Journal of Mathematical Sciences: Advances and Applications
Volume 36, 2015, Pages 7-27
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jmsaa_7100121556

RISK MANAGEMENT AND ALM: A THEORETICAL MODEL AND FOUR OPTION STRATEGIES

WERNER HÜRLIMANN

Independent Scholar
Swiss Mathematical Society
Switzerland
e-mail: whurlimann@bluewin.ch

Abstract

A specific actuarial protection model, which is based on a stable excess-of-loss reserve, is proposed as theoretical model of risk management for use in ALM. It is related to four option strategies, which are described and discussed. The first option strategy refers to a risky mean self-financing realization of the actuarial protection model. A second option strategy supposes that the hedging instrument is available in a global options market for actuarial and financial risks. The third option strategy is based on a risky mean self-financing dividend strategy. The last option strategy reproduces the financial gain by buying the dividend process in a global options market. Examples from financial markets illustrate some results.

1. Introduction

Risk management is an important component of modern corporate management, where the development of a holistic risk management comes to the force (e.g., Albrecht [1]). Risks should be looked at in their entirety and integrated in the corporate policy. Risk management should

---

2010 Mathematics Subject Classification: 62P05, 91B30, 91G10.
Keywords and phrases: strategic financial management, asset and lialibities, profit and loss, risk management, actuarial protection model, mean self-finance, option strategy.
Received October 19, 2015

not only recognize risk potentials in the sense of loss and capital consumption but also bear in mind chances in the sense of performance and profit-taking. Furthermore, individual corporate risks should be combined in a way that allows for compensation and diversification. Restricted to financial risks, there is an increased demand for appropriate mathematical tools of risk management that find applications in both the insurance and finance industry.

The theoretical model at hand for use in ALM (= asset and liability management) is a synthesis of considerations that can be found in numerous publications by the author (e.g., Hürlimann [15] and their references). It can possibly be used as a financial instrument to deal with the risk measurement of combined investment and insurance risks.

The following notations are used throughout:

$t \in [0, T]$: time parameter over a given finite time horizon.

$A(t)$: random accumulated value of the assets at time t.

$L(t)$: random accumulated value of the liabilities at time t.

$G(t) = A(t) - L(t)$: random profit at time t.

$V(t) = L(t) - A(t)$: random financial loss at time t.

$G(t)_+ = \max\{G(t), 0\}$: random positive profit at time t.

$V(t)_+ = \max\{V(t), 0\}$: random positive financial loss at time t.

We are interested in the following "AFIR problem" (AFIR = actuarial approach to financial risks). Give a comprehensive understanding of all risk management activities that can be performed in order to realize as closely as possible the expected future profit of a corporate firm at the future time T.

Starting point of the present theoretical model is the *profit and loss identity*:

$$G(t) + V(t)_+ = G(t)_+. \tag{1.1}$$

This equation is the core of portfolio insurance that has been founded by Leland in 1976 (see Luskin [20]) and also found in Bühlmann [7].

Following modern finance in continuous time (e.g., Merton [22]), it is assumed that a risk manager can continuously operate on the financial market in order to adapt his financial strategy to the daily, weekly, monthly or yearly profits and losses. The idea underlying our model is simple. To compensate a possible loss $V(t) > 0$ at time t, the risk manager puts aside a certain reserve denoted by $R(t)$ (similar to economic capital), called *excess-of-loss reserve*, which depends upon the information $F(t)$ available at time t, and which satisfies the constraint $0 \le R(t) \le G(t)_+$. The information set contains at least the value of the assets $A(0)$ at the starting time $t = 0$, and $L(0) = 0$ can be set without loss of generality. The quantity $A(0)$ is interpreted as discounted value at time $t = 0$ of the accumulated random value $L(T)$ of the liabilities at the terminal time T. For simplicity, we assume throughout that $F(t) = \{A(0)\}$. We suppose that a risk management action at time t is only possible if the undesirable event $V(t) > 0$ does not occur, that is equivalently if $G(t) > 0$.

Consider the variant of (1.1), called *risk adjusted profit and loss identity*, which reads

$$G(t) + \{V(t)_+ - R(t)\} = G(t)_+ - R(t). \tag{1.2}$$

The component $G_p(t) = R(t) - V(t)_+$ is interpreted as *actuarial protection model*, which is used to hedge the financial profit, and the component $D(t) = G(t)_+ - R(t) \ge 0$ is interpreted as *dividend*, which remains after the excess-of-loss reserve has been put aside. It is always assumed that the real-world *expected profit*, denoted by $E_G(t)$, is strictly positive, that is,

$$E_G(t) = E[G(t)|F(t)] > 0. \tag{1.3}$$

One says that the fluctuations of the financial profit are absorbed in the mean if one has

