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WEAKLY (REGULAR)*O* AND NOT-REGULAR PROPERTIES, AND EQUIVALENT SEPARATION AXIOMS

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Abstract

In a recent paper, a new category of topological properties called weakly *Po* properties were introduced and investigated. The search for a topological property that failed to be a weakly *Po* property led to the use of "not-T_0" within that paper, and the investigation of other "not-separation axioms" and other weakly *Po* properties in follow up papers. Within this paper, the study of weakly *Po* properties and "not-separation axioms" continues with the regular separation axiom and it is established that for regular spaces, T_0, T_1, T_2, Urysohn, and T_3 are equivalent.

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1. Introduction and Preliminaries

T_0-identification spaces were introduced in 1936 [11].

Definition 1.1. Let (X, T) be a space, R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, X_0 be the set of R equivalence classes of X, N be the nature map from X onto X_0, $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Within a 1975 paper [10], T_0-identification spaces were used to further characterize weakly Hausdorff spaces.

Theorem 1.1. *A space is weakly Hausdorff iff its T_0-identification space is Hausdorff.*

In the 1936 paper [11], T_0-identification spaces were used to further characterize pseudometrizable spaces.

Theorem 1.2. *A space is pseudometrizable iff its T_0-identification space is metrizable.*

As a result, the question of whether the process used to characterize pseudometrizable and weakly Hausdorff could be generalized to include additional topological properties arose leading to the introduction and investigation of weakly *Po* properties [2].

Definition 1.2. Let P be a topological property for which $Po = (P \text{ and } T_0)$ exists. Then (X, T) is weakly *Po* iff $(X_0, Q(X, T))$ has property P. A topological property *Po* for which weakly *Po* exists is called a *weakly Po property* [2].

Within the 2015 paper [2], it was proven that a space is weakly *Po* iff it T_0-identification space has property *Po*. Thus metrizable was the first known weakly *Po* property with weakly (metrizable) = pseudometrizable [11], with Hausdorff added to the weakly *Po* properties in 1975 [10].

In the 1975 paper [10], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.3. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\})$ and $Cl(\{Y\})$ are unequal, there exist disjoint open sets U and V such that x is in U and y is in V.

Thus Hausdorff is a weakly Po property with weakly (Hausdorff) = weakly $T_2 = R_1$.

Within the 1961 paper [1], the R_0 separation axiom was revisited and further investigated.

Definition 1.4. A space is R_0 iff for each open set O and each x in O, $Cl(\{x\})$ is a subset of O.

In the 2015 paper [2], it was shown that T_1 is a weakly Po property with weakly $T_1 = R_0$. Also, within the paper [2], it was shown that for a weakly Po property Qo, a space is weakly Qo iff its T_0-identification space is weakly Qo. Combining this result with the knowledge that other properties are simultaneously shared by both a space and its T_0-identification space led to the introduction of T_0-identification P properties [3].

Definition 1.5. Let Q be a topological property. Then Q is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property Q.

In the 2015 paper [3], it was shown that for a T_0-identification P property Q, Q = weakly Qo, tying T_0-identification P properties to weakly Po.

Within weakly Po properties, the T_0 separation axiom has a major role raising the questions of what would happen if T_0 in the definition of weakly Po was replaced by T_1 or T_2 and leading to the introduction of weakly $P1$ [4] and weakly $P2$ [5] properties.

Definition 1.6. Let P be a topological property for which $P1 = (P \text{ and } T_1)$ exists. Then a space (X, T) is weakly $P1$ iff $(X_0, Q(X, T))$ is $P1$. A topological property $P1$ for which weakly $P1$ exists is called a *weakly* $P1$ *property*.

Definition 1.7. Let P be a topological property for which $P2 = (P \text{ and } T_2)$ exists. Then a space (X, T) is weakly $P2$ iff $(X_0, Q(X, T))$ has property $P2$. A topological property for which weakly $P2$ exists is called a *weakly* $P2$ *property*.

