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AN IMPROVEMENT OF A BETA FUNCTION INEQUALITY

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Abstract

In this paper, we present a beta function inequality on $(0, 1] \times (0, 1]$, which improves an inequality of the author. A new inequality for the Ψ' function is also given, which improves an inequality of Gordon. Some elementary inequalities of one and two real variables are proved.

1. Introduction

For $x > 0$, the classical gamma function Γ and the Ψ function or digamma function are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{and} \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The derivatives Ψ', Ψ'', Ψ''', … are known as polygamma functions.

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Closely related to the gamma function is the beta function, which is the real function of two variables defined by

$$B(x,\ y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x > 0,\ y > 0.$$

A well-known equation connecting the beta and the gamma functions

$$B(x,\ y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1}$$

For a proof of (1.1) see, for example, [9] where a good reference for these functions is also given. Inequalities for the beta function can be found in the following literature [4, 5, 10, 11, 13, 14, 15, 16, 17, 18] and the references therein. Among the various kinds of inequalities concerning the beta function, we will select a special one which will be considered in detail on $(0, 1]\times(0, 1]$.

Dragomir et al. [11, p. 114, Theorem 3] established the relation

$$B(x,\ y) \le \frac{1}{xy} \quad \text{for} \quad 0 < x,\ y \le 1.$$

Recently, Alzer [5, p. 738, Theorem 3.1] has obtained the following improved results for all $x,\ y \in (0, 1]$:

$$\frac{1}{xy}\left(1-\alpha\frac{1-x}{1+x}\frac{1-y}{1+y}\right) \le B(x,\ y) \le \frac{1}{xy}\left(1-\beta\frac{1-x}{1+x}\frac{1-y}{1+y}\right), \tag{1.2}$$

with the best possible constants $\alpha = 2/3\pi^2 - 4 = 2.57973\ldots$ and $\beta = 1$, respectively. In [14, 15], the author presented the following inequalities for all $x,\ y \in (0, 1]$:

$$\frac{1}{xy}(x+y-xy) \le B(x,\ y) \le \frac{1}{xy}\frac{x+y}{1+xy}, \tag{1.3}$$

where the right side of (1.3) improves the right side of (1.2) while the left side in (1.2) and the left side in (1.3) are not comparable to each other.

The aim of this paper is to show that even the left side of (1.3) can be strengthened. More precisely, we show that for all $x,\ y \in (0,\ 1]$, we have

$$\frac{1}{xy}\frac{x + y - \frac{1}{2}xy}{1 + \frac{1}{2}xy} \leq B(x,\ y). \tag{1.4}$$

We also point out that the left side of (1.2) and the left side of (1.4) are not comparable to each other.

In connection with convex functions, we need *Jensen's inequality* for a real valued convex function $f : [a,\ b] \to \mathbb{R}$ which read as follows [18, p. 15]:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \lambda \in [0,\ 1], \tag{1.5}$$

for $x,\ y \in (a,\ b)$. For a concave function the opposite inequality holds. We also recall the *Hermite-Hadamard inequality* for a real valued convex function f as follows [19, p. 50]. Let $f : [a,\ b] \to \mathbb{R}$ be a convex function. Then we have

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a}\int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}, \tag{1.6}$$

with $a < b$. The numerical values given in the following sections have been calculated via Maple V. Release 10.0.

2. Lemmas

In order to establish the main theorem of this paper we need some lemmas, which we present in this section. The lemmas deal with some useful formulas and inequalities concerning the Ψ function and its derivative Ψ'. In the first lemma, we collect some useful formulas which can be found in [1, Chapter 6].

Lemma 2.1. *For all x, we have*

$$\Gamma(x+1) = x\Gamma(x). \tag{2.1}$$

For all $x \neq i$, $(i = 1, 2, \ldots)$, we have

$$\Psi(x) = -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty} \frac{1}{k(x+k)}, \quad (x \neq 0, -1, -2, \ldots), \tag{2.2}$$

$$\Psi(1/2) = -\gamma - 2\log 2,\ \Psi(1) = -\gamma, \tag{2.3}$$

$$\Psi^{(m)}(x) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}},\ (x \neq 0, -1, -2, \ldots \quad m = 1, 2, \ldots), \tag{2.4}$$

$$\Psi^{(n)}(x+1) = \Psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}}, \quad n \geq 0, \tag{2.5}$$

$$\Psi'(x) = \int_0^{\infty} \frac{te^{-xt}}{1-e^{-t}}\, dt, \tag{2.6}$$

where $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) = 0.57721\ldots$ *is the Euler-Mascheroni constant.*

In [12, p. 860, Theorem 4] – among other estimations – the following inequalities are given.