$$E\big[G_p(t)|F(t)\big] = 0. \tag{1.4}$$

We claim that this is the ideal goal that should be achieved through strategic financial management, that is in the long run. A statistical justification of the mean self-financing property is found in Hürlimann [16]. It is known that the optimal choice of the excess-of loss reserve is (e.g., Hürlimann [8, 9], see also Hürlimann [17], Proposition 3.2):

$$R^*(t) = \min\{B(t),\, G(t)_+\}, \tag{1.5}$$

where $B(t)$ is a deterministic amount. This so-called *stable excess-of-loss reserve* is optimal in the sense that it minimizes the fluctuations of the excess-of-loss reserve, such that $\mathrm{Var}[R(t)|F(t)] = \min$ among all possible excess-of-loss reserves, which satisfy the inequality constraints $0 \le R(t) \le G(t)_+$ and (1.4). The deterministic stable excess-of-loss amount $B(t)$ is solution of the equivalent implicit expected value equations

$$E[G(t)|F(t)] = E[(G(t) - B(t))_+|F(t)] \Leftrightarrow B(t) = E[(B(t) - G(t))_+|F(t)]. \tag{1.6}$$

Since $R^*(t) - V(t)_+ = B(t) - (B(t) - G(t))_+$, the identity (1.2) can be rewritten as

$$G^a(t) + V^a(t)_+ = G^a(t)_+, \tag{1.7}$$

where the quantities defined by

$$V^a(t) := B(t) - G(t), \quad G^a(t) := G(t) - B(t) = -V^a(t), \tag{1.8}$$

are viewed as *risk-adjusted loss*, respectively, *risk-adjusted profit*. With these definitions, the Equations (1.6) are equivalent with

$$E_G(t) = E[G(t)|F(t)] = E\left[G^{\alpha}(t)_+|F(t)\right] \Leftrightarrow B(t) = E\left[V^{\alpha}(t)_+|F(t)\right]. \quad (1.9)$$

With regard to applications the considered model of the stable excess-of-loss reserve leads to four different embedded option strategies as follows. The first strategy refers to a mean self-financing realization of the actuarial protection model. However, this strategy is not risk-free. The second option strategy envisages that this hedging instrument is available on the option market for finance and insurance risks. The third option strategy is based on a mean self-financing dividend strategy that is again not risk-free. The last option strategy consists of reproducing the financial profit through purchase of the dividend process in an option market for finance and insurance risks.

2. A Mean Self-Financing Hedging Strategy

At any time t, the stochastic process of the financial profit can be decomposed into two subprocesses $G(t) = G_d(t) + G_p(t)$ such that

$$G_d(t) = G^{\alpha}(t) + V^{\alpha}(t)_+ = G^{\alpha}(t)_+, \quad G_p(t) = B(t) - V^{\alpha}(t)_+. \quad (2.1)$$

The first component represents a guaranteed dividend process. The second one is identical with the introduced actuarial protection model for the choice of the stable excess-of-loss reserve (1.5). Table 1 is an accounting scheme for this option strategy.

With (1.9) one has the mean identity

$$E[G_d(t)|F(t)] = E[G(t)|F(t)], \quad (2.2)$$

which tells us that the expected guaranteed dividend equals the expected profit, and

$$E\left[G_p(t)|F(t)\right] = E\left[B(t) - V^{\alpha}(t)_+|F(t)\right] = 0, \quad (2.3)$$

which shows that the hedging process is mean self-financing. Together, these two properties fulfill the claimed ideal goal of strategic financial management. The profit chance and the risk of loss can be approximated in first order by the variance. One has the formula

$$\begin{aligned}\operatorname{Var}[G(t)|F(t)] &= \operatorname{Var}[G_d(t)|F(t)] + \operatorname{Var}[G_p(t)|F(t)] + 2\cdot\operatorname{Cov}[G_d(t),\, G_p(t)] \\ &= \operatorname{Var}[G_d(t)|F(t)] + \operatorname{Var}\left[G_p(t)|F(t)\right] + 2\cdot E_G(t)\cdot B(t), \quad (2.4)\end{aligned}$$

where the last equality follows from the calculation:

$$\begin{aligned}&\operatorname{Cov}\left[G_d(t),\, G_p(t)|F(t)\right] \\ &= E\left[G_d(t)_+\cdot\left(B(t)-V^a(t)_+\right)|F(t)\right] - E[G_d(t)_+|F(t)]\cdot\left(B(t)-E\left[V^a(t)_+|F(t)\right]\right) \\ &= B(t)\cdot E[G(t)_+|F(t)] = B(t)\cdot E_G(t).\end{aligned}$$

Since by assumption one has always $E[G(t)|F(t)] > 0$, the sum of the *profit chance* $\operatorname{Var}[G^a(t)_+|F(t)]$ (= variance of the guaranteed dividend) and the *hedging risk* $\operatorname{Var}[V^a(t)_+|F(t)]$ (= variance of the hedging payment) are strictly smaller than the *total risk* $\operatorname{Var}[G(t)|F(t)]$ (= variance of the financial profit & loss). The positive difference of amount

$$2\cdot E_G(t)\cdot B(t) \quad (2.5)$$

is due to the stochastic dependence between the dividend and hedging components. It equals twice the expected profit times the required stable excess-of-loss reserve. An effective implementation of this option strategy is always coupled with a variance reduction of the financial profit & loss, which is desirable in general (positive diversification effect).

