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PERIODIC SOLUTIONS OF A NEURAL NETWORK MODEL WITH DELAY

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Abstract

In this paper, we analyze a neural network system with delay. Firstly, by considering time delay as bifurcation parameter, we discuss the stability of the model and the conditions when Hopf bifurcation occurs. Finally, after complicated calculations with Lyapunov-Schmidt reduction, we obtain the approximate analytic expression of periodic solutions near the Hopf bifurcation points.

1. Introduction

The occuring of Hopf bifurcation is an important property of dynamical systems with delay, and it refers to a phenomenon that the systems have periodic solutions near the singularity when the stability of the singularity changes. After determining that a system exists Hopf

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bifurcation, under normal conditions, to determine the stability and direction of the periodic solutions [1-5], the delay differential equations need to be expressed as ordinary differential equations in abstract spaces. The system must has a 2-dimensional center manifold which is tangent to the central subspace near the constant solutions. Hence, we only need to considering the limitation of the original system on the center manifold, actually turning it into a 2-dimensional differential equation. As long as we study the stability and the direction of periodic solutions of the ordinary differential equation, we can obtain the properties related to the original system. For example, Faria and Magalh*á*es [6] and Liu et al. [7] all use this technique. Lyapunov-Schmidt reduction is an important method to investigate the bifurcation problems of ordinary differential equations [8]. Inspired by Yang and Guo [9], we investigate the periodic solutions of a neural network model with delay by using Lyapunov-Schmidt reduction [10-13] and obtain obvious expressions depending on the parameters of the original system. This method avoids numerous calculations of center manifold and normal form.

The model to study in this paper is

$$\dot{x} = -ax(t) + bx(t-\tau) + \frac{1}{6}x(t)^3, \tag{1.1}$$

where the dot denotes differentiation with respect to time, and $a > 0$, b are parameters.

Our paper is organized as follows: In Section 2, we discuss the stability of the model and the conditions when Hopf bifurcation occurs. In Section 3, we obtain the approximate analytic expression of periodic solutions near the Hopf bifurcation points with Lyapunov-Schmidt reduction.

2. Stability Analysis

From system (1.1), we can see that there exists an zero equilibrium and the characteristic equation of the linearized system at zero solution is

$$\lambda + a - be^{(-\lambda\tau)} = 0, \tag{2.1}$$

when $\tau = 0$, it is obvious that $b < a$, the root of the equation has always negative real parts. If $b < -a$, Equation (2.1) has one positive root $\omega = \sqrt{b^2 - a^2}$,

$$\tau_n = \frac{1}{\omega}(\pi - \arcsin(\frac{\omega}{-b})) + \frac{2n\pi}{\omega},\ n = 0, 1, 2, \cdots,\ sign\{\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}\}_{\tau=\tau_n}$$

$$= \frac{a}{b^2} > 0.$$

Theorem 2.1. *If* $b < -a$, *then the zero solution of system* (1.1) *is asymptotically stable for* $\tau \in [0, \tau_0)$, *and unstable when* $\tau > \tau_0$. *Furthermore, system* (1.1) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_n (n = 0, 1, \cdots)$.

3. Lyapunov-Schmidt Reduction Process

In this section, near the Hopf bifurcation, we obtain the approximate analytic periodic solutions bifurcated from the trivial solution of the neural network by Lyapunov-Schmidt reduction method. Now, we will mainly discuss how to get the periodic solutions near the Hopf bifurcation point τ_0.

First of all, by the transformation $\hat{s} = \frac{t}{\tau}$, $u(\hat{s}) = x(\tau\hat{s})$, $\tau = \tau_0 + \varepsilon$, $s = (1 + k)\hat{s}$, then system (1.1) is equivalent to

$$F(u, \varepsilon, k) = (1 + k)\frac{du}{ds} + (\tau_0 + \varepsilon)[au(s) - bu(s - (1 + k)) - \frac{1}{6}u(s)^3]. \quad (3.1)$$

Let

$$C_p^0[0, \frac{2\pi}{n}] = u \in C^0(-\infty, +\infty) \mid u(s) = u(s + \frac{2\pi}{n}),$$

$$C_p^1[0, \frac{2\pi}{n}] = u \in C^1(-\infty, +\infty) \mid u(s) = u(s + \frac{2\pi}{n}).$$

