SCHUR RING AND REDUCIBILITY MODULO \( p \)

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Abstract

Let \( R \) be a ring of algebraic integers of an algebraic number field \( F \) and let \( G \leq \text{GL}_n(R) \) be a finite group. In [11] was proved that the \( R \)-span of \( G \) is just the matrix ring \( M_n(R) \) of the \( n \times n \)-matrices over \( R \) if and only if the Brauer reduction of \( R^n \) modulo every prime is absolutely irreducible. In this paper, we show that \( \langle G \rangle_R = M_n(R) \) if and only if the Brauer reduction of \( R^n \) modulo a finite number of primes is absolutely irreducible. Moreover, we give conditions for \( n \), under which \( M_n(R) \) is a Schur ring.

1. Introduction

Let \( F \) be an algebraic number field with ring of algebraic integers \( R \) and let \( \pi = \{p_1, \ldots, p_t\} \) be a set of positive prime numbers. Assume that the \( I_i \) are maximal ideals of \( R \) such that \( p_i \in I_i, i = 1, \ldots, t \). Set

\[
L_\pi = \{ \frac{a}{b} | a, b \in R, b \notin I_i, i = 1, \ldots, t \}
\]

Then \( R_\pi \) denotes a localization
of $R$ at $L_{\pi}$. Thus $R_{\pi}$ is a principal ideal having quotient field of characteristic zero and containing a unique prime ideal $I_i$ such that $p_i \in I_i$, $i = 1, \ldots, t$. We denote the Jacobson radical of $R_{\pi}$ be $J(R_{\pi})$. Therefore, the residue ring $K = R_{\pi} / J(R_{\pi})$ is a semi-simple ring of characteristic $m = \prod_{i=1}^{t} p_i$. Thus, we may write

$$K = \bigoplus_{i=1}^{t} k_i,$$

where the $k_i$ are fields of characteristic $p_i (i = 1, \ldots, t)$. Let $G$ be a finite group. Then we have

$$KG = \bigoplus_{i=1}^{t} k_i G.$$  \hspace{1cm} (1.2)

From (1.2), it follows that

$$1 = f_1 + \cdots + f_t,$$

where the $f_i$ are orthogonal central idempotents in $KG$. Therefore,

$$KG = \bigoplus_{i=1}^{t} k_i G$$

$$= \bigoplus_{i=1}^{t} KGf_i,$$

with $k_i G = KGf_i$.

Now, $R_{\pi}$ is a Hausdorff space in its $J(R_{\pi})$-topology, i.e., that $\bigcap_{j=1}^{\infty} J(R)^j = (0)$, so the $J(R_{\pi})$-adic completion $\hat{R}_{\pi}$ of $R_{\pi}$ is a complete semi-local ring such that

$$K = \hat{R}_{\pi} / J(\hat{R}_{\pi}) = R_{\pi} / J(R_{\pi}),$$
and

\[ \hat{R}_\pi = \hat{R}_{\pi 1} \oplus \cdots \oplus \hat{R}_{\pi t}, \quad (1.4) \]

where the \( \hat{R}_{\pi i} \) are complete local rings such that \( \hat{R}_\pi / J(\hat{R}_\pi) = k_i \).

Therefore

\[ \hat{R}_\pi G = \hat{R}_{\pi 1}G \oplus \cdots \oplus \hat{R}_{\pi t}G. \quad (1.5) \]

From (1.5), we obtain

\[ 1 = \hat{f}_1 + \cdots + \hat{f}_t, \quad (1.6) \]

where the \( \hat{f}_i \) are orthogonal central idempotents in \( \hat{R}_\pi G \).

In the study of the Schur subgroup of the Brauer group of a commutative ring \( R \), one is interested in finding the Azumaya algebras over \( R \) that are epimorphic images of a group ring \( RG \) for some finite group \( G \).

For a commutative ring \( R \), the matrix ring \( M_n(R) \) of the \( n \times n \)-matrices over \( R \) is called a Schur ring if there exists a finite group \( G \leq GL_n(R) \) such that the \( R \)-span of \( G \) is just \( M_n(R) \), i.e., in such case we have \( \langle G \rangle_R = M_n(R) \).