In case the distribution of the profit is known, or much weaker, if the probabilities

$$F_G(t) := \Pr(G(t) \le B(t)|F(t)), \quad \overline{F}_G(t) := 1 - F_G(t), \quad (2.6)$$

are known, then one has the following bounds for the profit chance and the hedging risk (generalized inequality of Kremer after Hürlimann [10], Proposition 4.2):

$$\frac{F_G(t)}{\overline{F}_G(t)} \cdot B(t)^2 \leq \mathrm{Var}[G_d(t)|F(t)] \leq \mathrm{Var}[G(t)|F(t)] - 2 \cdot B(t) \cdot E_G(t) - \frac{\overline{F}_G(t)}{F_G(t)} \cdot E_G(t)^2, \tag{2.7}$$

$$\frac{\overline{F}_G(t)}{F_G(t)} \cdot E_G(t)^2 \leq \mathrm{Var}\left[G_p(t)|F(t)\right] \leq \mathrm{Var}[G(t)|F(t)] - 2 \cdot B(t) \cdot E_G(t) - \frac{F_G(t)}{\overline{F}_G(t)} \cdot B(t)^2. \tag{2.8}$$

In practice, it is realistic to assume that probabilities cannot be estimated with enough precision, which means incomplete information. However, if the expected value $E_G(t)$, the stable excess-of-loss amount $B(t)$ and the variance $V_G(t) := \mathrm{Var}[G(t)|F(t)]$ are known, then one obtains from the generalized inequality of Schmitter the following best bounds (see Hürlimann [12, 13]):

$$\begin{aligned} &\frac{1}{2}\left[V_G(t) - 2 \cdot B(t) \cdot E_G(t) - \sqrt{V_G(t)^4 - 4 \cdot B(t) \cdot E_G(t) \cdot V_G(t)^2}\right] \\ &\quad \leq \mathrm{Var}[G_d(t)|F(t)],\ \mathrm{Var}\left[G_p(t)|F(t)\right] \\ &\quad \leq \frac{1}{2}\left[V_G(t) - 2 \cdot B(t) \cdot E_G(t) + \sqrt{V_G(t)^4 - 4 \cdot B(t) \cdot E_G(t) \cdot V_G(t)^2}\right]. \end{aligned} \tag{2.9}$$

In this situation, the profit chance and the hedging risk lie in the same interval. A practical implementation of the accounting scheme for this first option strategy, including control of the profit chance and financial risks, depends upon the development of statistical methods to determine the quantities $E_G(t)$, $B(t)$, and $V_G(t)$. Distribution-free methods for the evaluation of $B(t)$, given $E_G(t)$ and $V_G(t)$, are developed in Hürlimann [15] (see also Hürlimann [14]).

Table 1. Accounting scheme for the first option strategy

	Guaranteed dividend process	**Hedging process**	**Profit & loss option strategy**
Income			
Assets	$A(t)$		$A(t)$
Stable excess-of-loss amount		$B(t)$	$B(t)$
Hedging payment	$V^a(t)_+$		$V^a(t)_+$
Outcome			
Liabialities	$L(t)$		$L(t)$
Stable excess-of-loss reserve	$B(t)$		$B(t)$
Hedging payment		$V^a(t)_+$	$V^a(t)_+$
Financial result			
Guaranteed dividend	$G^a(t)_+$		
Hedging profit & loss		$B(t) - V^a(t)_+$	
Profit & loss option strategy			$G(t) = A(t) - L(t)$

3. Purchase of the Hedging Strategy on the Financial Market

At the starting time $t = 0$, the risk manager buys the hedging instrument in a (yet to be installed) global option and reinsurance market. For simplicity, we assume that the required financial instrument is of European type, that is the exercise of the rights and obligations of the instrument occur solely at the terminal time T. The passage to a hedging instrument of American type is formally the same but requires advanced financial mathematics for a concrete implementation.

