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# A MODIFIED BROYDEN-LIKE METHOD WITH 3-1 PIECEWISE NCP FUNCTION FOR GENERALIZED NONLINEAR COMPLEMENTARITY PROBLEM

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## Abstract

In this paper, we propose an improved Broyden-like method for nonlinear complementarity problems by the 3-1 piecewise NCP function. In order to reduce the amount of calculation, we introduce a parameter to avoid compute function of $F(x)$, and give the update formula of the parameter. Moreover, a new nonmonotone line search is adopted, so that only one system of equation is needed to solve and one matrix is needed to update at each iteration. In the end, under suitable conditions, the global convergent properties are proved.

## 1. Introduction

The generalized nonlinear complementarity problem (GNCP): Find a vector $x \in R^n$ such that

$$F(x) \geq 0,\ G(x) \geq 0,\ F(x)^{\top} G(x) = 0, \tag{1.1}$$

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where $F, G : R^n \to R^n$ is continuously differentiable. When $G(x) = x$, GNCP reduces to the classic nonlinear complementarity problem (NCP). Moreover, when $F(x) = Mx + q$, with $M \in R^{n\times n}$ and $q \in R^n$, NCP becomes a linear complementarity problem (LCP). The generalized nonlinear complementarity problem has many important applications in engineering and economics [1, 2], and its solution has been considered extensively by using special techniques in some literatures [3-8]. In the past few years, the GNCP also had been devoted a growing attention. Zhang et al. [9] reformulated the GNCP over a polyhedral as a smoothing system of equations, and then developed a smoothing Newton-type method for solving it. And, Chen and Ma [10] introduced a smoothing Broyden-like algorithm for the GNCP over a polyhedral cone based on the smoothing Fischer-Burmeister function. Their methods also need to introduce the smoothing parameter μ to Fischer-Burmeister function. However, Pu and Zhou [11], by using the piecewise NCP function, proposed a QP-free feasible method. Motivated by the above idea, we will introduce a smoothing Broyden-like algorithm for the GNCP based on 3-1 piecewise NCP function in this paper. Firstly, we transform the generalized nonlinear complementarity problem to smooth equations by 3-1 piecewise NCP function. Then, a nonmonotone line search which is based on Broyden-like algorithm be adopt to solve it. Furthermore, in Algorithm, we let $s = F(x), t = G(x)$ as the independent variable to make the algorithm run easily. And it is a solution of GNCP from any accumulation point of $x$ which is the iteration sequence generated by our algorithm.

This paper is organized as follows. In the next section, we introduce the piecewise NCP function and transform GNCP into a equivalent system of smooth equations. In Section 3, we introduce the algorithm in detail. In Section 4, we prove the algorithm to be well-defined and present the convergent properties of the presented algorithm.

## 2. Preliminaries

**Definition 2.1** ($P_0$ - function). $F : R^n \to R^n$ is a continuously differetiable $P_0$-function, i.e., for all $x, y \in R^n$ with $x \neq y$, there exists an index $i_0$ such that

$$x_{i_0} \neq y_{i_0}, (x_{i_0} - y_{i_0})[F_{i_0}(x) - F_{i_0}(y)] \geq 0. \tag{2.1}$$

**Definition 2.2** (NCP pair and NCP function). We call a pair $(a, b) \in R^2$ to be an NCP pair if $a \geq 0, b \geq 0$ and $ab = 0$; a function $\Phi : R^2 \to R$ is called an NCP function if $\Phi(a, b) = 0$ if and only if $(a, b)$ is an NCP pair.

The 3-1 piecewise linear NCP function is defined as:

$$\Phi(a, b) = \begin{cases} 3a - \dfrac{a^2}{b}, & \text{if } b \geq a > 0, \text{ or } 3b > -a \geq 0, \\ 3b - \dfrac{b^2}{a}, & \text{if } a > b > 0, \text{ or } 3a > -b \geq 0, \\ 9a + 9b, & \text{if } a \leq 0 \text{ and } -a \geq 3b, \text{ or } b \leq -3a \leq 0. \end{cases} \tag{2.2}$$