Within the 2015 paper [2], the search for a topological property that failed to be weakly Po focused attention on the "not-T_0" separation axiom leading to two investigations of "not-separation axioms" [6] and [7]. In this paper, the investigation of weakly Po properties and "not-separation axioms" continues with the regular separation axiom.

2. More Weakly *P* Properties

Regular spaces were introduced in 1921 [13].

Definition 2.1. A space (X, T) is regular iff for each closed set C and each x not in C, there exist disjoint open sets U and V such that x is in U and C is a subset of V. A regular T_1 space is denoted by T_3.

In the 1961 paper [1], it was proven that a space is T_i iff it is $(T_{i-1} \text{ and } R_{i-1})$, $i = 1, 2,$ and R_1 implies R_0. In the 1975 paper [10], it was shown that regular implies R_1. These results proved to be useful in this paper.

Theorem 2.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is regular* T_0, (b) (X, T) *is regular* $T_1 = T_3$, *and* (c) (X, T) *is regular* T_2.

Proof. (a) implies (b): Since (X, T) is regular, then (X, T) is R_1, which implies (X, T) is R_0. Thus (X, T) is $(R_0$ and $T_0)$, which implies (X, T) is T_1. Hence (X, T) is regular $T_3 = T_3$.

Since (regular and $T1$) $= T_3$ implies (regular and T_2), then (b) implies (c) and clearly (c) implies (a).

In a 1977 paper [8], it was proven that, for a space (X, T), the following are equivalent: (a) (X, T) is regular, (b) $(X_0, Q(X, T))$ is regular, and (c) $(X_0, Q(X, T))$ is T_3, which is used along with Theorem 2.1 to obtain the next result.

Theorem 2.2. *T_3 is a weakly $P2$, weakly $P1$, and weakly Po property with weakly T_3 = weakly (regular)2 = weakly (regular)1 = weakly (regular)o = T_0-identification (regular) = regular.*

Proof. Let (X, T) be a regular space. Since (X, T) is regular iff $(X_0, Q(X, T))$ is regular, then regular = T_0-identification (regular) = weakly (regular)*o*. Since regular is a weakly *Po* property, then $(X_0, Q(X, T))$ is (regular)*o*, which is equivalent to each of the following: $(X_0, Q(X, T))$ is (regular)1 and $(X_0, Q(X, T))$ is (regular)2. Thus, weakly (regular)*o* = weakly (regular)1 = weakly (regular)2 and (regular)*o* $= T_3$ is a weakly *Po*, weakly $P1$, and weakly $P2$ property.

In the weakly *Po* paper [2], topological properties which failed to be weakly *Po* properties were sought, leading to the discovery that neither T_0 nor "not-T_0" are weakly *Po* properties. Within the paper [4], it was shown that for each weakly *Po* property *Qo*, weakly *Qo* = (*Qo* and (weakly *Qo* and "not-T_0")) can be decomposed into two topological properties neither of which are weakly *Qo*, which, when combined with the results above, give the following result.

Corollary 2.1. Regular = weakly (regular)*o* can be decomposed into two topological properties neither of which are weakly (regular)*o*, which for this case are T_3 and (regular and "not-T_0").

Thus two more topological properties that fails to be weakly *Po* are known.

Given a topological property, questions naturally arise about product properties and subspace properties. Since regular is both a product property and a subspace property, then weakly (regular)*o* = regular is both a product property and a subspace property.

In the section below, the study of "not-separation axioms" and weakly *P* properties continues with the investigation of the "not-regular" axiom.

3. More "Not-Separation Axioms"

Within the papers [6] and [7], "not-T_i", $i = 0, 1, 2,$ and "not-R_i", $i = 0, 1,$ were investigated. Below "not-regular" is defined and investigated.

Definition 3.1. A space (X, T) is "not-regular" iff there exist a closed set C and an element x not in C such that for each open set U containing x and each open set V containing C, U and V are not disjoint.

Theorem 3.1. "*Not-regular*" = "*not*-(*weakly* (*regular*)*o*)" = "*not*-(*weakly* (*regular*)1)" = "*not*-(*weakly* (*regular*)2)".