Let \( R \) be a ring of algebraic integers of an algebraic number field \( F \) and let \( G \leq GL_n(R) \) be a finite group. In [11], it was proved that \( \langle G \rangle_R = M_n(R) \) if and only if \( R^n \) is globally irreducible, i.e., the Brauer reduction of \( R^n \) modulo every prime is absolutely irreducible. In this paper, we show that \( \langle G \rangle_R = M_n(R) \) if and only if the Brauer reduction of \( R^n \) modulo a finite number of primes is absolutely irreducible. Moreover, we give conditions for \( n \), under which \( M_n(R) \) is a Schur ring.
2. Preliminary Results

**Definition 2.1.** Let $\mathbb{C}$ denote the field of complex numbers and $G \leq GL_n(\mathbb{C})$. We call $G$ globally irreducible if for every prime $p$ the reduction of $\mathbb{C}^n$ modulo $p$ is absolutely irreducible.

Let $G$ be a finite group, $p$ be a prime divisor of $|G|$, and $R$ be a complete discrete valuation ring with quotient field $F$ of characteristic 0. We assume that the residue field $k = R / J(R)$ has characteristic $p$, where $J(R)$ denotes the Jacobson radical of $R$. With this assumption, we refer to the triple $(F; R; k)$ as a splitting $p$-modular system.

Recall that the Brauer reduction of a modulo for a natural prime $p$ is defined as follows. If $V$ is a $FG$-module, then there exists a full $RG$-lattice $L \subseteq V$. The $kG$-module $L / J(R)L = U$ is called a reduction of $V$ modulo $p$. Moreover, in such case, we say also that $U$ is the reduction modulo $p$ of the $RG$-lattice $L$.

According to Definition 2.1, the linear group $G$ is globally irreducible if for every prime $p$ the reduction of $\mathbb{C}^n$ modulo $p$ is absolutely irreducible. We know that in such case is sufficient the study of the reduction modulo $p$, for every prime $p$ divisor of $|G|$. Therefore, result evident the necessity of to study the following problem:

**Problem 2.2.**

When the reduction modulo $p$ of the an absolutely irreducible $FG$-module $V$ is an absolutely irreducible $kG$-module, being $k$ a field of characteristic $p$?

3. Some Properties of Certain Induced Modules

Let $G$ be a finite group with splitting field $k$, and let $Q$ be a $p$-subgroup of $G$. Assume that $s = |G : Q|$ and let $X^+ = \{x_1, \ldots, x_s\}$ be a full set of representatives in $G$ of the cosets in $G / Q$. Then $Ind_Q^G(k)$ is isomorphic to $kGQ^+$ as left $kG$-module, where $Q^+ = \{ \sum_{x \in X^+} ax \in kG\}$. 
Set $X = \{x_i - x_j, y \in Q\}$. We denote the left ideal generated by $X$ in $kG$ by $I_Q(G)$. We claim that

\[
\text{rank}_k(I_Q(G)) = |G : Q|(|Q| - 1)
\]

\[
= |G : P|\frac{|P|}{|Q|}(|Q| - 1).
\]

Thus, we have

\[
kG / I_Q(G) \cong kGQ^+,
\]

as $k$-modules. We now assume that $Q < Q'$, where $Q'$ is also a $p$-subgroup of $G$. Set $X^Q_Q = \{x_i - x_j, x_j = yxy', y \in Q$ and $y' \in Q'\}$.

Then $kG / I_Q(G)$ contains a left ideal isomorphic to the left ideal generated by $X^Q_Q$. We denote this ideal by $I^Q_Q$. Observe that

\[
\text{rank}_k(I^Q_Q) = |G : P|\frac{|P|}{|Q|}(|Q| - 1).
\]

Let us write $T_Q$ by $kG / I_Q(G)$. Thus we have

\[
T_Q / I^Q_Q \cong kGQ^+.
\]

It is well know that

\[
kG = \bigoplus_{j=1}^r F^\dim S_j,
\]

where $P_{S_j}$ is the projective cover of the simple $kG$-module $S_j$ and $r$ is the number of conjugacy classes of $p$-regular elements of $G$.