Lemma 2.2. *For all $t > 0$, we have*

$$\frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6\left(t+\frac{1}{14}\right)^3} < \Psi'(t) < \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3}, \tag{2.7}$$

and

$$\frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \frac{1}{30t^5} < \Psi'(t) < \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \frac{1}{30\left(t+\frac{1}{8}\right)^5}. \tag{2.8}$$

Lemma 2.3. *For all* $t > 0$, *we have*

$$1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} < \frac{t}{1 - e^{-t}} < 1 + \frac{t}{2} + \frac{t^2}{12}. \tag{2.9}$$

Proof. Writing the left side of (2.9) in an equivalent form as $h_0(t) < 0$, where

$$h_0(t) := 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} - \left(1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720}\right)e^{-t} - t,$$

we obtain after differentiation

$$h_0'(t) = -\frac{1}{720}\, e^{-t} p_0(t),$$

where $p_0(t) := e^t(4t^3 - 120t + 360) + (t^4 - 4t^3 - 60t^2 - 240t - 360)$. Now we show that $p_0(t) > 0$. Since

$$\begin{aligned} p_0'(t) &= e^t(4t^3 + 12t^2 - 120t + 240) + (4t^3 - 12t^2 - 120t - 240) \\ &> 24(t^2 + 20) > 0, \end{aligned}$$

this implies that p_0 is strictly increasing. Hence $p_0(t) > p_0(0) = 0$, i.e., $h_0'(t) < 0$, so we find $h_0(t) < h_0(0) = 0$.

On the other hand, we can write the right side of (2.9) also in an equivalent form as $h_1(t) < 0$, where

$$h_1(t) := \log \frac{1 + \frac{t}{2} + \frac{t^2}{12}}{1 - \frac{t}{2} + \frac{t^2}{12}} - t.$$

Clearly, we have $h_1(0) = 0$. Since $h_1'(t) = -t^4 / (t^4 - 12t^2 + 144) < 0$, therefore $h_1(t)$ is strictly decreasing, i.e., $h_1(t) < h_1(0) = 0$. □

The next lemma gives a somewhat better estimation for Ψ' than given in (2.7).

Lemma 2.4. *For all* $t > 0$, *we have*

$$\frac{1}{t^2} + \frac{30t^4 + 135t^3 + 230t^2 + 175t + 49}{30(t+1)^5} < \Psi'(t) < \frac{1}{t^2} + \frac{6t^2 + 15t + 10}{6(t+1)^3}. \tag{2.10}$$

Proof. Inequaliy (2.10) is a consequence of (2.5), (2.6), and (2.9). In fact, we have for the left side of (2.10)

$$\Psi'(t) - \frac{1}{t^2} = \Psi'(t+1) > \int_0^\infty \left(1 + \frac{u}{2} + \frac{u^2}{12} - \frac{u^4}{720}\right) e^{-(t+1)u} du$$

$$= \frac{30t^4 + 135t^3 + 230t^2 + 175t + 49}{30(t+1)^5}.$$

On the other hand, we obtain

$$\Psi'(t) - \frac{1}{t^2} = \Psi'(t+1) < \int_0^\infty \left(1 + \frac{u}{2} + \frac{u^2}{12}\right) e^{-(t+1)u} du = \frac{6t^2 + 15t + 10}{6(t+1)^3},$$

which completes the proof of the lemma. □

Lemma 2.5. *For all* $0 < t \le 1/2$, *we have*

$$\Psi'(t) < \frac{1}{t^2} + \frac{4t + 2}{t(4-t)(t^2+2)}. \tag{2.11}$$

Proof. We show that (2.11) is a consequence of (2.10). In fact, we have

$$\frac{6t^2 + 15t + 10}{6(t+1)^3} - \frac{4t + 2}{t(4-t)(t^2+2)} = -\frac{p_1(t)}{6(4-t)t(t+1)^3(t^2+2)},$$