If $(a, b) \neq (0, 0)$, then

$$\nabla\Phi(a, b) = \begin{cases} \begin{pmatrix} 3 - \dfrac{2a}{b} \\ \dfrac{a^2}{b^2} \end{pmatrix}, & \text{if } b \geq a > 0, \text{ or } 3b > -a \geq 0, \\ \begin{pmatrix} \dfrac{b^2}{a^2} \\ 3 - \dfrac{2b}{a} \end{pmatrix}, & \text{if } a > b > 0, \text{ or } 3a > -b \geq 0, \\ \begin{pmatrix} 9 \\ 9 \end{pmatrix}, & \text{if } 0 \geq a \text{ and } -a \geq 3b, \text{ or } b \leq -3a \leq 0. \end{cases}$$

(2.3)

Denote $H : R^{3n} \to R^{3n}$

$$H(s, t, x) = \begin{pmatrix} s - F(x) \\ t - G(x) \\ \Phi(s, t) \end{pmatrix}, \tag{2.4}$$

where $s$ approaches $F(x)$, $t$ approaches $G(x)$ and $s = F(x)$, $t = G(x)$ at the solution. $\Phi(s, t)$ is 3-1 piecewise linear NCP function.

Therefore, we can reform the GNCP (1.1) to the following minimization problem:

$$\min \Psi(s, t, x)$$

$$\Psi(s, t, x) = \|H(s, t, x)\|. \tag{2.5}$$

## 3. A modified Broyden-like Algorithm with 3-1 Piecewise NCP Function

For solving (2.5), we need to introduce the following symbols:

$$(\xi_i^k, \eta_i^k) = \begin{cases} (1, 1) & (x, s) = (0, 0), \\ \nabla\Phi(x, s) & \text{otherwise}, \end{cases} \tag{3.1}$$

$i = 1, 2, \cdots n$, obviously, $\xi_i^k > 0$ and $\eta_i^k > 0$.

Compute the Jacobian matrix $V(s^k, t^k, x^k)$ of $H(s^k, t^k, x^k)$, we get

$$V(s^k, t^k, x^k) = \begin{pmatrix} I & 0 & -\nabla F(x^k) \\ 0 & I & -\nabla G(x^k) \\ \text{diag}(\xi_i^k) & \text{diag}(\eta_i^k) & 0 \end{pmatrix},$$

where $I$ is identity matrix of $n \times n$; $\nabla F(x^k)$, $\nabla G(x^k)$ are gradient matrix of $F(x^k)$ and $G(x^k)$, respectively, $\text{diag}(\xi_i^k)$ or $\text{diag}(\eta_i^k)$ denotes the diagonal matrix whose $i$-th diagonal element is $\xi_i^k$ or $\eta_i^k$, respectively.

**Algorithm 3.1**

**Step 0.** Initialization

Given initial point $(s^0, t^0, x^0) \in R^{3n}$, $\tau \in (0, 1)$, $0 < \theta, \bar{\theta} < 1$, $B_0 = V(s^0, t^0, x^0)$, $k = 0$.

**Step 1.** If $\Psi(s^k, t^k, x^k) = 0$, then stop. Otherwise, calculation of the search direction: Calculate $u^k$, $v^k$, and $d^k$ by solving the following linear system in $(u, v, d)$ :

$$B_k \begin{pmatrix} u \\ v \\ d \end{pmatrix} = \begin{pmatrix} F(x^k) - s^k \\ G(x^k) - t^k \\ -\Phi(s^k, t^k) \end{pmatrix}. \tag{3.2}$$

**Step 2.** Nonmonotone line search.

**Step 2.1.** If

$$\Psi(s^k + u^k, t^k + v^k, x^k + d^k) \le \theta\Psi(s^k, t^k, x^k), \tag{3.3}$$

$$\|\Phi(s^k + u^k, t^k + v^k)\| \le \theta \max_{0\le r\le m(k)-1} \|\Phi^{k-r}\|, \tag{3.4}$$

where $m(0) = 0$, $0 \le m(k) \le \min\{m(k-1)+1, M\}$, $M$ is a positive constant. Then let

$$s^{k+1} = s^k + u^k, \quad t^{k+1} = t^k + v^k, \quad x^{k+1} = x^k + d^k \tag{3.5}$$

go to Step 3, otherwise go to Step 2.2.