The proof follows immediately by the equivalent contrapositive of Theorem 2.2.

Theorem 3.2. "*Not-regular*" *is a* T_0*-identification* P *property*.

Proof. Since a space is regular iff its T_0-identification space is regular, then a space is "not-regular" iff its T_0-identification space is "not-regular" and "not-regular" is a T_0-identification P property.

By continued use of contrapositives, "not-R_1" implies "not-regular", "not-T_2" implies "not-T_3", "not regular" implies "not-T_3", and "not-R_1" implies "not-T_3". Within the 1970 book [14], an example of a T_2 space that is not regular was given. Thus, the converses of the statements above in this paragraph fail to be true.

In the 1925 paper [12], the Urysohn separation axiom, which is stronger than T_2 and weaker than regular, was introduced.

Definition 3.2. A space (X, T) is Urysohn iff for distinct elements *x* and *y* in *X*, there exist open sets *U* and *V* such that *x* is in *U*, *y* is in *V*, and *Cl*(*U*) and *Cl*(*V*) are disjoint.

Contrapositives give the following results.

Corollary 3.1. *"Not-T_2" implies "not-Urysohn", which implies "not-regular".*

In the 2016 paper [7], it was proven that for a topological property *P* for which weakly *Po* exists, "not-(weakly *Po*)" exists and is a topological property, ("not-*P*")*o* = ("not-(*Po*)")*o* exists, weakly (("not-*P*")*o*) exists, and weakly (("not-*P*")*o*) = "not-(weakly *Po*)", which is combined with the results above to give the following result.

Corollary 3.2. *"Not-(weakly regular)o)" exists and is a topological property, ("not-regular")o = ("not-((regular)o)")o exists, weakly (("not-regular")o) exists, weakly (("not-regular")o) = "not-(weakly (regular)o)", and ("not-regular")o is a weakly Po property.*

The study of "not-*P*", where *P* is a topological property and "not-*P*" exists, led to the discovery of the least of all topological properties $L = (T_0$ or "not-T_0"$)$, which is also given by *L* = (*P* or "not-*P*"), where *P* is a topological property and "not-*P*" exists [9] , which is used below.

Theorem 3.3. *Regular = (T_3 or (regular and "not-T_3")), where (T_3 and (regular and "not-T_3")) does not exist and neither T_3 nor (regular and "not-T_3") are weakly Po.*

Proof. Since regular = (regular and L) = (regular and (T_3 or "not-T_3")), then regular = ((regular and T_3) or (regular and "not-T_3") = (T_3 or (regular and "not-T_3")). By the results above, T_3 is not weakly *Po* and since weakly (regular)*o* = regular, (regular and "not-T_3") is not weakly *Po*.

Theorem 3.4. *Let* (X, T) *be regular. Then* (X, T) *is "not-*T_0*" iff* (X, T) *is "not-*T_3*".*

Proof. Since regular = (T_3 or (regular and "not-T_0")) = (T_3 of (regular and "not-T_3")), then "not-T_0" = regular$\setminus T_3$ = "not-T_3". Thus (X, T) is "not-T_0" iff (X, T) is "not-T_3".

Corollary 3.3. *Let* (X, T) *be a regular space. Then* (X, T) *is "not-Urysohn" iff* (X, T) *is "not-*T_2*" iff* (X, T) *is "not-*T_1*" iff* (X, T) *is "not-*T_0*" iff* (X, T) *is "not-*T_3*".*

Corollary 3.4. *Let* (X, T) *be a regular space. Then* (X, T) *is Urysohn iff* (X, T) *is* T_2 *iff* (X, T) *is* T_1 *iff* (X, T) *is* T_0 *iff* (X, T) *is* T_3.

In topological studies prior to weakly *Po* spaces and properties, the focus was on the weakly *Po* properties: T_0, T_1, T_2, T_3, Urysohn, T_3, etc. Weakly *Po* shifted the focus from the weakly *Po* properties to weakly *Po*: R_0, R_1, regular, etc., and, as a result, long overlooked, fundamental properties in the study of topology have been discovered.

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