where $p_1(t) := 6t^6 - 9t^5 - 14t^4 + 26t^3 + 8t^2 - 20t + 12$. We derive that $p_1(t) > 0$. Taking $l(t) := 1/32(217 - 38t)$, it is clear, that $l(t) > 0$, for all $0 \le t \le 1/2$. Next we show that $p_1(t) > l(t)$ holds which proves (2.11). Now we find

$$p_1(t) - l(t) = \frac{1}{32}(2t-1)^2(48t^4 - 24t^3 - 148t^2 + 66t + 167)$$

$$= \frac{1}{32}(2t-1)^2 k(t).$$

Since $k''(t) = 576t^2 - 144t - 296 < 0$ on $0 \le t \le t_1 = 1/24(3 + \sqrt{305}) = 0.852\ldots$, therefore $k(t)$ is concave on this interval. On account of *Jensen's inequality* (1.5), the reversed inequality holds for the concave function k. In fact we have by $[a, b] = [0, 1/2]$,

$$k(\lambda a + (1-\lambda)b) \ge \lambda k(a) + (1-\lambda)k(b), \quad \lambda \in [0, 1],$$

where $k(0) = 167$ and $k(1/2) = 163$. This clearly forces $k(t) > 0$. □

Lemma 2.6. *For all* $x, y \in [0, 1/2]$, *we have*

$$p_2(x, y) := x^4y^3 + 3x^3y^3 - 3x^2y^3 - 4xy^3 + 4y^3 - x^4y^2$$
$$- 2x^3y^2 + 5x^2y^2 + 4xy^2 - 4y^2 - 2x^4y - 8x^3y$$
$$+ 2x^2y + 12xy - 8y - 8x^2 - 8x + 8 > 0. \tag{2.12}$$

Proof. Building the derivative, we obtain

$$p_3(x, y) := \frac{\partial p_2(x, y)}{\partial x} = 4x^3y^3 + 9x^2y^3 + 10xy^2 + 4y^2 + 4xy$$
$$+ 12y - 6xy^3 - 4y^3 - 4x^3y^2 - 6x^2y^2$$
$$- 8x^3y - 24x^2y - 16x - 8.$$

It is easy to check that

$$p_3(x, y) = (4x^3 + 9x^2 - 6x - 4)y^3 + 2(-2x^3 - 3x^2 + 5x + 2)y^2$$
$$+ 4(-2x^3 - 6x^2 + x + 3)y - 8(2x + 1)$$
$$< -4y^3 + 7y^2 + 13y - 8 < 0,$$

holds for all $x, y \in [0, 1/2]$. This means $p_2(x, y)$ is strictly decreasing in x, i.e., $p_2(x, y) > p_2(1/2, y)$. After some computations, we obtain

$$p_2(1/2,\ y) = \frac{1}{16}(y-1)(27y^2+10y-32) > 0. \qquad \square$$

The next result is due to Alzer [3, p. 382, Theorem 7].

Lemma 2.7. *Let* $n \geq 0$ *be an integer and let* $x > 0$ *and* $s \in (0, 1)$ *be real numbers. Then we have*

$$\frac{1-s}{x+s+n} + (1-s)\sum_{i=0}^{n-1}\frac{1}{(x+i+1)(x+i+s)} < \Psi(x+1) - \Psi(x+s).$$

Lemma 2.8. *For all* $0 < x < y \leq 1/2,$ *we have*

$$g(x,\ y) := \frac{\Psi(y)-\Psi(x)}{y-x} - \left[\frac{1}{xy} + \frac{2(x+y+1)}{(2x+2y-xy)(xy+2)}\right] < 0. \qquad (2.13)$$