**Step 2.2.** Let

$$s^{k+1} = s^k + \alpha_k u^k, \quad t^{k+1} = t^k + \alpha_k v^k, \quad x^{k+1} = x^k + \alpha_k d^k, \tag{3.6}$$

where $\alpha_k = \tau^j$ $(0 < \tau < 1)$ and $j$ is the smallest non-negative integer and satisfied:

$$\|\Phi(s^{k+1}, t^{k+1})\| \le \theta \max_{0\le r\le m(k)-1} \|\Phi^{k-r}\|, \tag{3.7}$$

$$\|s^{k+1} - F(x^{k+1}) + t^{k+1} - G(x^{k+1})\| \le \theta \max_{0\le r\le m(k)-1} \|s^{k-r} - F(x^{k-r}) + t^{k-r} - G(x^{k-r})\|. \tag{3.8}$$

**Step 3.** Update $B_k$ to get $B_{k+1}$ :

$$B_{k+1} = B_k + \theta_k \frac{(y^k - B_k z^k)(z^k)^\top}{\|z^k\|^2}, \tag{3.9}$$

where

$$z^k = \begin{pmatrix} s^{k+1} \\ t^{k+1} \\ x^{k+1} \end{pmatrix} - \begin{pmatrix} s^k \\ t^k \\ x^k \end{pmatrix}, \quad y^k = H(s^{k+1}, t^{k+1}, x^{k+1}) - H(s^k, t^k, x^k),$$

select $\theta_k$ to satisfy $|\theta_k - 1| \le \bar{\theta}$ and matrix $B_{k+1}$ is nonsingular.

**Step 4.** Let $k = k + 1$ and go to Step 1.

**Remark 3.1.** In the above algorithm, there are two types of successive iteration: (3.5) and (3.6).

## 4. Convergent Properties

In this section, we discuss the global convergence properties of a modified Broyden-like algorithm with 3-1 piecewise NCP function.

**Assumption 4.1**

(a) $F, G : R^n \to R^n$ is continuously differentiable $P_0$-function.

(b) For initial point $(s^0, t^0, x^0) \in R^{3n}$, denote the level set of $\Psi(s, t, x)$ by

$$L(s^0, t^0, x^0) = \{(s, t, x) \in R^{3n} | \Psi(s, t, x) \le \Psi(s^0, t^0, x^0)\}. \tag{4.1}$$

Defined by (4.1), the level set of $\Psi(s, t, x)$ is bounded.

**Remark 4.1.** $F(x)$, $G(x)$ are $P_0$-function, then $\nabla F(x)$, $\nabla G(x)$ is positive semidefinite.

**Lemma 4.1.** *If* $H(s^0, t^0, x^0) \neq 0$, *then* $B_0 = V(s^0, t^0, x^0)$ *is nonsingular.*

**Proof.** Assume $H(s^0, t^0, x^0) \neq 0$. If $V(s^0, t^0, x^0) \cdot (u, v, d)^\top = 0$ for some $(u, v, d) \in R^{3n}$, where $u = (u_1 \cdots u_n)^\top$, $v = (v_1 \cdots v_n)^\top$, $d = (d_1 \cdots d_n)^\top$, then

$$Iu - \nabla F(x^0)d = 0, \tag{4.2}$$

$$Iv - \nabla G(x^0)d = 0, \tag{4.3}$$

$$\operatorname{diag}(\xi^0)u + \operatorname{diag}(\eta^0)v = 0. \tag{4.4}$$

From the definitions of $\xi_i^0$ and $\eta_i^0$, we know that $\xi_i^0 > 0$ and $\eta_i^0 > 0$ for all $i$, $i = 1 \cdots n$. So, $\operatorname{diag}(\eta^0)$ is nonsingular. By (4.4), we have

$$v = -(\operatorname{diag}(\eta^0))^{-1}\operatorname{diag}(\xi^0)u. \tag{4.5}$$

Substitute $v$ in (4.5) by (4.3), we have

$$d = -(\nabla G(x^0))^{-1}(\operatorname{diag}(\eta^0))^{-1}\operatorname{diag}(\xi^0)u. \tag{4.6}$$

Substitute $d$ in (4.6) by (4.2), and multiplying by $u^\top$, we have

$$u^\top Iu + u^\top \nabla F(x^0)(\nabla G(x^0))^{-1}(\operatorname{diag}(\eta^0))^{-1}\operatorname{diag}(\xi^0)u = 0. \tag{4.7}$$

The fact that $F(x)$, $G(x)$ is the $P_0$-function, so all principal minor determinant of the $\nabla F(x)$, $\nabla G(x)$ is non-negative, that is to say, $\nabla F(x^0)$, $(\nabla G(x^0))^{-1}$ is positive semidefinite. And matrix $(\operatorname{diag}(\eta^0))^{-1}\operatorname{diag}(\xi^0)$ is positive definite. So $u = 0$. By (4.4) and (4.5), we know $v = 0$, $d = 0$. Hence, $B_0$ is nonsingular.