Then $F(u, \varepsilon, k) : C_p^1[0, \frac{2\pi}{n}] \times R \times R \to C_p^0[0, \frac{2\pi}{n}]$, the linearized equation of Equation (3.1) at zero solution is

$$L_0 v = F_u^-(0, 0, 0)v = \frac{dv}{ds} + \tau_0 av - \tau_0 bv(s-1) = 0, \tag{3.2}$$

the adjoint equation is

$$L_0^* v = F_u^+(0, 0, 0)v = -\frac{dv}{ds} + \tau_0 av - \tau_0 bv(s+1) = 0, \tag{3.3}$$

obviously

$$N(L_0) = span\{\sin ns, \cos ns\}, \quad N(L_0^*) = span\{\sin ns, \cos ns\}.$$

We have the following decomposition:

$$C_p^1[0, \frac{2\pi}{n}] = N(L_0) \oplus M, \quad C_p^0[0, \frac{2\pi}{n}] = N(L_0^*) \oplus R(L_0).$$

Define

$$Q : C_p^0[0, \frac{2\pi}{n}] \to N(L_0^*), \; P : C_p^0[0, \frac{2\pi}{n}] \to R(L_0),$$

$$Qv = < v, \sin ns > \sin ns + < v, \cos ns > \cos ns, \; Pv = v - Qv,$$

define an inner product is given by

$$< v_1, v_2 >= \frac{n}{\pi} \int_0^{\frac{2\pi}{n}} v_1 v_2 ds.$$

Let

$$u(s) = x \sin ns + y \cos ns + \omega(s),$$

where $\omega(s) \in M$. By respectively projecting system (3.1) onto $R(L_0)$ and $N(L_0^*)$, we obtain

$$PF(x \sin ns + y \cos ns + \omega(s), \varepsilon, k) = 0, \tag{3.4}$$

$$QF(x \sin ns + y \cos ns + \omega(s), \varepsilon, k) = 0. \tag{3.5}$$

Rewrite (3.4) as

$$\begin{aligned}\widetilde{F}(\omega;\, x,\, y,\, \varepsilon,\, k) = P[&(1+k)(nx\cos ns - ny\sin ns + \frac{d\omega(s)}{ds}) \\ &+ a(\tau_0 + \varepsilon)(x\sin ns + y\cos ns + \omega(s)) \\ &- b(\tau_0 + \varepsilon)(x\sin n(s-(1+k)) + y\cos n(s-(1+k))) \\ &+ \omega(s-(1+k)) - \frac{1}{6}(\tau_0+\varepsilon)(x\sin ns + y\cos ns + \omega(s))^3\,] = 0.\end{aligned} \tag{3.6}$$

We know $\widetilde{F}_\omega(0;\, 0,\, 0,\, 0,\, 0) = PL_0 = L_0,$ and L_0 is regular on M, by the implicit function theorem, it is easy to solve from system (3.6) that

$$\omega(s) = \omega(s;\, x,\, y,\, \varepsilon,\, k). \tag{3.7}$$

Obviously, $\omega(s;\, 0,\, 0,\, 0,\, 0) \equiv 0,$ plug the system (3.7) into the system (3.5), we can obtain the equivalent bifurcation equations are

$$g(x,\, y,\, \varepsilon,\, k) =< F(x\sin ns + y\cos ns + \omega(s;\, x,\, y,\, \varepsilon,\, k)),\, \sin ns >= 0, \tag{3.8}$$

$$h(x,\, y,\, \varepsilon,\, k) =< F(x\sin ns + y\cos ns + \omega(s;\, x,\, y,\, \varepsilon,\, k)),\, \cos ns >= 0. \tag{3.9}$$

Secondly, to obtain the further information respect to the periodic solutions, we need to calculate the approximate analytic expressions of system (3.7), (3.8), and (3.9).