Proof. Applying Lemma 2.7 with $n = 2$ and $s = y - x,$ as well as (2.5) we get

$$\begin{aligned} g(x,\ y) &< \frac{1}{y-x}\left\{\frac{1}{x} - (1+x-y)\left[\frac{1}{y+2} + \frac{1}{(x+1)y} + \frac{1}{(x+2)(y+1)}\right]\right\} \\ &\quad - \left[\frac{1}{xy} + \frac{2(x+y+1)}{(2x+2y-xy)(xy+2)}\right] \\ &= -\frac{p_4(x,\ y)}{(x+1)(x+2)(y+1)(y+2)(2x+2y-xy)(xy+2)}, \end{aligned}$$

where

$$\begin{aligned} p_4(x,\ y) := {} & x^4y^3 + 3x^3y^3 - 3x^2y^3 - 4xy^3 + 4y^3 - x^4y^2 - 2x^3y^2 \\ & + 5x^2y^2 + 4xy^2 - 4y^2 - 2x^4y - 8x^3y + 2x^2y \\ & + 12xy - 8y - 8x^2 - 8x + 8. \end{aligned}$$

In view of Lemma 2.6, we have $p_4(x,\ y) > 0,$ which implies $g(x,\ y) < 0.$

$\square$

Lemma 2.9. *For all* $t \in [0, 1/2]$, *we have*

$$p_5(t) := 24t^{18} + 24t^{17} + 804t^{16} + 11966t^{15} + 46887t^{14} + 48798t^{13}$$
$$- 27275t^{12} + 435897t^{11} + 2637222t^{10} + 5071265t^9 + 940026t^8$$
$$- 13772406t^7 - 28854908t^6 - 29111892t^5 - 15482520t^4$$
$$- 1935112t^3 + 3639024t^2 + 2863776t + 740032 > 0.$$

In order to check that $p_5(t)$ has no real roots in $[0, 1/2]$ we used the built - in functions: **sturmseq** and **sturm** of Maple V. Release 10.0. Since $p_5(0) = 740032 > 0$, thus $p_5(t) > 0$. □

Lemma 2.10. *Consider the following transcendental equation*:

$$h_2(t) := \Psi(2t) - \Psi(1 + t) + \frac{2(t^2 + t - 2)}{t(t - 4)(2 + t^2)} = 0. \tag{2.14}$$

Then (2.14) *has on* $(0, 1/2)$ *precisely one zero, namely,* $t_1 = 0.207\ldots$. *Moreover* $h_2(t)$ *is a concave function on that interval.*

Proof. Building the first two derivatives, we obtain

$$h_2'(t) = 2\Psi'(2t) - \Psi'(1 + t) + \frac{2(-2t^5 + t^4 + 6t^3 - 34t^2 + 8t - 16)}{(t - 4)^2 t^2 (2 + t^2)^2},$$

and

$$h_2''(t) = 4\Psi''(2t) - \Psi''(t + 1)$$
$$+ \frac{4(3t^8 - 6t^7 - 38t^6 + 222t^5 - 324t^4 + 148t^3 - 216t^2 + 96t - 127)}{(t - 4)^3 t^3 (2 + t^2)^3},$$

respectively. Next we establish $h_2''(t) < 0$ on $0 < t \le 1/2$. Applying (2.4) for $m = 2$, we obtain

$$4\Psi''(2t) - \Psi''(t+1) = \sum_{k=0}^{\infty} \frac{2}{(t+1+k)^3} - \sum_{k=0}^{\infty} \frac{8}{(2t+k)^3}$$

$$= -2\sum_{k=0}^{\infty} \frac{q(k,\, t)}{(t+1+k)^3(2t+k)^3},$$

where

$$q(k,\, t) := -4t^3 + 12t^2 + 6k^2t + 24kt + 12t + 3k^3 + 12k^2 + 12k + 4 > 0,$$

for all $k \geq 0$ and $0 \leq t \leq 1/2$. So we get

$$4\Psi''(2t) - \Psi''(t+1) < -2\sum_{k=0}^{2} \frac{q(k,\, t)}{(t+1+k)^3(2t+k)^3},$$

and

$$h_2''(t) < -2\sum_{k=0}^{2} \frac{q(k,\, t)}{(t+1+k)^3(2t+k)^3}$$

$$+ \frac{4(3t^8 - 6t^7 - 38t^6 + 222t^5 - 324t^4 + 148t^3 - 216t^2 + 96t - 127)}{(t-4)^3t^3(2+t^2)^3}$$

$$= -\frac{p_6(t)}{(4-t)^3(t+1)^3(t+2)^3(t+3)^3(2t+1)^3(t^2+2)^3},$$

where

$$p_6(t) := 24t^{18} + 24t^{17} + 804t^{16} + 11966t^{15} + 46887t^{14} + 48798t^{13}$$