The hedging payment $V^a(T)_+ = (B(T) - G(T))_+$ corresponds to the payment of a put option on the financial profit with the stable excess-of-loss amount as exercise price. The algebraic manipulation $V^a(T)_+ =$

$(L(T) - [A(T) - B(T)])_+$ shows that this instrument can be interpreted as exchange option of the type introduced first by Margrabe [21]. It is an option to exchange the liabilities $L(t)$ against the risk-adjusted assets $A(T) - B(T)$. Let $B^*(T) = H[V^a(T)_+|F(T)]$ be the market premium of this hedging instrument with option payment $V^a(T)_+$, called *hedging premium*, where $H[\cdot]$ is market price principle. Examples of classical price principles include the variance and standard deviation principles, the distortion price principle, the Choquet price principle, the martingale principle, the CAPM principle from Borch [3], the Esscher principle from Bühlmann [5, 6], etc. More recently, the author has considered alternative option pricing methods through state-price deflators, which can also be applied in the present context (see Hürlimann [18] and their references). Since the hedging process is not risk-free (e.g., a lower bound for the variance risk is found in (2.8)), the price principle fulfills the minimal condition

$$B^*(T) = H[V^a(T)_+|F(T)] > E[V^a(T)_+|F(T)] = B(T). \tag{3.1}$$

The difference $RP(T) = B^*(T) - B(T)$ is called *hedging risk premium*. For the purpose of financial analysis, the financial profit is again decomposed in two components $G(T) = G_d^*(T) + G_p^*(T)$ such that

$$G_d^*(T) = G_d(T) - RP(T), \quad G_p^*(T) = G_p(T) + RP(T). \tag{3.2}$$

The hedging payment is allocated to the dividend process respectively its proceeds are credited to the (external) hedging process. The accounting scheme of this alternative option strategy is summarized in Table 2.

Table 2. Accounting scheme for the second option strategy

	Dividend process	**Hedging process (external)**	**Profit & loss option strategy**
Income			
Assets	$A(T)$		$A(T)$
Hedging premium		$B^*(T)$	$B^*(T)$
Hedging payment	$V^a(T)_+$		$V^a(T)_+$
Outcome			
Liabilities	$L(T)$		$L(T)$
Stable excess-of-loss amount	$B(T)$		$B(T)$
Hedging risk premium	$RP(T)$		$RP(T)$
Hedging payment		$V^a(T)_+$	$V^a(T)_+$
Financial result			
Dividend	$G^a(T)_+ - RP(T)$		
Profit & loss hedging		$B^*(T) - V^a(T)_+$	
Profit & loss option strategy			$G(T)$

In the mean one has again from (1.9):

$$E[G_d^*(t)_+ | F(t)] = E[G(t) | F(t)] - RP(T), \tag{3.3}$$

which means that the expected dividend is equal to the expected financial profit less the hedging premium, and

$$E[G_p^*(t)_+ | F(t)] = E[B(T) + RP(T) - V^a(T)_+ | F(T)] = RP(T) > 0, \tag{3.4}$$

that is, the external hedging process is closing with an expected financial result equal to the hedging risk premium. A market participant, which chooses the second option strategy, operates risk-free in the sense that the hedging risk is completely transferred to the global option market. However, the ideal goal of strategic financial management is only fulfilled in a reduced extent, that is up to the cost of hedging. In contrast to this, the statements about the profit chance and hedging risk are preserved in case the variance is used as a measure of risk.

4. Mean Self-Financing Dividend Strategy

The risk-adjusted profit and loss identity (1.7) can be rewritten as

$$B(t) + G^{a}(T)_{+} = G(t) + V^{a}(T)_{+}. \tag{4.1}$$

The financial profit can be achieved in the mean by a risk manager by investing the stable excess-of-loss amount $B(t)$ and hedging through the dividend process taking into account the restriction

$$B(t) \geq E[G(t)|F(t)], \tag{4.2}$$

as displayed in Table 3. Since no risky investment is dealt with, this third option strategy must be thought of as synthetic or virtual reproduction process of the financial profit.

With (1.9) one sees that the mean of the reproduction process and option strategy is equal to $B(t) - E[G(t)|F(t)]$, which is non-negative under the condition (4.2). The hedging process is trivially mean self-financing. Securing the financial profit and loss implies that the ideal goal of strategic financial management is fulfilled (similarly to the first option strategy). However, the expected profit of the reproduction process is risky. When based on the variance measure, the risk is identical with the hedging risk of the first option strategy. As described in Section 1, these risks can be estimated and controlled.