From (3.9), the following holds:

\[
kGQ^+ = \bigoplus_{j=1}^r M^Q_j,
\]

where $M^Q_j = F^\dim S_j / F^\dim S_j I_Q(G)$.
The following two lemmas are easy but useful to our results:

**Lemma 3.3.** Let $G$ be a finite group with splitting field $k$ of characteristic $p$. Fixed $P \in \text{Syl}_p(G)$. Then $I_P(G)$ is an annihilator of the trivial $kG$-module.

**Proof.** Since for all finite group $G$, the trivial module is a $kG / I_P(G)$-module, the assertion follows. □

**Lemma 3.4.** Let $G$ be a finite group with splitting field $k$ of characteristic $p$. Fixed $P \in \text{Syl}_p(G)$. We denote the Jacobson radical of $kG$ by $J(G)$. Then $J(G) \subseteq I_P(G)$ if and only if $kG / I_P(G)$ is semisimple.

**Proof.** Since every indecomposable $kG$-module, direct summand of $kG / I_P(G)$, is annihilated by $I_P(G)$, the result follows.

The converse implication is trivial. □

**Lemma 3.5.** Let $G$ be a finite group with splitting field $k$ of characteristic $p$, and let $S$ be a simple $kG$-module. Set $P \in \text{Syl}_p(G)$ fixed. Then $M_P^j$ is a projective $kG$-module if and only if $P_{S_j}$ is a blocks of defect zero.

**Proof.** Let $J(G)$ be the Jacobson radical of $kG$. We to check two cases.

**Case 1.** $J(G) \subseteq I_P(G)$.

In this case, the assertion follows by Lemma 3.4.

**Case 2.** $J(G) \not\subseteq I_P(G)$.

Assume that $M_P^Q \cong P_{S_j}^l$ is a projective $kG$-module, where $l$ is the multiplicity of $P_{S_j}$ as direct summand of $M_P^Q$. We show that $P_{S_j}$ is a simple $kG$-module.
Since $I_P(G)$ is left ideal of $\mathcal{k}G$ from (3.9) is follows that
\[ I_P(G) = P_{S_1}^{\dim S_1} I_P(G) \oplus \cdots \oplus P_{S_r}^{\dim S_r} I_P(G). \tag{3.11} \]

We have $P_j I_P(G) = 0$ by assumption, so we deduce that $P_{S_j}^{\dim S_j} I_P(G)$ is a projective $\mathcal{k}G$-module, where the multiplicity of $P_{S_j}$ is equal to $\dim(S_j) - l$, i.e., we have
\[ P_{S_j}^{\dim S_j} I_P(G) = P_{S_j}^{\dim S_j - l}. \]

Therefore, we may assert that $P_{S_j} I_P(G)$ is a right indecomposable $I_P(G)$-module such that
\[ (P_{S_j} I_P(G))^{\dim S_j} = P_{S_j}^{\dim S_j - l}. \tag{3.12} \]

We assume that $\alpha = \dim(P_{S_j} I_P(G))$ and $\beta = \dim(P_{S_j})$. According to (3.12), we may write the following equality:
\[ \alpha \dim(S_j) = \beta(\dim(S_j) - l). \tag{3.13} \]

From (3.13), it follows that
\[ \frac{\alpha}{\dim(S_j) - l} = \frac{\beta}{\dim S_j}. \tag{3.14} \]

We now claim that the Equality (3.14) is true if and only if $\frac{\alpha}{\dim(S_j) - l} = \frac{\beta}{\dim S_j} = 1$. Thus, the following holds $\dim S_j = \dim P_{S_j}$, which is what we need to prove.

Conversely, by assumption it follows that
\[ P_{S_j}^{\dim S_j} I_P(G) = (P_{S_j} I_P(G))^{\dim S_j} \tag{3.15} \]
where \( \dim(P_{S_j} I_P(G)) = \dim(S_j) - l \) with \( l = \dim S_{jp'} \), being \( \dim S_{jp'} \)
the \( p' \)-part of \( \dim S_j \). Thus, we deduce that \( P_{S_j}^{\dim S_j} I_P(G) = P_{S_j}^{\dim(S_j)-l} \).