$$- 27275t^{12} + 435897t^{11} + 2637222t^{10} + 5071265t^9 + 940026t^8$$

$$- 13772406t^7 - 28854908t^6 - 29111892t^5 - 15482520t^4$$

$$- 1935112t^3 + 3639024t^2 + 2863776t + 740032.$$

From Lemma 2.9, we imply $h_2''(t) < 0$, i.e., $h_2(t)$ is a concave function on $[0, 1/2]$. Moreover $h_2'(t)$ is strictly decreasing and since $h_2'(2/5) = 0.015$

$\ldots > 0$ so we conclude $h_2'(t) > 0$ in $[0, 2/5]$. Hence $h_2(t)$ is strictly increasing there. After some computations, we obtain $h_2(1/10) = -0.043\ldots < 0$ and $h_2(2/5) = 0.022\ldots > 0$. Since $h_2(t)$ is continuous there exits a unique $1/10 < t_1 < 2/5$ for which $h_2(t_1) = 0$. A more precise calculation gives $t_1 = 0.207\ldots$. In order to complete the proof, we must show that h_2 has no further roots in $[2/5, 1/2]$. Since h_2 is concave there, *Jensen's inequality* (1.5) with opposite sign gives with $[c, d] = [2/5, 1/2]$, $h_2(2/5) = 0.022\ldots > 0$ and $h_2(1/2) = 0.021\ldots > 0$,

$$h_2[\lambda c + (1-\lambda)d] \geq \lambda h_2(c) + (1-\lambda)h_2(d) > 0.$$

Thus the proof of the lemma is complete. □

Lemma 2.11. *For all* $1/2 \leq x \leq 1$ *and all* $0 \leq y \leq 1$, *we have*

$$p_7(x, y) := a_1(x)y^4 + a_2(x)y^3 + a_3(x)y^2 + a_4(x)y + a_5(x) > 0, \qquad (2.15)$$

where

$$a_1(x) := 3(1-x)^2,$$

$$a_2(x) := 3(1-x)^2(4x+1),$$

$$a_3(x) := 12x^4 - 30x^3 + 14x^2 + 14x - 9,$$

$$a_4(x) := -(24x^4 - 12x^3 - 32x^2 + 2x + 11),$$

$$a_5(x) := 2(2x+3).$$

Proof. It is easy to see, that $a_1(x) \geq 0$, $a_2(x) \geq 0$ and also $a_5(x) > 0$. It is also easy to verify that $a_3(x)$ has only one positive real root which is approximately $x_{3,0} = 0.6613\ldots$ therefore on $[1/2, x_{3,0})$ we have $a_3(x) < 0$ and on $(x_{3,0}, 1]$ we have $a_3(x) > 0$. Applying *Descartes' rule of signs* implies that $a_4(x)$ has at most two positive real roots where the first one is approximately $x_{4,0} = 0.6418\ldots$, whilst the second one is

greater than 1. We have within $[1/2, x_{4,0})\, a_4(x) < 0$ whilst within $(x_{4,0}, 1]$ we have $a_4(x) > 0$. For $\max(x_{3,0}, x_{4,0}) \le x \le 1$ and $0 \le y \le 1$, we infer $p_7(x, y) > 0$. Now, it remains to show that (2.15) holds also for $1/2 \le x \le \max(x_{3,0}, x_{4,0})$ and $0 \le y \le 1$. Consider the following polynomial:

$$p_8(x, y) := a_3(x)y^2 + a_4(x)y + a_5(x).$$

Building the following partial derivatives, we obtain:

$$p_9(x, y) := \frac{\partial p_8(x, y)}{\partial x} = (48x^3 - 90x^2 + 28x + 14)y^2 - (96x^3 - 36x^2 - 64x + 2)y + 4,$$

$$p_{10}(x, y) := \frac{\partial^2 p_8(x, y)}{\partial x \partial y} = 2(48x^3 - 90x^2 + 28x + 14)y - (96x^3 - 36x^2 - 64x + 2),$$

and

$$p_{11}(x, y) := \frac{\partial^3 p_8(x, y)}{\partial x \partial y^2} = 2(48x^3 - 90x^2 + 28x + 14) = 4(x - 1)(24x^2 - 21x - 7),$$