Remarks 4.1. (i) The optimal choice of the excess-of-loss reserve is the one with minimum cost. In the long term equilibrium, one has herewith from (4.2) that

$$B(t) = E[G(t)|F(t)], \tag{4.3}$$

which is interpreted as a financial economics *risk premium*. Taking into account (1.9), one obtains herewith

$$B(t) = E[G(t)|F(t)] = E[(G(t) - E[G(t)|F(t)])_+|F(t)]. \tag{4.4}$$

For example, if $G(t)$ given $F(t)$ is normally distributed with standard deviation σ, then one has

$$E[G(t)|F(t)] = \sqrt{\frac{\mathrm{Var}[G(t)|F(t)]}{2\pi}} \approx \sigma \cdot \sqrt{\frac{t}{2\pi}} \cdot A(0). \tag{4.5}$$

If $G(t)$ given $F(t)$ is log-normally distributed with volatility ν (Black-Scholes-Merton model), then one has

$$E[G(t)|F(t)] = \left[2 \cdot \Phi\left(\frac{1}{2}\nu\sqrt{t}\right) - 1\right]. A(0), \tag{4.6}$$

where $\Phi(x)$ is the standard normal distribution. In both cases, the expected profit is proportional to the volatility in the long term equilibrium.

In practice, a long term one-year volatility can be statistically estimated. For example, in the normally distributed case, one obtains for the yearly standard deviation $\sigma = 0.169285$ a yearly return rate $\frac{1}{\sqrt{2\pi}} \cdot \sigma \approx 0.067535.$ This correspond to the approximate performance of the Swiss Market Index (SMI) in the 25Y period from 30 June 1988 to 28 June 2013. Similar figures hold in the log-normally distributed case.

The above explanation justifies again the choice of (4.5) for a "mean-variance portfolio selection under portfolio insurance", an extension of the standard Markowitz portfolio selection, which has been presented in Hürlimann [11].

(ii) It is possible to modify the third option strategy by securing a guaranteed dividend of amount $D(t) = V^a(t)_+$ in the reproduction process (exchange the dividend payment with the hedging payment in the first option strategy). For this one must increase the invested amount by the expected dividend payment, such that the total cost of this modified option strategy reads

$$K[B(t)] := B(t) + E[(G(t) - B(t))_+ | F(t)]. \tag{4.7}$$

In Table 3, the expected dividend payment entry does not appear any more under the outcome entry of the reproduction process, whose financial result closes with a guaranteed dividend payment of amount $V^a(t)_+$. Under the condition (4.2), the mean self-financing property is preserved. To determine the minimum cost under the side condition (4.2), consider the derivative

$$\frac{dK[B(t)]}{dB(t)} = 1 + \frac{dE[(G(t) - B(t))_+ | F(t)]}{dB(t)} = \Pr(G(t) \le B(t) | F(t)) > 0. \tag{4.8}$$

Since $K[B(t)]$ is monotone increasing, the minimum is attained provided (4.3) holds.

(iii) The relationships (4.4) have two immediate and simple applications. Following the classical insurance approach, set $G(t) = P(t) - S(t)$, where $P(t)$ represents the deterministic accumulated insurance premiums, and $S(t)$ stands for the accumulated stochastic insurance claims. Then, one obtains from (4.4) the optimal premium formula:

$$P(t) = E[S(t) | F(t)] + E[(S(t) - E[S(t)])_+ | F(t)]. \tag{4.9}$$

The financial economics risk premium $B(t) = E[(S(t) - E[S(t)])_+ | F(t)]$, which is called *loading* in insurance terminology, is a net stop-loss premium with a deductible set at the expected insurance claims. Similarly, following the classical finance approach, set $G(t) = A(t) - FW(t)$, where $A(t)$ represents the accumulated value of the initial investment, and $FW(t)$ is the forward price of the investment (e.g., forward contract in Merton [22], pp. 347-349). Then, one obtains from (4.4) the optimal forward price formula:

$$FW(t) = E[A(t) | F(t)] - E[(A(t) - E[A(t)])_+ | F(t)]. \tag{4.10}$$

Table 3. Accounting scheme for the third option strategy

	Profit & loss reproduction process	**Hedging process**	**Profit & loss option strategy**
Income			
Stable excess-of-loss amount	$B(t)$		$B(t)$
Expected dividend proceeds		$E[G^a(t)_+ \| F(t)]$	$E[G^a(t)_+ \| F(t)]$
Dividend proceeds	$G^a(T)_+$		$G^a(t)_+$
Outcome			
Profit & loss	$G(t)$		$G(t)$
Dividend payment		$G^a(t)_+$	$G^a(t)_+$
Expected dividend payment	$E[G^a(t)_+ \| F(t)]$		$E[G^a(t)_+ \| F(t)]$
Financial result			
Profit & loss reproduction process	$V^a(t)_+ - E[G^a(t)_+ \| F(t)]$		
Profit & loss hedging process		$E[G^a(t)_+ \| F(t)] - G^a(t)_+$	
Profit & loss option strategy			$B(t) - G(t)$