So we are done.

\[ \square \]

**Lemma 3.6.** Let \( G \) be a finite group with splitting field \( k \) of characteristic \( p \) and let \( P \in \text{Syl}_p(G) \) fixed. Then every indecomposable \( kG \)-module direct summand of \( kGP^+ \) has a radical vertex.

**Proof.** Let \( N_G(P) \) be the normalizer of \( P \). According to the Green correspondence, every indecomposable \( kG \)-module direct summand of \( kGP^+ \cong \text{Ind}_P^{N_G(P)} \text{Ind}_G^{N_G(P)}(k) \) has vertex \( P \) or a vertex in \( P \cap P^g, \)
g \( \in G - N_G(P) \). Observe that if \( P \) is a normal subgroup of \( G \), then \( kGP^+ \)
is semi-simple, so every indecomposable \( kG \)-module direct summand of \( kGP^+ \) is a simple \( kP \)-module with vertex \( P \). Therefore, we now consider the case where \( P \) is not a normal subgroup of \( G \). Assume that \( U \) is an indecomposable \( kG \)-module with vertex \( Q \leq P \), being \( U \) a direct summand of \( kGP^+ \). We to check two cases.

- **Case 1.** \( Q = 1 \) or \( Q = P \).

The assertion results trivially by assumption.

- **Case 2.** \( 1 < Q < P \).

In this case \( Q = P \cap P^g \), for some \( g \in G - N_G(P) \). Let \( N_P(Q) \) be the normalizer of \( Q \) in the Sylow \( p \)-subgroup \( P \). Since \( P \cap N_G(Q) = N_P(Q) \) and \( P^g \cap N_G(Q) = N_P^g(Q) \) are Sylow \( p \)-subgroups of \( N_G(Q) \), we deduce that \( g \in N_G(Q) - N_P(Q) \). We now shows that \( N_P(Q) \) is not a normal subgroup of \( N_G(Q) \).
Let us write \( P \) for \( N_p(Q) \). Conversely, we assume that \( P \) is a normal subgroup of \( N_G(Q) \). In such case \( P = P \cap P^G = Q \), which is a contradiction. Now, since \( Q = P \cap P^G \) is follows that
\[
Q \supseteq O_p(N_G(Q)).
\]
But on the other hand, \( Q \) is a normal \( p \)-subgroup of \( N_G(Q) \), and so is contained in \( O_p(N_G(Q)) \). Thus we have equality.

\[\square\]

Many of the properties of the \( kG \)-modules with trivial source was studied by several authors. In particular, Okuyama’s obtained the following results (see [9]).

**Lemma 3.7.** Let \( S \) be a simple \( kG \)-module with vertex \( Q \) and trivial source. Then the Green correspondent \( f(S) \) of \( S \) is a simple projective \( k[N_G(Q)/Q] \)-module.

**Lemma 3.8.** Let \( S \) be a simple \( kG \)-module with vertex \( Q \) and trivial source. Then the \( p \)-part of \( \dim S \) is equal to \( \frac{|P|}{[Q]} \), where \( P \in \text{Syl}_p(G) \).

Alperin’s obtained the following result (see [1]):

**Lemma 3.9.** Let \( P \) be a Sylow \( p \)-subgroup of \( G \). If \( W \) is a simple projective \( k[N_G(Q)/Q] \)-module, then its Green correspondent of \( W \) is isomorphic to an indecomposable direct summand of \( kGP^+ \).

4. **\( G \)-Weight and Main Proprieties**

In the rest of this paper, we will assume the notations and terminologies used in the last section.
Lemma 4.10. Let $G$ be a finite group, $k$ be a splitting field for $G$, and $Q \neq 1$ be a radical $p$-subgroup of $G$. If $M_j^Q$ has an indecomposable non-projective $kG$-module as direct summand, then it is unique, up to isomorphism.

Proof. By the Krull-Schmidt theorem, each left $kG$-module $M_j^Q$ can be decomposed of unique manner as a direct sum of indecomposable $kG$-modules, i.e., we may write

$$M_j^Q = \bigoplus_{\gamma=1}^{\mu} U_{\gamma},$$

where the $U_{\gamma}$ are indecomposable $kG$-modules.