respectively. Clearly, $p_{11}(x, y) > 0$, thus $p_{10}(x, y)$ is strictly increasing in y so we imply $p_{10}(x, y) > p_{10}(x, 0) = -(96x^3 - 36x^2 - 64x + 2) > 0$ on $x \in [1/2, 1]$. This yields that $p_9(x, y)$ is strictly increasing in y. Hence $p_9(x, y) > p_9(x, 0) = 4 > 0$. Moreover $p_8(x, y)$ is strictly increasing in x which implies $p_8(x, y) > p_8(1/2, y) = -3/2y^2 - 4y + 8 > 0$ and the lemma follows. □

Lemma 2.12. *For all* $t > 0$, *we have*

$$p_{12}(t) := 2940t^6 + 12166t^5 + 16147t^4 + 1103t^3 - 2961t^2 + 675t + 15 > 0.$$

Proof. It is sufficient to show that $p_{12}(t)$ is strictly increasing. Differentiation gives

$$p'_{12}(t) = 17640t^5 + 60830t^4 + 64588t^3 + 3309t^2 - 5922t + 675$$

$$> 64588t^3 + 3309t^2 - 5922t + 675 =: p_{13}(t).$$

Since $p'_{13}(t) = 193764t^2 + 6618t - 5922 = 0$ has two real zeros, which are $t_1 = -0.192\ldots < 0$ and $t_2 = 0.158\ldots > 0$, respectively, we have $p''_{13}(t_2) > 0$. This clearly forces that $p_{13}(t)$ has a unique minimum at $t = t_2$ and we obtain $p_{13}(t) \geqslant p_{13}(t_2) = 76.67\ldots > 0$. □

3. Main Result

Now we give the main result of this paper.

Theorem. *For all real numbers* $x, y \in (0, 1]$, *we have*

$$\frac{1}{xy}\,\frac{x + y - \frac{1}{2}xy}{1 + \frac{1}{2}xy} \leq B(x, y), \tag{3.1}$$

with equality if and only if $x = 1$ *and* $0 < y \leq 1$ *or* $y = 1$ *and* $0 < x \leq 1$.

Proof. Consider the function G defined by

$$G(x, y) := \log\frac{x + y - 1/2xy}{1 + 1/2xy} - \log\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y)}.$$

After applying partial differentiation, we get

$$\frac{\partial G(x, y)}{\partial x} = \Psi(x + y) - \Psi(x + 1) + \frac{2(y^2 + y - 2)}{(-2x - 2y + xy)(2 + xy)}, \tag{3.2}$$

$$\frac{\partial G(x,\ y)}{\partial y} = \Psi(x + y) - \Psi(y + 1) + \frac{2(x^2 + x - 2)}{(-2x - 2y + xy)(2 + xy)}, \tag{3.3}$$

$$\frac{\partial^2 G(x,\ y)}{\partial x^2} = \psi'(x + y) - \psi'(x + 1) + \frac{4(y^2 + y - 2)(-xy^2 + y^2 + 2xy - y + 2)}{(-2x - 2y + xy)^2(2 + xy)^2}. \tag{3.4}$$

and

$$\frac{\partial^2 G(x,\ y)}{\partial x \partial y} = \Psi'(x + y) - \frac{2(3x^2y^2 - 4xy^2 - 4x^2y + 16xy + 4x^2 + 4y^2 + 8)}{(-2x - 2y + xy)^2(2 + xy)^2}, \tag{3.5}$$

respectively. The proof will be divided into three steps.

Case 1. $1/2 \le x < 1$ and $0 < y < 1$.