For the convenience of narrative, we make some notes

$$\omega^0(s) = \omega(s;\, 0,\, 0,\, 0,\, 0) \equiv 0,\ \omega_x^0(s) = \omega_x(s;\, 0,\, 0,\, 0,\, 0),\ \omega_y^0(s) = \omega_y(s;\, 0,\, 0,\, 0,\, 0),$$

$$\omega_\varepsilon^0(s) = \omega_\varepsilon(s;\, 0,\, 0,\, 0,\, 0),\ \omega_k^0(s) = \omega_k(s;\, 0,\, 0,\, 0,\, 0),\ g_x^0(s) = g_x(s;\, 0,\, 0,\, 0,\, 0),$$

$$h_x^0(s) = h_x(s;\, 0,\, 0,\, 0,\, 0),\ g_y^0(s) = g_y(s;\, 0,\, 0,\, 0,\, 0),\ h_y^0(s) = h_y(s;\, 0,\, 0,\, 0,\, 0),$$

$$g_\varepsilon^0(s) = g_\varepsilon(s;\, 0,\, 0,\, 0,\, 0),\ h_\varepsilon^0(s) = h_\varepsilon(s;\, 0,\, 0,\, 0,\, 0),\ g_k^0(s) = g_k(s;\, 0,\, 0,\, 0,\, 0),$$

$$h_k^0(s) = h_k(s;\, 0,\, 0,\, 0,\, 0)\cdots.$$

Take the derivative of system (3.6) with respect to x, we have

$$\frac{d\omega_x^0(s)}{ds} + a\tau_0\omega_x^0(s) - b\tau_0\omega_x^0(s-1) = 0,$$

and

$$< \omega_x^0(s), \sin ns >= 0, < \omega_x^0(s), \cos ns >= 0.$$

We can solve that

$$\omega_x^0(s) \equiv 0.$$

Analogously, we can obtain

$$\omega_y^0(s) = 0,\ \omega_\varepsilon^0(s) = 0,\ \omega_k^0(s) = 0,\ \omega_{xx}^0(s) = 0,\ \omega_{yy}^0(s) = 0,\ \omega_{xy}^0(s) = 0,$$

$$\omega_{x\varepsilon}^0(s) = 0,\ \omega_{xk}^0(s) = 0,\ \omega_{y\varepsilon}^0(s) = 0,\ \omega_{yk}^0(s) = 0,\ \omega_{\varepsilon\varepsilon}^0(s) = 0,\ \omega_{kk}^0(s) = 0.$$

Take three derivatives of system (3.6) with respect to x, we have

$$\frac{d\omega_{xxx}^0(s)}{ds} + a\tau_0\omega_{xxx}^0(s) - b\tau_0\omega_{xxx}^0(s-1) - \tau_0 P[(\sin ns)^3] = 0, \tag{3.10}$$

$$P[(\sin ns)^3] = -\frac{1}{4}\sin 3ns,$$

let

$$\omega_{xxx}^0(s) = c_1 \sin 3ns + c_2 \cos 3ns + c_3, \tag{3.11}$$

plug the system (3.11) into the system (3.10) , we can calculate that

$$c_1 = \frac{-\frac{1}{4}\tau_0(a\tau_0 - b\tau_0\cos 3n)}{(a\tau_0 - b\tau_0\cos 3n)^2 + (3n + b\tau_0\sin 3n)^2},$$

$$c_2 = \frac{\frac{1}{4}\tau_0(3n + b\tau_0\sin 3n)}{(a\tau_0 - b\tau_0\cos 3n)^2 + (3n + b\tau_0\sin 3n)^2},\ c_3 = 0.$$

In the same way, we can obtain

$$\omega^0_{yyy}(s) = c_2 \sin 3ns - c_1 \cos 3ns,\ \omega^0_{xxy}(s) = -c_2 \sin 3ns + c_1 \cos 3ns,$$

$$\omega^0_{xyy}(s) = -c_1 \sin 3ns - c_2 \cos 3ns,\ \omega^0_{xx\varepsilon}(s) = 0,\ \omega^0_{xxk}(s) = 0,$$

$$\omega^0_{yy\varepsilon}(s) = 0,\ \omega^0_{yyk}(s) = 0,\ \omega^0_{xy\varepsilon}(s) = 0,\ \omega^0_{xyk}(s) = 0.$$

