinality

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of $\mathcal{C}(H, i)$. The polynomial $\sum_{i=\tau(H)}^{|V(H)|} c(H, i)x^i$ is defined as vertex cover polynomial of H. For a graph $G = (V, E)$, H_G denotes the hypergraph with vertex set V and edge set $\{N_G(x)|x \in V\}$. In this paper, we prove that the total domination polynomial of a connected graph G is the vertex cover polynomial of H_G. Using this result, we determine total domination polynomials of cartesian products of certain classes of graphs.

1. Introduction

All graphs considered in this paper are simple and connected unless otherwise stated. Notations and definitions not given here can be found in [2, 5, 9]. A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set and $E(G)$ is a collection of 2-point subsets of V. The sets $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. The *cartesian product* of two graphs G and H, denoted $G\Box H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices (a_1, b_1) and (a_2, b_2) if and only if either $a_1 = a_2$ and $b_1b_2 \in E(H)$ or $b_1 = b_2$ and $a_1a_2 \in E(G)$. The *open neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V|uv \in E(G)\}$. The *closed neighbourhood* of a vertex $v \in V(G)$ is $N_G[v] = N_G(v) \cup \{v\}$. If the graph G is clear from the context, we write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$. A *dominating set*, of a graph $G = (V, E)$ with no isolated vertex is a set S of vertices of G such that every vertex in $G \setminus S$ is adjacent to a vertex in S. If no proper subset of S is a dominating set of G, then S is a *minimal dominating set* of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. Let $\mathcal{D}(G, i)$ be the family of dominating sets of G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The polynomial $\mathcal{D}(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$ is defined as *domination polynomial*

of G. A *total dominating set*, abbreviated TD-set, of a graph $G = (V, E)$ with no isolated vertex is set S of vertices of G such that every vertex of G is adjacent to a vertex in S. If no proper subset of S is a TD-set of G, then S is a *minimal TD-set* of G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial $\mathcal{D}_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$ is defined as *total domination polynomial* of G. A *hypergraph* $H = (V, E)$ is a finite nonempty set $V = V(H)$ of elements called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyper edges* or simply *edges*. The *order* and *size* of H are $|V|$ and $|E|$, respectively. A *k-edge* in H is an edge of size k. A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover*) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. For further information on hypergraphs refer [4]. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of H with cardinality i and let $c(H, i) = |\mathcal{C}(H, i)|$. The polynomial $\mathcal{C}(H, x) = \sum_{i=\tau(H)}^{|V(H)|} c(H, i)x^i$ is defined as *vertex cover polynomial* of H. For a graph $G = (V, E)$, the $ONH(G)$ or H_G is the *open neighbourhood hypergraph* of G; $H_G = (V, C)$ is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = C = \{N_G(x) | x \in V\}$ consisting of the open neighbourhoods of vertices of V in G. Let Γ be a finite group with identity e. Let $S \subseteq \Gamma$ such that $e \notin S$ and $S = S^{-1}$, that is, S is inverse closed. Then the *Cayley graph* $G = Cay(\Gamma, S)$, is defined as a graph with vertex set $V(G) = \Gamma$ and edge set $E(G) = \{(a, b) | ab^{-1} \in S\}$.

We need the following theorems to prove the main results of this paper.

Theorem 1.1 ([8]). *The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected.*

Theorem 1.2 ([9]). *If G is a graph with no isolated vertex and* H_G *is the ONH of G, then* $\gamma_t(G) = \tau(H_G)$.

Theorem 1.3 ([1]). *If a graph G consists of m components* $G_1, G_2, \ldots, G_m$, *then* $D(G, x) = D(G_1, x) \ldots D(G_m, x)$.

Theorem 1.4 ([1]). *For every* $n \geq 4$,

$$D(P_n, x) = x[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)],$$

with initial values $D(P_1, x) = x$, $D(P_2, x) = x^2 + 2x$, $D(P_3, x) = x^3 + 3x^2 + x$.

Theorem 1.5 ([1]). *For every* $n \geq 4$,

$$D(C_n, x) = x[D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)],$$

with initial values $D(C_1, x) = x$, $D(C_2, x) = x^2 + 2x$, $D(C_3, x) = x^3 + 3x^2 + 3x$.