**Lemma 4.2.** *Suppose the Assumption* 4.1 *holds,* $\Phi(s^k, t^k) \to 0$, *as* $k \to \infty$.

**Proof.** In view of convenience, if for all sufficiently large $k$ (3.4), (3.8) holds, define $\|\Phi^{l(k)}\| = \max\limits_{0\le r\le m(k)-1}\|\Phi^{k-r}\|$, where $k - m(k) + 1 \le l(k) \le k$.

Since $m(k+1) \le m(k) + 1$, then

$$\begin{aligned}\|\Phi^{l(k+1)}\| &= \max_{0\le r\le m(k+1)-1}\|\Phi^{k+1-r}\| \\ &\le \max_{0\le r\le m(k)}\|\Phi^{k+1-r}\| \\ &= \max\{\|\Phi^{l(k)}\|, \|\Phi^{k+1}\|\} \\ &= \|\Phi^{l(k)}\|.\end{aligned}$$

So, $\|\Phi^{l(k)}\|$ is monotone decreasing, which implies that the $\{\|\Phi^{l(k)}\|\}$ converges. It follows from (3.4), (3.8) that $\|\Phi^{l(k)}\| \le \theta\|\Phi^{l(l(k)-1)}\|$.

Since $\theta \in (0, 1)$, therefore $\{\|\Phi^{l(k)}\|\} \to 0 (k \to \infty)$.

Therefore $\|\Phi^{k+1}\| \le \theta\|\Phi^{l(k)}\| \to 0 (k \to \infty)$ holds by the Algorithm 3.1. That is, $\lim\limits_{k\to\infty}\|\Phi^k\| = 0$.

**Lemma 4.3.** *Suppose the Assumption* 4.1 *holds,* $[s^k - F(x^k) + t^k - G(x^k)] \to 0$, *as* $k \to \infty$.

The proof is similar to Lemma 4.2.

**Lemma 4.4.** *Suppose the Assumption* 4.1 *holds,* $u^k \to 0$, $v^k \to 0$, $d^k \to 0$, $H(s^k, t^k, x^k) \to 0$, *as* $k \to \infty$.

**Proof.** In Algorithm 3.1, there are two types of successive iteration: (3.5) and (3.6).

For iteration (3.5), it satisfied (3.3). So, $\Psi(s^k, t^k, x^k)$ is monotone decreasing. And $\Psi(s, t, x) \geq 0$. Hence, $\Psi(s^k, t^k, x^k) \to 0$, as $k \to \infty$. Namely, $H(s^k, t^k, x^k) \to 0$, as $k \to \infty$.

For iteration (3.6), it satisfied (3.7) and (3.8). By Lemmas 4.2 and 4.3, we know $\Phi(s^k, t^k) \to 0$, $[s^k - F(x^k) + t^k - G(x^k)] \to 0$, as $k \to \infty$. So $H(s^k, t^k, x^k) \to 0$, as $k \to \infty$.

Therefore,

$$B_k \begin{pmatrix} u^k \\ v^k \\ d^k \end{pmatrix} = \begin{pmatrix} F(x^k) - s^k \\ G(x^k) - t^k \\ -\Phi(s^k, t^k) \end{pmatrix} \to 0, \text{ as } k \to \infty.$$

In Algorithm 3.1, $B_k$ is nonsingular. So, $u^k \to 0$, $v^k \to 0$, $d^k \to 0$, as $k \to \infty$.

**Theorem 4.1.** *Suppose the Assumption* 4.1 *holds*, *Equation* (3.2) *is solvable*, *Algorithm* 3.1 *is well-defined.*

**Proof.** We know $B_0$ is nonsingular by Lemma 4.1. And produced $B_k$ of Algorithm 3.1 is nonsingular. So the Equation (3.2) has one and only one solution. Otherwise, we know $H(s^k, t^k, x^k) \to 0$, as $k \to \infty$ by Lemma 4.4. So, Algorithm 3.1 is well-defined.

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