We can calculate that

$$g^0_x(s) =< F_x(\omega;\ x,\ y,\ \varepsilon,\ k),\ \sin ns >= a\tau_0 - b\tau_0 \cos n.$$

In the same way, we can obtain

$$g^0_y(s) = -b\tau_0 \sin n - n,\ h^0_x(s) = b\tau_0 \sin n + n,\ h^0_y(s) = a\tau_0 - b\tau_0 \cos n,$$

$$g^0_\varepsilon(s) = 0,\ g^0_k(s) = 0,\ h^0_\varepsilon(s) = 0,\ h^0_k(s) = 0,\ g^0_{\varepsilon\varepsilon}(s) = 0,\ g^0_{kk}(s) = 0,\ g^0_{xx}(s) = 0,$$

$$g^0_{yy}(s) = 0,\ h^0_{\varepsilon\varepsilon}(s) = 0,\ h^0_{kk}(s) = 0,\ h^0_{xx}(s) = 0,\ h^0_{yy}(s) = 0,\ g^0_{x\varepsilon}(s) = a - b\cos n,$$

$$g^0_{y\varepsilon}(s) = -b\sin n, g^0_{xk}(s) = nb\tau_0 \sin n, g^0_{yk}(s) = -n - b\tau_0 \cos n,\ h^0_{x\varepsilon}(s) = b\sin n,$$

$$h^0_{y\varepsilon}(s) = a - b\cos n, h^0_{xk}(s) = n + b\tau_0 n\cos n, h^0_{yk}(s) = b\tau_0 n \sin n, g^0_{xxy}(s) = 0,$$

$$h^0_{xxy}(s) = -\frac{\tau_0}{4}, g^0_{xyy}(s) = -\frac{\tau_0}{4}, h^0_{xyy}(s) = 0, g^0_{xx\varepsilon}(s) = 0, h^0_{xx\varepsilon}(s) = 0, g^0_{xxk}(s) = 0,$$

$$h^0_{xxk}(s) = 0,\ g^0_{yy\varepsilon}(s) = 0,\ h^0_{yy\varepsilon}(s) = 0,\ g^0_{yyk}(s) = 0,\ h^0_{yyk}(s) = 0,\ g^0_{xy\varepsilon}(s) = 0,$$

$$h^0_{xy\varepsilon}(s) = 0,\ g^0_{xyk}(s) = 0,\ h^0_{xyk}(s) = 0.$$

By Taylor expansion, we know

$$g(x,\ y,\ \varepsilon,\ k) \doteq [H + Jk + L\varepsilon + Nx^2 + Ny^2]x + [I + Pk + M\varepsilon]y,$$

$$h(x,\ y,\ \varepsilon,\ k) \doteq [I + Pk + M\varepsilon](-x) + [H + Jk + L\varepsilon + Nx^2 + Ny^2]y,$$

where

$$H = a\tau_0 - b\tau_0 \cos n,\ J = nb\tau_0 \sin n,\ L = a - b\cos n,\ N = -\frac{1}{8}\tau_0,$$

$$I = -b\tau_0 \sin n - n,\ P = -n - b\tau_0 n\cos n,\ M = -b\sin n.$$

From what has been discussed above, we have

Theorem 3.1. *For system* (1.1), *near the Hopf bifurcation point* τ_0, *there exist periodic solutions of which the period is* $(\tau_0 + \varepsilon)\frac{2\pi}{n}$, *and the approximate analytic expression is*

$$\begin{aligned}\bar{x}(s) = {} & x\sin ns + y\cos ns + (c_1\sin 3ns + c_2\cos 3ns)\frac{x^3}{6} \\ & + (-c_2\sin 3ns + c_1\cos 3ns)\frac{x^2 y}{2} + (-c_1\sin 3ns - c_2\cos 3ns)\frac{xy^2}{2} \\ & + (c_2\sin 3ns - c_1\cos 3ns)\frac{y^3}{6},\end{aligned}$$

where

$$s = s(t) = \frac{t}{\tau}(1+k), \qquad \varepsilon \doteq \frac{PH - IJ - PN(x^2+y^2)}{JM - PL},$$

$$k \doteq \frac{LI - MH - MN(x^2+y^2)}{JM - PL}.$$

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