We show that $\partial G(x,\ y)/\partial x > 0$. Since $\Psi'(t)$ is convex [cf. (2.4)] and applying the *Hermite-Hadamard* inequality (1.6), we obtain

$$(1 - y)\Psi'\left(x + \frac{1 + y}{2}\right) \le \int_{x+y}^{x+1} \Psi'(t)dt. \tag{3.6}$$

In view of (3.2), (3.6) and using the right side of (2.7), we claim

$$\begin{aligned}
\frac{\partial G(x,\ y)}{\partial x} &\ge -(1 - y)\Psi'\left(x + \frac{1 + y}{2}\right) + \frac{2(y^2 + y - 2)}{[x(y - 2) - 2y](xy + 2)} \\
&> -(1 - y)\left[\frac{1}{x + \frac{1+y}{2}} + \frac{1}{2(x + \frac{1+y}{2})^2} + \frac{1}{6(x + \frac{1+y}{2})^3}\right] \\
&\quad + \frac{2(y^2 + y - 2)}{[x(y - 2) - 2y](xy + 2)} \\
&= \frac{2(1 - y)}{3(y + 2x + 1)^3(2x + 2y - xy)(xy + 2)} P(x,\ y),
\end{aligned}$$

where

$$\begin{aligned} P(x, y) &:= 3x^2y^4 - 6xy^4 + 3y^4 + 12x^3y^3 - 21x^2y^3 + 6xy^3 + 3y^3 \\ &\quad + 12x^4y^2 - 30x^3y^2 + 14x^2y^2 + 14xy^2 - 9y^2 - 24x^4y + 12x^3y \\ &\quad + 32x^2y - 2xy - 11y + 4x + 6 \\ &= 3(x-1)^2y^4 + 3(x-1)^2(4x+1)y^3 \\ &\quad + (12x^4 - 30x^3 + 14x^2 + 14x - 9)y^2 \\ &\quad - (24x^4 - 12x^3 - 32x^2 + 2x + 11)y + 2(2x+3). \end{aligned}$$

Utilizing Lemma 2.11, we infer $P(x, y) > 0$, i.e., $\partial G(x, y)/\partial x > 0$ therefore $G(x, y)$ is strictly increasing in x thus $G(1/2, y) < G(x, y) < G(1, y) = 0$, as was stated in (3.1).

Case 2. $0 < x < 1$ and $1/2 \le y < 1$.

According to the symmetry of (3.1) in x and y changing x to y we get by a similar argument that $\partial G(x, y)/\partial y > 0$ which implies that $G(x, y)$ is strictly increasing in y. Hence we obtain $G(x, 1/2) < G(x, y) < G(x, 1) = 0$, and (3.1) follows.

Case 3. $0 \le x \le 1/2$ and $0 \le y \le 1/2$.

It is easy to see that on the boundary of the given square we have $G(x, 0) = 0$, $G(1/2, y) < 0$, $G(x, 1/2) < 0$, and $G(x, 1/2) < 0$. Since $G(x, y)$ is continuous on the closed square it takes its maximum and also its minimum value. Denote by $P(x_0, y_0)$, the point within the square where the function takes its maximum value and suppose that there we have $G(x_0, y_0) > 0$, then a necessary condition for that maximum value leads to the following system of equations:

$$\frac{\partial G(x,\ y)}{\partial x} = \Psi(x + y) - \Psi(x + 1) + \frac{2(y^2 + y - 2)}{[x(y - 2) - 2y](xy + 2)} = 0, \quad (3.7)$$

and

$$\frac{\partial G(x,\ y)}{\partial y} = \Psi(x + y) - \Psi(y + 1) + \frac{2(x^2 + x - 2)}{[x(y - 2) - 2y](xy + 2)} = 0, \quad (3.8)$$

respectively. Subtracting (3.8) from (3.7) gives – after some computations –

$$F(x,\ y) := (y - x)\left[\frac{\Psi(y) - \Psi(x)}{y - x} - \frac{1}{xy} - \frac{2(x + y + 1)}{(2x + 2y - xy)(xy + 2)}\right] = 0,$$

$$(x \neq y). \quad (3.9)$$

Now, we determine all the $(x,\ y)$ values with $x,\ y \in [0,\ 1\ /\ 2]$ for which the equation $F(x,\ y) = (y - x)F_1(x,\ y) = 0$ holds. We consider two cases.

Case a. $x \neq y$ and $0 < x < y \leq 1\ /\ 2$.

Applying Lemma 2.8, it is clear that $F(x,\ y) \neq 0$ holds.