We now assume that $U_{\gamma}$ is direct summand in (4.16), which is an indecomposable non-projective $kG$-module. Firstly, we show that $P_{S_j}$ is the projective cover of $U_{\gamma}$.

Since $P_{S_j} / \text{Rad}(P_{S_j}) \cong U_{\gamma} / \text{Rad}(U_{\gamma}) \cong S_j$, we deduce that there is an epimorphism $P_{S_j} \rightarrow U_{\gamma}$, which necessarily is essential by Nakayama's lemma.

We now show that $U_{\gamma}$ is unique, up to isomorphism.

Suppose that $U_{\gamma'}$ is other indecomposable non-projective $kG$-module in the decomposition (4.16). Since $P_{S_j}$ is projective cover of $U_{\gamma}$ and $U_{\gamma'}$ we assert that there are two essential epimorphisms $\theta_1 : P_{S_j} \rightarrow U_{\gamma}$ and $\theta_2 : P_{S_j} \rightarrow U_{\gamma'}$. We define the epimorphism $\sigma : U_{\gamma} \rightarrow U_{\gamma'}$ given by $\sigma(\theta_1(a)) = \theta_2(a), a \in P_{S_j}$. Let $\Omega(U_{\gamma})$ and $\Omega(U_{\gamma'})$ be the Heller operators of $U_{\gamma}$ and $U_{\gamma'}$, respectively. Then we may write

$$\ker \sigma = \{ \theta_1(a) \in U_{\gamma} : a \in \Omega(U_{\gamma'}) \}. $$
Thus we may assert that $\ker \sigma \cong \Omega(U_\gamma)$ by assumption. Hence, we have

$$P_{S_j} / \Omega(U_\gamma) \cong U_\gamma / \ker \sigma \cong U_{\gamma'}.$$  \hspace{1cm} (4.17)

We claim that the relation (4.17) is true if and only if $P_{S_j} \cong U_\gamma$ or $\ker \sigma = 0$. By assumption, we may deduce that the unique case possible is $\ker \sigma = 0$. Therefore $U_\gamma \cong U_{\gamma'}$, which is what we need to prove. \hfill \Box

According to the Lemmas 3.5 and 4.10 takes place the following definition:

**Definition 4.11.** A $G$-weight for $G$ is a pair $(U, Q)$, where $U$ is a direct summand of $kGP^+$ with vertex $Q$, which is simple or an indecomposable non-projective $kG$-module.

The following theorem is fundamental in our investigation.

**Theorem 4.12.** Let $G$ be a finite group with splitting field $k$ of characteristic $p$. Then the number of non-isomorphic $G$-weights equals the number of conjugacy classes of $p$-regular elements of $G$.

**Proof.** From (3.10), we have

$$kGP^+ = \bigoplus_{j=1}^{r} M_j^P,$$  \hspace{1cm} (4.18)

where $r$ is the number of conjugacy classes of $p$-regular elements of $G$ and the $M_j^P$ are left $kG$-modules such that

$$M_j^P \cong P_{S_j}^{\dim S_j} / P_{S_j}^{\dim S_j} I_P(G),$$

for some simple $kG$-module $S_j$.

We check two cases.

**Case 1.** $M_j^P$ is projective.

In such case we have $M_j^P = \oplus P_{S_j}$. According to Lemma 3.5, we obtain $P_{S_j} \cong S_j$. Therefore, the assertion follows by assumption.
Case 2. $M_j^P$ is not projective.

If $M_j^P$ is not projective, then has a unique indecomposable non-projective $kG$-module as direct summand, by Lemma 4.10. Thus the result follows by assumption. 

5. Conditions for the Reducibility Modulo $p$ of an Irreducible Brauer $p$-Character

Let $G$ be a finite group, $p$ be a prime divisor of $|G|$, and $R$ be a complete discrete valuation ring with quotient field $F$ of characteristic 0. We assume that the residue field $k = R / J(R)$ has characteristic $p$, where $J(R)$ denotes the Jacobson radical of $R$. With this assumption, we refer to the triple $(F; R; k)$ as a splitting $p$-modular system.