Theorem 1.6 ([1]). $D(K_{m,n}, x) = ((1+x)^m - 1)((1+x)^n - 1) + x^m + x^n$.

Theorem 1.7 ([10]). *Let* L_n *be the graph* $P_n \square K_2$. *The domination polynomial for* L_n *satisfies the recurrence*: $D(L_n, x) = x(x+2)D(L_{n-1}, x) + x(x+1)D(L_{n-2}, x) + x^2(x+1)D(L_{n-3}, x) - x^3 D(L_{n-5}, x)$ *with initial values,*

n	$P_n \square K_2$
1	$x^2 + 2x$
2	$x^4 + 4x^3 + 6x^2$
3	$x^6 + 6x^5 + 15x^4 + 16x^3 + 3x^2$
4	$x^8 + 8x^7 + 28x^6 + 52x^5 + 48x^5 + 47x^4 + 2x^3$
5	$x^{10} + 10x^9 + 45x^8 + 116x^7 + 178x^6 + 148x^5 + 47x^4 + 2x^3$
6	$x^{12} + 12x^11 + 66x^{10} + 216x^9 + 453x^8 + 604x^7 + 470x^6 + 168x^5 + 17x^4$

2. Main Results

Theorem 2.1. *The total domination polynomial of a connected bipartite graph G is the product of the vertex cover polynomials of the two components of* H_G, *while the total domination polynomial of a connected graph that is not bipartite is the vertex cover polynomial of* H_G.

Proof. The proof follows immediately from the definitions of total dominating set of G and vertex cover polynomial of H_G. □

Theorem 2.2. *For a bipartite graph G,*

$$D_t(K_2 \square G, x) = [D(G, x)]^2.$$

Proof. Let $X = \{x_1, x_2, x_3, \ldots, x_m\}$, $Y = \{y_1, y_2, y_3, \ldots, y_n\}$ be the bipartition of G. If $V(K_2) = \{a, b\}$, then the bipartition of $K_2 \square G$ is $\{(a, x_1), (a, x_2), \ldots, (a, x_m), (b, y_1), (b, y_2), \ldots, (b, y_n)\}$, $\{(b, x_1), (b, x_2), \ldots, (b, x_m), (a, y_1), (a, y_2), \ldots, (a, y_n)\}$. For $i = 1, 2, \ldots, m$ and for $j = 1, 2, \ldots, n$, we have

$$N((a, x_i)) = \{(a, y) / y \sim x_i \text{ in } G\} \cup \{(b, x_i)\},$$

$$N((b, x_i)) = \{(b, y) / y \sim x_i \text{ in } G\} \cup \{(a, x_i)\},$$

$$N((a, y_j)) = \{(a, x) / x \sim y_j \text{ in } G\} \cup \{(b, y_j)\},$$

$$N((b, y_j)) = \{(b, x) / x \sim y_j \text{ in } G\} \cup \{(a, y_j)\}.$$

Let G_1 be a graph with bipartition $\{(b, x_1), (b, x_2), \ldots, (b, x_m)\}$, $\{(a, y_1), (a, y_2), \ldots, (a, y_n)\}$, such that $(b, x_i) \sim (a, y_j)$ if and only if $(a, x_i) \sim (a, y_j)$ in $K_2 \square G$. Similarly we construct another graph G_2 with bipartition $\{(a, x_1), (a, x_2), \ldots, (a, x_m)\}$, $\{(b, y_1), (b, y_2), \ldots, (b, y_n)\}$, such that $(a, x_i) \sim (b, y_j)$ if and only if $(b, x_i) \sim (b, y_j)$ in $K_2 \square G$. It can be observed that both G_1 and G_2 is isomorphic to G and a set N is an open neighbourhood of a vertex in $K_2 \square G$ if and only if N is a closed neighbourhood of a vertex in G_1 or G_2. Therefore, a set S is a total dominating set of $K_2 \square G$ if and only if S is a dominating set of $G_1 \cup G_2$. So,

$$\begin{aligned} D_t(K_2 \square G, x) &= D(G_1 \cup G_2, x) \\ &= D(G_1, x).D(G_2, x) \\ &= [D(G, x)]^2. \end{aligned}$$

This completes the proof. □

Corollary 2.3. *From Theorem* 2.2, *we can infer the following results*:

(1) $D_t(K_2 \square P_n, x) = [D(P_n, x)]^2$.