5. Purchase of the Dividend Strategy on the Financial Market

Consider the modified version of the third option strategy, where now the guaranteed dividend $V^a(T)_+$ with exercise price $V^a(T)_+$ at the terminal time T (European option type) is purchased in a global option and reinsurance market. The terminal value of the *acquisition premium* for our fourth option strategy at time T reads

$$P(T) = B(T) + H\left[G^a(t)_+ | F(t)\right], \tag{5.1}$$

where $H[\cdot]$ is market price principle for the valuation of the certainty equivalent of $G^a(T)_+$. The quantity $H[G^a(T)_+|F(T)]$ is called *dividend premium*. By means of (4.1), one obtains taking expected values the alternate formula

$$P(T) = E[G(T)|F(T)] + E[V^a(T)_+|F(T)] + \left(H[G^a(T)_+|F(T)] - E[G^a(T)_+|F(T)]\right). \tag{5.2}$$

The operational implementation of this option strategy is summarized in Table 4 (generalized version of Table 1 in Hürlimann [10]).

From Table 4, one sees that this option strategy reproduces the value of the financial profit respectively loss plus the guaranteed dividend $V^a(T)_+$. The mean financial result equals

$$E\left[G(T) + V^a(T)_+|F(T)\right] = B(T) + E\left[G^a(T)_+|F(T)\right]. \tag{5.3}$$

In terms of the variance measure, the risk of this option strategy is given by

$$\mathrm{Var}\left[G(T) + V^a(T)_+|F(T)\right] = \mathrm{Var}\left[B(T) + G^a(T)_+|F(T)\right] = \mathrm{Var}\left[G^a(T)_+|F(T)\right]. \tag{5.4}$$

Similarly to the first option strategy, the risk can be easily estimated and controlled using the generalized inequalities of Kremer and Schmitter. In the special case that $B(T)$ is chosen as stable excess-of-loss amount (validity of Equation (1.9)), the risk is identical to the profit chance of the first option strategy.

Table 4. Accounting scheme for the fourth option strategy

Income	
Acquisition premium	$E[G(T)\|F(T)] + E[V^a(T)_+\|F(T)]$ $+ \left(H\left[G^a(T)_+\|F(T)\right] - E\left[G^a(T)_+\|F(T)\right]\right)$
Dividend proceeds	$G^a(T)_+$
Outcome	
Dividend premium	$H[G^a(T)_+\|F(T)]$
Financial result	$G(T) + V^a(T)_+ + E\left[G(T) + V^a(T) - G^a(T)_+\|F(T)\right]$ $-\left(G(T) + V^a(T) - G^a(T)_+\right)$ $= G(T) + V^a(T)_+ + E[B(T)\|F(T)] - B(T) = G(T) + V^a(T)_+$

One can ask for an optimal choice of the parameters, that is an optimal price for the purchase of this option strategy. Critical are the choices of $B(T)$ and the price principle $H[\cdot]$.

We begin with a possible choice of $H[\cdot]$. The reproduction of a random profit $G(T)$ depends upon the future profit $G_M(T)$ that can be achieved on the financial market. A popular model for this is the CAPM (= capital asset pricing model) of Sharpe [23] and Lintner [19] (see also the modified versions by Black [2] and Hürlimann [9]):

$$H[G(T)|F(T)] = E[G(T)|F(T)]$$

$$+ \frac{\mathrm{Cov}[G(T), G_M(T)|F(T)]}{\mathrm{Var}[G_M(T)|F(T)]} \cdot (H[G_M(T)|F(T)] - E[G_M(T)|F(T)]). \quad (5.5)$$

In particular, one sees that if, at some appropriate date $T_0 < T$, the future profit $G_M(T)$ can be purchased at the expected (or fair) price $H[G_M(T_0)|F(T_0)] = E[G_M(T_0)|F(T_0)]$, then the future profit $G(T)$ of any other risky investment can also be acquired at the expected price $H[G(T_0)|F(T_0)] = E[G(T_0)|F(T_0)]$. To illustrate what is meant, consider the example of the Swiss Market Index on 28 June 2013. This index could be purchased for the fair price of 7685 points at this date (under the assumption of a future yearly return rate of 6.7535% that has been observed since 30 June 1988). Now, in our option strategy one should purchase $G^a(T)_+$ and not $G(T)$. According to the CAPM of Borch [3] (see also Hürlimann [10], p.180), one has

$$H\left[G^a(T)_+ | F(T)\right] = E\left[G^a(T)_+ | F(T)\right] + \frac{\operatorname{Cov}\left[G^a(T)_+,\, G(T) | F(T)\right]}{\operatorname{Var}[G(T) | F(T)]} \cdot (H[G(T) | F(T)] - E[G(T) | F(T)]), \tag{5.6}$$

where (5.5) has been used.