Case b. If x tends to y the second factor in $F(x,\ y)$ implies

$$\lim_{x \to y} F_1(x,\ y) = \Psi'(y) - \frac{1}{y^2} - \frac{4y + 2}{y(y - 4)(y^2 + 2)} =: f(y).$$

Lemma 2.5 shows $f(y) \neq 0$ for all $0 < y \leq 1\ /\ 2$. Hence $F(x,\ y) = 0$ holds if and only if $x = y$. The task is now to find the solution of the equation

$$\Psi(2x) - \Psi(x + 1) + \frac{2(x^2 + x - 2)}{x(x - 4)(x^2 + 2)} = 0, \quad (3.10)$$

on $(0,\ 1\ /\ 2]$. From Lemma 2.10, we get the only zero $x_1 = 0.207\ldots$. The sufficient condition for the extreme value leads now to the following function:

$$\Delta(x,\ y) := \frac{\partial^2 G}{\partial x^2}\frac{\partial^2 G}{\partial y^2} - \left(\frac{\partial^2 G}{\partial x \partial y}\right)^2.$$

We have to discuss the sign of $\Delta(x, y)$ with respect to the special solution of (3.10). If $x_1 = y_1 = 0.207\ldots$, then we get $\Delta(x_1, y_1) = 0.132\ldots > 0$. On the other hand, we find

$$\left[\frac{\partial^2 G(x, y)}{\partial x^2}\right]_{x=x_1,\, y=y_1} = 0.384\ldots > 0,$$

so we claim that at $x_1 = y_1 = 0.207\ldots$ the function G attains its minimum value which is $G(x_1, y_1) = -0.019\ldots < 0$. Therefore, there is no $P(x_0, y_0)$, where G has a maximum value with $G(x_0, y_0) > 0$. This completes the proof of the theorem.

4. Concluding Remarks

Now we show that the left side of (3.1) improves the left side of (1.3).

Corollary 4.1. *For all real numbers* $x, y \in [0, 1]$, *we have*

$$x + y - xy \le \frac{x + y - 1/2xy}{1 + 1/2xy}. \tag{4.1}$$

The proof is trivial, we omit details.

Finally, we show, that (2.9) improves (2.7) and the left side of (2.9) improves the lower bound given in (2.8), i.e., the inequalities of Gordon.

Corollary 4.2. *For the lower bounds in* (2.7) *and* (2.8), *we have for all* $t > 0$,

$$\frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6(t + \frac{1}{14})^3} < \frac{1}{t^2} + \frac{30t^4 + 135t^3 + 230t^2 + 175t + 49}{30(t+1)^5}, \tag{4.2}$$

and

$$\frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \frac{1}{30t^5} < \frac{1}{t^2} + \frac{30t^4 + 135t^3 + 230t^2 + 175t + 49}{30(t+1)^5}, \tag{4.3}$$

respectively. For the upper bound in (2.7), *we have similarly*

$$\frac{1}{t^2}+\frac{6t^2+15t+10}{6(t+1)^3}<\frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6t^3}. \tag{4.4}$$

Proof. First we consider (4.2). Building the following difference, we get:

$$\Delta(t):=\frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6(t+\frac{1}{14})^3}-\left(\frac{1}{t^2}+\frac{30t^4+135t^3+230t^2+175t+49}{30(t+1)^5}\right)$$

$$=-\frac{P_0(t)}{30t^2(t+1)^5(14t+1)^3},$$

where $P_0(t):=2940t^6+12166t^5+16147t^4+1103t^3-2961t^2+675t+15.$ Utilizing Lemma 2.12 gives $\Delta(t)<0$.

Next, to prove (4.3), we have

$$\frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6t^3}-\frac{1}{30t^5}-\left(\frac{1}{t^2}+\frac{30t^4+135t^3+230t^2+175t+49}{30(t+1)^5}\right)$$

$$=-\frac{5t^7+5t^2+5t+1}{30t^5(t+1)^5}<0,$$

which we had to show.

In order to verify (4.4) a short computation reveals

$$\frac{1}{t^2}+\frac{6t^2+15t+10}{6(t+1)^3}-\left(\frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6t^3}\right)=-\frac{1}{6t^3(t+1)^3}<0,$$

and the corollary follows. □

It is worth pointing out that the lower bounds in (2.7) and (2.8) are not comparable to each other on the whole $(0,\infty)$ interval. Moreover, the right side of (2.8) and the right side of (2.9) are also not comparable to each other on the whole $(0,\infty)$ interval.

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