(2) $\gamma_t(K_2 \square P_n) = 2.\gamma(P_n) = 2.\left\lceil \frac{n}{3} \right\rceil$.

(3) $D_t(K_2 \square C_{2n}, x) = [D(C_{2n}, x)]^2$.

(4) $\gamma_t(K_2 \square C_{2n}) = 2.\gamma(C_{2n}) = 2.\left\lceil \frac{2n}{3} \right\rceil$.

(5) $D_t(K_2 \square K_{m,n}, x) = [D(K_{m,n}, x)]^2$.

(6) $\gamma_t(K_2 \square K_{m,n}) = 4$.

(7) $D_t(K_2 \square B_{m,n}, x) = [D(B_{m,n}, x)]^2$, *where* $B_{m,n}$ *is the bi-star graph*.

(8) *If* T *is a tree, then* $D_t(K_2 \square T, x) = [D(T, x)]^2$.

Theorem 2.4. *A set S is a dominating set of a cycle C_n if and only if it has non empty intersection with the set of vertices of all paths of length two in C_n.*

Proof. Let S be a dominating set and $P = uvw$ be a path of length two in C_n. If $S \cap \{u, v, w\} = \phi$, then the vertex v is not adjacent to any vertex in S and so S cannot be a dominating set of C_n. The converse is obvious. □

Theorem 2.5. *If n is odd, then $D_t(K_2 \square C_n, x) = D(C_{2n}, x)$.*

Proof. Let the vertices of $K_2 \square C_n$ be labelled as shown in Figure 1.

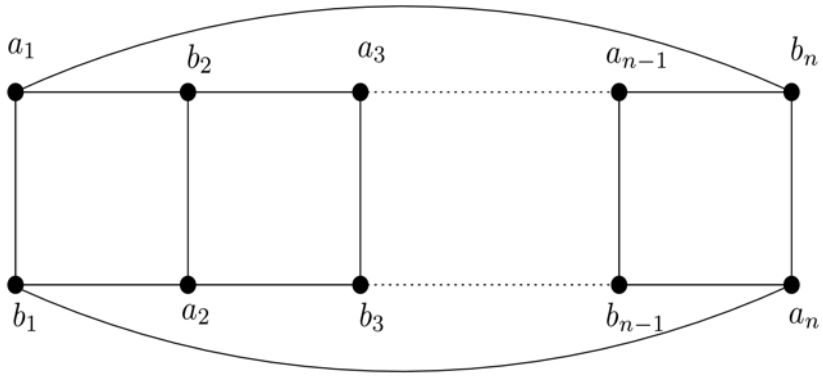

Figure 1. The graph $K_2 \square C_n$.

Let H_G be the open neighbourhood hypergraph of $K_2 \square C_n$. Let C_{2n} be the cycle, $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, a_1)$. It can be observed that if $v \in V(K_2 \square C_n)$, then the vertices of $N(v)$ form a path of length two in C_{2n}. Also the set of vertices of any path of length two in C_{2n} is an edge in H_G. Since $\mathcal{C}(C_{2n}, x) = D(C_{2n}, x)$, the proof follows from Theorems 2.1 and 2.4. □

Theorem 2.6. *Let* $G = Cay(\mathbb{Z}_n, S)$, *where* $S = \{a, b, b^{-1}\}$ *is a generating set of* $\mathbb{Z}_n$ *such that* $a^{-1} = a$ *and* $a \notin < b >$. *Then,*

$$D_t(G, x) = \begin{cases} [D(C_{\frac{n}{2}}, x)]^2, & \text{if } n/2 \text{ is even,} \\ D(C_n, x), & \text{otherwise.} \end{cases}$$

Proof. Here $G \cong K_2 \square C_{\frac{n}{2}}$. So the proof follows from Corollary 2.3(3) and Theorem 2.5.

Theorem 2.7. *Let* $G = Cay(\mathbb{Z}_n, S)$, *where* $S = \{a, b, b^{-1}\}$ *such that* $a^{-1} = a$ *and* b *is a generator of* $\mathbb{Z}_n$. *Then,*

$$D_t(G, x) = \begin{cases} D(C_n, x), & \text{if } n \text{ is a multiple of } 4, \\ [D(C_{\frac{n}{2}}, x)]^2, & \text{otherwise.} \end{cases}$$

Proof. Since b generates $\mathbb{Z}_n$, $G = Cay(\mathbb{Z}_n, S)$ can be labelled as shown in Figure 2.