**Lemma 5.13.** Let $G$ be a finite group and $k$ be a splitting field for $G$. Let $U_1, ..., U_r$ be a complete list of non-isomorphic $G$-weights, with projective covers $P_{S_1}, ..., P_{S_r}$, respectively. Then the Brauer characters $\phi_{U_1}, ..., \phi_{U_r}$ of the $G$-weights form a basis in the space $\mathbb{C}^{p-\text{reg}(G)}$ of class functions on the $p$-regular elements of $G$.

**Proof.** Everything follows from the formula

$$
\tau = \langle \phi_{P_{S_i}}, \phi_{U_j} \rangle = \begin{cases} 
\tau = 0, & \text{if } i \neq j; \\
\tau = 1, & \text{if } i = j \text{ and } U_j \cong S_i; \\
\tau > 1, & \text{if } i = j \text{ and } U_j \not\cong S_i,
\end{cases}
$$

and the fact that the number of non-isomorphic $G$-weights modules equals the number of $p$-regular conjugacy classes of $G$. Thus if $\sum_{i=1}^{r} \lambda_i \phi_{U_i} = 0$, we have $\langle \phi_{P_{S_i}}, \phi_{U_i} \rangle \lambda_i = 0$, so $\lambda_i = 0$, which shows that the are independent, and hence form a basis. 

\qed
Theorem 5.14. Let \((F; R; k)\) be a splitting \(p\)-modular system for the finite group \(G\). The simple \(kG\)-module \(S\) is the reduction modulo \(p\) of an \(RG\)-lattice if and only if \(S\) is a \(G\)-weight.

Proof. Let \(S\) be a simple \(kG\)-module with projective cover \(P_S\), and let \(U_i\) be a \(G\)-weight such that \(U_i / \text{Rad}(U_i) \cong S\). Assume that \(S\) is the reduction modulo \(p\) of an \(RG\)-lattice. According to the Lemma 5.13, we may write

\[
\sum_{i=1}^{r} \lambda_i \phi_{U_i} = \phi_S. \tag{5.19}
\]

From (5.19), we may write

\[
\langle \phi_S, \phi_{U_i} \rangle \lambda_i = \langle \phi_S, \phi_S \rangle. \tag{5.20}
\]

Since \(S\) and \(U_i\) are liftable to one \(RG\)-lattice, and \(S\) is the radical quotient of \(U_i\), it follows that \(\langle \phi_S, \phi_{U_i} \rangle = \langle \phi_S, \phi_S \rangle\), so \(\lambda_i = 1\).

Conversely, since \(kGP^+\) is the reduction modulo \(p\) of the \(RG\)-lattice \(RGP^+\), the result follows.

Theorem 5.15. Let \(G\) be a finite group with \(P \in \text{Syl}_p(G)\) fixed, and let \(S\) be a simple \(kG\)-module with radical vertex \(Q\) and trivial source \(k\). Then \(S\) is a \(G\)-weight.

Proof. Combining the Lemmas 3.7 and 3.9, we deduce that \(S\) is a direct summand of \(kGP^+\). Therefore, by assumption, the assertion follows.

Combining the Theorems 5.14 and 5.15, we deduce the following result:

Corollary 5.16. Let \((F; R; k)\) be a splitting \(p\)-modular system for the finite group \(G\). The simple \(kG\)-module \(S\) has trivial source if and only if \(S\) is the reduction modulo \(p\) of an \(RG\)-lattice.
6. Schur Ring and Globally Simple Modules

Definition 6.17. Let $R$ be a commutative ring and let $G \leq GL_n(R)$ be a finite group. Then the matrix ring $M_n(R)$ is called Schur ring (more briefly $S$-ring) if $(G)_R = M_n(R)$.

Lemma 6.18. Let $k$ be a field of characteristic $p$ and let $G \leq GL_n(k)$ be a finite group with splitting field $k$. Let us write $U$ for $k^n$. Then $U$ is a simple $kG$-module if and only if $(G)_k = M_n(k)$.