Next, let us find a possibly optimal choice of $B(T)$. Suppose that in first approximation the variance is a good risk measure. Then, the variance principle can be used as a good comparative price principle to the displayed market principles. Under the assumption that the above quantities can be determined, one finds variance loadings θ_R and θ_M for the risk manager and the financial market, such that

$$H[G(T) | F(T)] = E[G(T) | F(T)] + \theta_R \cdot \operatorname{Var}[G(T) | F(T)], \tag{5.7}$$

$$H[G_M(T) | F(T)] = E[G_M(T) | F(T)] + \theta_M \cdot \operatorname{Var}[G_M(T) | F(T)]. \tag{5.8}$$

In case the ideal goal of strategic financial management should be achieved, then the guaranteed dividend $V^a(T)_+$ in the financial result position of Table 4 should disappear. For this, the required comparative price must be equal to

$$P\left[V^a(T)_+;\, T\right] = \left[B(T) - V^a(T)_+\right] + E\left[G^a(T)_+ | F(T)\right] + \theta_M \cdot \operatorname{Var}\left[G^a(T)_+ | F(T)\right]. \tag{5.9}$$

But, the risk manager applies a loading $\theta_R \neq \theta_M$ and obtains for the certainty equivalent of (5.9) the amount

$$\begin{aligned} P(T) &= E\left[P\left[V^a(T)_+;\, T\right]\right] + \theta_R \cdot \mathrm{Var}\left[P\left[V^a(T)_+;\, T\right]\right] \\ &= E_G(T) + \theta_R \cdot R_G\left[V^a(T)_+,\, G^a(T)_+\right] - (\theta_R - \theta_M)\cdot \mathrm{Var}\left[G^a(T)_+ | F(T)\right] \\ &= E_G(T) + \theta_M \cdot R_G\left[V^a(T)_+,\, G^a(T)_+\right] + (\theta_R - \theta_M)\cdot \mathrm{Var}\left[V^a(T)_+ | F(T)\right], \end{aligned} \tag{5.10}$$

where the quantity

$$\begin{aligned} R_G\left[V^a(T)_+,\, G^a(T)_+ | F(T)\right] &= \mathrm{Var}\left[V^a(T)_+ | F(T)\right] + \mathrm{Var}\left[G^a(T)_+ | F(T)\right] \\ &= \mathrm{Var}[G(T) | F(T)] - 2\cdot \mathrm{Cov}\left[B(T) - V^a(T)_+,\, G^a(T)_+ | F(T)\right] \\ &= \mathrm{Var}[G(T) | F(T)] - 2\cdot E\left[V^a(T)_+ | F(T)\right]\cdot E\left[G^a(T)_+ | F(T)\right], \end{aligned} \tag{5.11}$$

plays the role of a risk measure (with respect to the variance criterion) for the option strategy. Intuitively, the risk aversion of a risk manager is bigger than the risk aversion of the market, hence $\theta_R \geq \theta_M$. In the extreme case $\theta_R = \theta_M$, one sees that the minimum required amount in (5.10) is attained when (5.11) is minimum, that is when $E\left[V^a(T)_+ | F(T)\right] \cdot E\left[G^a(T)_+ | F(T)\right]$ is maximum. To achieve the ideal mean self-financing goal of strategic financial management, the condition (4.2) must be fulfilled, that is $B(T) \geq E_G(T)$. One must solve the following optimization problem:

$$E[(B(T) - G(T))_+ | F(T)] \cdot E[(G(T) - B(T))_+ | F(T)] = \max \tag{5.12}$$

under the side condition

$$B(T) \geq E_G(T) = E[G(T) | F(T)]. \tag{5.13}$$

The next result provides a partial solution.

Theorem 5.1 (Local minimum hedging premium)**.** *Let* $G(T)$ *be a random profit with nonnegative conditional expected value* $E_G(T) = E[G(T)|F(T)] > 0$ *and distribution function* $F(x) = \Pr(G(T) \le x|F(T)) = \int_{-\infty}^{x} f(t)dt$, *and set* $S_G = \{x : 0 < F(x) < 1\}$. *Then, the optimization problem* (5.12), (5.13) *has a local minimum at* $B_0(T) \in S_G$ *if, and only if, the following conditions are fulfilled*:

$$E[G(T) - B_0(T)|G(T) > B_0(T),\ F(T)] = E[B_0(T) - G(T)|G(T) \le B_0(T),\ F(T)], \tag{5.14}$$

$$2 \cdot F[B_0(T)] \cdot \overline{F}[B_0(T)]$$
$$> f[B_0(T)] \cdot \left(E\left[(G(T) - B_0(T))_+|F(T)\right] + E\left[(B_0(T) - G(T))_+|F(T)\right]\right). \tag{5.15}$$

Proof. The details are found in Hürlimann [16], Theorem 3. □

A necessary condition for the existence of a local minimum is the existence of a stationary point of the function (5.12). In certain cases, this condition is always fulfilled.