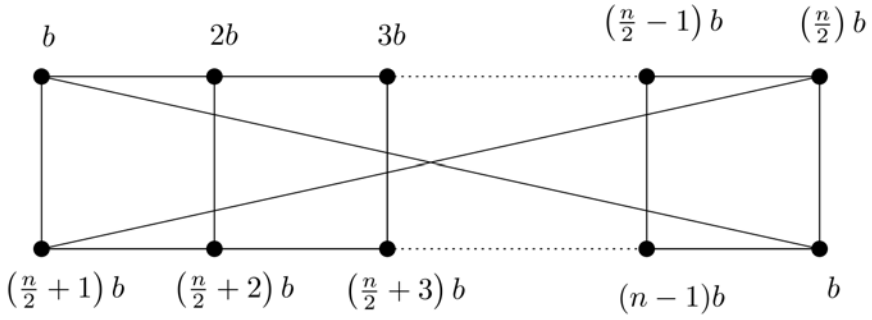

Figure 2. The graph $G = Cay(\mathbb{Z}_n, S)$.

Case 1. When n is a multiple of 4.

Consider the cycle $C_n = (nb, (\frac{n}{2}+1)b, 2b, (\frac{n}{2}+3)b, 4b, (\frac{n}{2}+5)b, 6b, \ldots, (n-2)b, (\frac{n}{2}-1)b, nb)$. It can be observed that if $v \in V(G)$, then

the vertices of $N(v)$ form a path of length two in C_n and any path of length two in C_n is an open neighbourhood of a vertex in G. Therefore by Theorem 2.4, $D_t(G, x) = D(C_n, x)$.

Case 2. When n is not a multiple of 4.

In this case the graph $G = Cay(\mathbb{Z}_n, S)$ is bipartite. Let $X = \{kb / k \text{ is odd}\}$ and $Y = \{sb / s \text{ is even}\}$ be the bipartition. Let $H_X = (Y, \{N(x) / x \in X\})$ and $H_Y = (X, \{N(x) / x \in Y\})$ be the components of the open neighbourhood hypergraph of G. Since H_X is isomorphic to H_Y, by Theorem 2.1, $D_t(G, x) = [C(H_X, x)]^2$. Also $E(H_X) = \{N(b), N((\frac{n}{2}+2)b),$ $N(3b), \ldots, N((\frac{n}{2})b)\} = \{\{nb, (\frac{n}{2}+1)b, 2b\}, \{(\frac{n}{2}+1)b, 2b, (\frac{n}{2}+3)b\},$ $\{2b, (\frac{n}{2}+3)b, 4b\}, \ldots, \{(\frac{n}{2}-1)b, nb, (\frac{n}{2}+1)b\}\}$.

Consider the cycle, $C_{\frac{n}{2}} = (nb, (\frac{n}{2}+1)b, 2b, (\frac{n}{2}+3)b, 4b, \ldots, (\frac{n}{2}-1)$ $b, nb)$, with vertex set Y. It is obvious that if $v \in Y$, then the vertices of $N(v)$ form a path of length two in $C_{\frac{n}{2}}$. Also any path of length two in $C_{\frac{n}{2}}$ is an open neighbourhood of a vertex in Y. Therefore by Theorem 2.4, $C(H_X, x) = D(C_{\frac{n}{2}}, x)$. This completes the proof. □

Lemma 2.8. *If G is an $n-1$ regular bipartite graph, then*

$$D(G, x) = nx^2\left[1 + \binom{2n-2}{1}x + \binom{2n-2}{2}x^2 + \ldots + \binom{2n-2}{n-3}x^{n-1}\right]$$

$$+\left[n\binom{2n-2}{n-2} + 2\right]x^n + \binom{2n}{n+1}x^{n+1} + \binom{2n}{n+2}x^{n+2} + \ldots + \binom{2n}{2n}x^{2n}.$$