Proof. If $U$ is a simple $kG$-module, then the assertion follows by Burnside’s theorem. Conversely, since $U^n \cong M_n(k)$ the result follows by assumption. □

Let $G$ be a finite group, $\pi = \{p_1, \ldots, p_t\}$ be a finite set of prime numbers and $R_\pi$ be a localization of $R$ at $L_\pi$, where $R$ is the ring of integers of $F$, being $F$ the quotient field of $R$ and splitting field of $G$. We assume that the residue field $K = R_\pi / J(R_\pi)$ has characteristic $m = \prod_{i=1}^{t} p_i$, where $J(R)$ denotes the Jacobson radical of $R_\pi$. With this assumption, we refer to the triple $(F; R; K)$ as a splitting $\pi$-modular system.

The reduction modulo $\pi$ of an $RG$-module is defined as follows. If $V$ is a $RG$-module, then there exists a full $R_\pi G$-lattice $\mathcal{L} \subseteq V$. The $KG$-module $\mathcal{L} / J(R)\mathcal{L} = U$ is called a reduction of $V$ modulo $\pi$. Moreover, in such case, we say also that $U$ is the reduction modulo $\pi$ of the $R_\pi G$-lattice $\mathcal{L}$.

Let $G \leq GL_n(R)$ be a finite group. We consider the natural projection $\sigma : GL_n(R_\pi) \to GL_n(K)$. Then $\sigma(G) = \overline{G}$ is called reduction of $G$ modulo $\pi$. 
In such case, we way write
\[ \overline{G} = \overline{G}_1 \otimes \cdots \otimes \overline{G}_t, \]
where \( \overline{G}_i(i = 1, \ldots, t) \) is the reduction of \( G \) modulo \( p_i \).

**Definition 6.19.** Let \((F, R, K)\) be an \( m \)-modular system, where \( \pi = \{p_1, \ldots, p_t\} \) is a set of prime numbers. Assume that \( G \leq GL_n(R) \) is a finite group. Let us write \( V \) for \( R^n \) and we write \( U \) for \( K^n \). Then \( V \) is called \( \pi \)-quasi-simple if each direct summand \( Uf_i \) is a simple \( \overline{G}_i \)-weight.

**Lemma 6.20.** Let \((F, R, K)\) be an \( m \)-modular system, and let \( G \leq GL_n(R) \) be a finite group. Let us write \( V \) for \( R^n \) and we write \( U \) for \( K^n \). Then \( \langle \overline{G} \rangle_K = M_n(K) \) if and only if \( V \) is \( \pi \)-quasi-simple.

**Proof.** We have
\[
\langle \overline{G} \rangle_K = \langle \overline{G} \rangle_K f_1 \oplus \cdots \oplus \langle \overline{G} \rangle_K f_t
\]
\[
= \langle \overline{G}_1 \rangle_{k_1} \oplus \cdots \oplus \langle \overline{G}_t \rangle_{k_t}
\]
\[
= M_n(k_1) \oplus \cdots \oplus M_n(k_t)
\]
\[
= M_n(K)f_1 \oplus \cdots \oplus M_n(K)f_t
\]
\[
= M_n(K).
\] (6.21)

From (6.21), it follows that \( \langle \overline{G}_i \rangle_{k_i} = M_n(k_i) \) for all \( i(1 \leq i \leq t) \). Therefore, applying the last lemma we assert that \( Uf_i \) is a simple \( k_i \overline{G}_i \)-module for every \( i \). Hence, by Theorem 5.14, the result follows. On the other hand, by assumption and applying again the Lemma 6.18, we deduce that \( \langle \overline{G}_i \rangle_{k_i} = M_n(k_i)(i = 1, \ldots, t) \). Therefore, the equality follows. \( \square \)
Definition 6.21. Let \((F, R, K)\) be a \(m\)-modular system, and let \(G \leq GL_n(R_{\pi})\) be a finite group. Assume that \(\pi\) is the set of the positive prime divisors of \(|G|\). If \(R^n\) is a \(\pi\)-quasi-simple, then we say that \(G\) is \(\pi\)-globally simple.

Lemma 6.22. Let \((F, R, K)\) be an \(m\)-modular system, and let \(G \leq GL_n(R)\) be a finite group. Assume that \(\pi\) is a set of the prime divisors of \(|G|\). Then \(\langle G \rangle_{R_{\pi}} = M_n(R_{\pi})\) if and only if \(G\) is \(\pi\)-globally simple.