Theorem 5.2. *Let* $G(T)$ *be a random profit that fulfills the assumptions of Theorem* 5.1, *and assume a finite variance of the profit* $\mathrm{Var}[G(T)|F(T)] < \infty$. *If* $F[E_G(T)] \ge \frac{1}{2}$, *then there exists at least one stationary point* $B(T) \in [E_G(T), \infty) \cap S_G$ *such that*

$$\frac{d}{dB(T)}\left(E\left[(B(T) - G(T))_+|F(T)\right] \cdot E\left[(G(T) - B(T))_+|F(T)\right]\right) = 0. \tag{5.16}$$

Proof. This is shown in Hürlimann [16], Theorem 4. □

Let us conclude with a striking example.

Example 5.1. In case the random profit $G(T)$ given $F(T)$ is normally distributed, then $B(T) = E[G(T)|F(T)]$ is a local maximum for the risk measure $R_G\left[V^a(T)_+, G^a(T)_+|F(T)\right]$ under the condition

$$\sqrt{\mathrm{Var}[G(T)|F(T)]} < \frac{\pi}{2} = 1.5708. \tag{5.17}$$

It is remarkable that this choice imples minimum investment costs in the fourth option strategy, and at the same time it is the optimal excess-of-loss amount in the long term equilibrium (see Equation (4.3)). This simple result (obtained under simplifying assumptions) is at the interface between probability and statistics, actuarial science and finance.

References

[1] A. Albrecht, Auf dem Weg zu einem holistischen Risikomanagement? Versicherungswirtschaft 19 (1999), 1404-1409.

[2] F. Black, Capital market equilibrium with restricted borrowing, Journal of Business (1972), 445-455.

[3] K. Borch, Additive insurance premiums: A note, The Journal of Finance 11295-98. Reprinted in Borch (1990) (1982), 192-197.

[4] K. H. Borch, Economics of Insurance, Advanced Textbooks in Economics 29, North-Holland, 1990.

[5] H. Bühlmann, An economic premium principle, ASTIN Bulletin 11 (1980), 52-60.

[6] H. Bühlmann, The general economic premium principle, ASTIN Bulletin 14 (1984), 13-21.

[7] H. Bühlmann, Life Insurance with Stochastic Interest Rates, In: G. Ottaviani (Ed.), Financial Risk in Insurance, Springer-Verlag, 1995.

[8] W. Hürlimann, Stochastic tariffing in life insurance, Proc. Int. Colloquium Life, disability and pensions: Tomorrow's challenge, Paris, 3, 203-12. Tarification stochastique en assurance-vie, French translation, same 3 (1991a), 81-90.

[9] W. Hürlimann, Absicherung des Anlagerisikos, Diskontierung der Passiven und Portfoliotheorie, Bulletin Swiss Assoc. Actuaries (1991b), 217-250.

[10] W. Hürlimann, Splitting risk and premium calculation, Bulletin Swiss Assoc. Actuaries (1994), 167-97.

[11] W. Hürlimann, Mean-variance portfolio selection under portfolio insurance, Proceedings of the 6-th International AFIR-Colloquium, Verlag Versicherungswirtschaft, Karlsruhe 1 (1996), 347-374.

[12] W. Hürlimann, An elementary unified approach to loss variance bounds, Bulletin Swiss Assoc. Actuaries (1997a), 73-88.

[13] W. Hürlimann, Fonctions extrémales et gain financier, Elemente der Mathematik 52 (1997b), 152-168.

[14] W. Hürlimann, Bounds for expected positive differences by means of Hoeffding-Fréchet extremal distributions, Proc. 28th Int. ASTIN Colloquium, Cairns, Australia, (1997c).

[15] W. Hürlimann, Distribution-free excess-of-loss reserves for some actuarial protection models, Trans. 26th Int. Congress of Actuaries, Birmingham 4 (1998), 291-317.

[16] W. Hürlimann, Non-optimality of a linear combination of proportional and non-proportional reinsurance, Insurance Math. Econom. 24(3) (1999), 219-227.

[17] W. Hürlimann, On the accumulated aggregate surplus of a life portfolio, Insurance Math. Econom. 30(1) (2002), 27-35.

[18] W. Hürlimann, Normal variance-mean mixtures (III) Option pricing through state-price deflators, J. Math. Sci.: Adv. and Appl. 25(1) (2014), 13-41.

[19] J. Lintner, The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets, Review of Economics and Statistics 47 (1965), 13-27.

[20] D. L. Luskin, (Editor), Portfolio Insurance: A Guide to Dynamic Hedging, J. Wiley, New York, 1988.

[21] W. Margrabe, The value of an option to exchange one asset for another, Journal of Finance 33 (1978), 177-186.

[22] R. C. Merton, Continuous-time Finance, Basil Blackwell, 1990.

[23] W. F. Sharpe, A simplified model for portfolio analysis, Management Science 9 (1963), 277-293.

■