Proof. Let $X = \{a_1, a_2, \ldots, a_n\}$ and $Y = \{b_1, b_2, \ldots, b_n\}$ be the bipartition of G. Assume that $a_i \not\sim b_i$ for all i. If $a_r, b_r \in S$ for some r, then S is a dominating set of G. Since there are n pairs, one pair can be selected in n ways. So when $k < n$, the coefficient of x^k in $D(G, x)$ is $n\binom{2n-2}{k-2}$. When $k = n$, since X and Y are dominating sets of G, the coefficient of x^n is $n\binom{2n-2}{n-2} + 2$. When $k > n$, any subset of the vertices of G contains a pair a_i, b_i. So the coefficient of x^k is $\binom{2n}{k}$. This completes the proof. □

Theorem 2.9. *$D_t(K_2 \square K_n, x) = D(G, x)$, where G is an $n-1$ regular bipartite graph on $2n$ vertices.*

Proof. Let $V(K_2) = \{a, b\}$ and $V(K_n) = \{1, 2, 3, \ldots, n\}$. Let $A = \{(a, 1), (a, 2), \ldots, (a, n)\}$ and $B = \{(b, 1), (b, 2), \ldots, (b, n)\}$. Then $V(K_2 \square K_n) = A \cup B$. Also $N((a, i)) = \{(b, i)\} \cup A \setminus \{(a, i)\}$ and $N((b, i)) = \{(a, i)\} \cup B \setminus \{(b, i)\}$. We construct an $(n-1)$-regular bipartite graph G in which the open neighbourhoods $N((a, i))$ and $N((b, i))$ are represented as star graphs with root vertices (b, i) and (a, i), respectively.

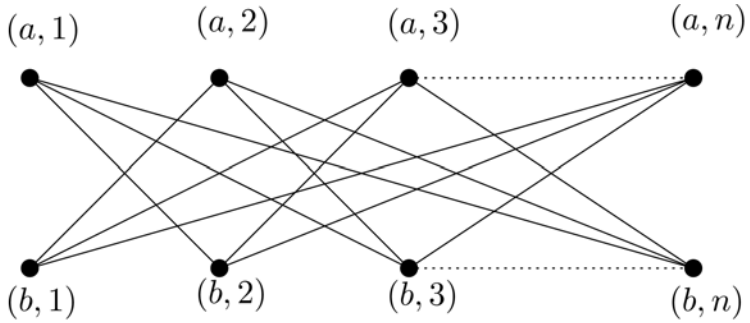

Figure 3. The graph G.

Let S be a dominating set of G. Then for all i, either $(a, i) \in S$ or (a, i) is adjacent to some vertex in S. If $(a, i) \in S$ for all i, then $S \cap N((b, i)) \neq \phi$ for all i. If $(a, r) \notin S$ for some r, then there exists $(b, s) \in S$ for some $s \neq r$. Therefore $S \cap N((b, i)) \neq \phi$ for all i. Similarly, we can prove that $S \cap N((a, i)) \neq \phi$ for all i. Therefore S is a vertex covering set of $H_{K_2 \square K_n}$. Conversely, let S is a vertex covering set of $H_{K_2 \square K_n}$. Consider a vertex (a, i). If $(a, i) \in S$, then there is nothing to prove. If $(a, i) \notin S$, then $(b, j) \in S$ for some $j \neq i$. So $(a, i) \sim (b, j)$ in G. Similarly we can prove the case of (b, j) also. Therefore S is a dominating set of G. Thus the result follows from Theorem 2.1. □

Next, we determine the total domination polynomials of cartesian product of certain classes of graphs with C_4.

Theorem 2.10. *For a bipartite graph G,*

$$D_t(C_4 \square G, x) = [D(K_2 \square G, x)]^2.$$

Proof. Since the graph $K_2 \square K_2 \square G$ is isomorphic to $C_4 \square G$, the result follows from Theorems 2.1 and 2.2. □

Corollary 2.11. *From Theorem* 2.10, *we can infer the following results*:

(1) $D_t(C_4 \square P_n, x) = [D(L_n, x)]^2$.

(2) $\gamma_t(C_4 \square P_n) = 2\left\lceil \frac{n+1}{2} \right\rceil$.

(3) $D_t(C_4 \square C_{2n}, x) = [D(K_2 \square C_{2n}, x)]^2$.

(4) $\gamma_t(C_4 \square C_{2n}) = \begin{cases} 2n, & \text{if } n \text{ is even}, \\ 2n + 2, & \text{otherwise.} \end{cases}$

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