Proof. By assumption, we may write
\[
\langle \overline{G} \rangle_K = M_n(K). \tag{6.22}
\]
Therefore, we may assert that \(R^n\) is \(\pi\)-quasi-simple by Lemma 6.20. Thus, by assumption, the result follows. On the other hand, by assumption and applying the Lemma 6.18, we obtain
\[
\langle \overline{G} \rangle_K = M_n(K). \tag{6.23}
\]
From (6.23), it follows that
\[
R_{\pi} \otimes_K \langle \overline{G} \rangle_K = R_{\pi} \otimes_K M_n(K).
\]
Since \(\langle G \rangle_{R_{\pi}} \cong R_{\pi} \otimes_K \langle \overline{G} \rangle_K\) and \(M_n(R_{\pi}) \cong R_{\pi} \otimes_K M_n(K)\) the assertion follows. \qed

Theorem 6.23. Let \(R\) be a ring of algebraic integers and let \(G \leq GL_n(R)\) be a finite group. Then \(\langle G \rangle_R = M_n(R)\) if and only if \(G\) is \(\pi\)-globally simple.

Proof. From \(\langle G \rangle_R = M_n(R)\), it follows that \(\langle G \rangle_{R_{\pi}} = M_n(R_{\pi})\).

Hence, the result follows by Lemma 6.22. Conversely, according to the Lemma 6.22, we have
\[
\langle G \rangle_{R_{\pi}} = M_n(R_{\pi}).
\]
Therefore, the assertion follows by assumption. \qed
**Theorem 6.24.** Let \( R \) be a ring of algebraic integers with quotient field \( F \) and let \( G \leq \text{GL}_n(R) \) be a finite group with splitting field \( F \). Assume that \( \pi = \{p_1, \ldots, p_t\} \) is the set of prime divisors of \( |G| \) and we denote the \( \pi \)-part of \( n \) by \( n_\pi \). If \( \langle G \rangle_R = M_n(R) \), then \( n_\pi = \prod_{i=1}^{t} \frac{|P_i|}{|Q_i|} \), where \( P_i \in \text{Syl}_{p_i}(G) \) and \( Q_i \) is the vertex of the \( k_iG \)-module \( K_n^i \).

**Proof.** According to Theorem 6.23, each \( K_n \) is a simple \( G \)-weight. Hence, the \( p_i \)-part of \( \dim K_n \) is \( \frac{|P_i|}{|Q_i|} \) by Lemma 3.8. Since \( n = \dim R^n = \dim K_n = \dim K_n^i (i = 1, \ldots, t) \) the assertion follows.

**Theorem 6.25.** Let \( R \) be a ring of algebraic integers with quotient field \( F \) and let \( G \) be a finite group with splitting field \( F \). Assume that \( \pi = \{p_1, \ldots, p_t\} \) is the set of prime divisors of \( |G| \), and \( H \) is a \( \pi \)-subgroup of \( G \) such that \( N_G(Q_i) \leq H \), where \( Q_i \in \text{Syl}_{p_i}(H) (i = 1, \ldots, t) \). If \( \text{Ind}_H^G(R) \) is a simple \( FG \)-module, then \( M_n(R) \) is an \( S \)-ring, for \( n = \frac{|G|}{|H|} \).

**Proof.** Let \( U \) be the reduction modulo \( \pi \) of \( \text{Ind}_H^G(R) \). Since \( Q_i \) is the vertex of the trivial \( k_iH \)-module we deduce that \( \text{Ind}_H^G(k_i) \cong Uf_i \) has vertex \( Q_i \) and trivial source. Therefore, we may assert that \( Uf_i \) is a \( G_i \)-weight for all \( i = 1, \ldots, t \). Since \( Uf_i (i = 1, \ldots, t) \) is the reduction modulo \( p_i \) of the simple \( FG \)-module \( \text{Ind}_H^G(R) \) we deduce that \( Uf_i \) is a simple \( k_iG \)-module. Therefore, by assumption, we may assert that \( \text{Ind}_H^G(R) \) is \( \pi \)-globally simple. Since \( \dim \text{Ind}_H^G(R) = \frac{|G|}{|H|} \) the assertion follows by Theorem 6.23.
References


