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UNIFORM APPROXIMATION ON RIEMANNIAN MANIFOLDS BY SOLUTIONS OF THE EQUATION $\Delta u = f$

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Abstract

This paper presents several results of Runge type on uniform approximation by solutions of the equation $\Delta u = f$ on C^∞ Riemannian manifolds.

1. Introduction

Let Ω be an orientable Riemannian manifold and let Δ be the Laplacian of Ω. In this paper, we prove several approximation theorems for solutions of the equation $\Delta u = f$, where $f \in C^\infty(\Omega)$. Since f is C^∞, any solution u of $\Delta u = f$ is necessarily C^∞ by elliptic regularity theory.

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We say that a continuous function $u : \Omega \to [-\infty, \infty]$ is **Newtonian** provided that there is a discrete set $A \subset \Omega$ with the following properties:

(i) $u|_{\Omega \setminus A}$ is harmonic, and

(ii) for each point $a \in A$, there is a regular subregion R satisfying $R \cap A = \{a\}$ and a constant c, such that the function $u - c\, G_R(\bullet, a)$ can be defined at a so as to be harmonic on R, where $G_R(\bullet, a)$ is the Green function of R.

We say that a continuous function $u : \Omega \to [-\infty, \infty]$ is a **singular solution** of the equation $\Delta u = f$ provided $u = v + w$, where v is Newtonian on Ω and where w is a regular (C^∞) solution of $\Delta w = f$ on Ω. Thus $\Delta u = f$ on Ω except possibly for isolated singularities.

Lemma 1. *Let Ω be an orientable compact C^∞ Riemannian manifold and let $f \in C^\infty(\Omega)$. There exists $u \in C^\infty(\Omega)$ such that $\Delta u = f$ on Ω if and only if $\int f d\lambda = 0$. The solution u is unique up to a constant.*

Lemma 1 is proved in [1, Theorem 4.7, page 104].

Theorem 1. *Let Ω be an orientable compact C^∞ Riemannian manifold and let $f \in C^\infty(\Omega)$ such that $\int f d\lambda = 0$. Let F be a closed subset of Ω, and let u be a solution of $\Delta u = f$ on an open set containing F. If $\varepsilon > 0$, then there exists a singular solution v of $\Delta v = f$ on Ω such that* $\sup_F |u - v| < \varepsilon$.

Proof. We may assume that F is not all of Ω, and that u is a solution of $\Delta u = f$ on an open set $N \supset F$, which is not dense in Ω. By Lemma 1, there exists $u_1 \in C^\infty(\Omega)$ such that $\Delta u_1 = f$ on Ω. Since $\Delta(u - u_1) = \Delta u - \Delta u_1 = f - f = 0$ on N we see that $u - u_1$ is harmonic on N. By [2, Theorem 5.1], there exists a Newtonian function h on Ω such that $\sup_F |u - u_1 - h| < \varepsilon$. We set $v = h + u_1$. Thus v is a singular solution of the equation $\Delta v = f$ on Ω which has the required property.

Theorem 2. *Let* Ω *be an orientable compact* C^{∞} *Riemannian manifold and let* $f \in C^{\infty}(\Omega)$ *such that* $\int f d\lambda = 0$. *Let F be a closed subset of* Ω, *and let* u *be a singular solution of* $\Delta u = f$ *on an open set containing F. If* $\varepsilon > 0$, *then there exists a singular solution* v *of* $\Delta v = f$ *on* Ω *such that* $\sup_F |u - v| < \varepsilon$.

Proof. We may assume that F is not all of Ω, and that u is a singular solution of $\Delta u = f$ on an open set $N \supset F$, which is not dense in Ω. Then we may write $u = v + w$, where v is Newtonian on N and where w is a regular C^{∞} solution of $\Delta w = f$ on N. Therefore, there exists a Newtonian function v_1 on Ω such that $v - v_1$ is harmonic on N. By Lemma 1, there exists $w_1 \in C^{\infty}(\Omega)$ such that $\Delta w_1 = f$ on Ω. Since $\Delta(w - w_1) = f - f = 0$ on N we see that $w - w_1$ is harmonic on N. By [2, Theorem 5.1], there exists a Newtonian function h_1 on Ω such that $\sup_F |v - v_1 + w - w_1 - h_1| < \varepsilon$. We set $h = h_1 + v_1 + w_1$. Since the sum of two Newtonian functions is Newtonian we see that h is a singular solution of $\Delta h = f$ on Ω such that $\sup_F |u - h| < \varepsilon$. This completes the proof of Theorem 2.

Lemma 2. *Let* Ω *be an orientable noncompact* C^{∞} *Riemannian manifold and let* $f \in C^{\infty}(\Omega)$. *There exists* $u \in C^{\infty}(\Omega)$ *such that* $\Delta u = f$ *on* Ω.

Lemma 2 is proved in [3, page 268].

Theorem 3. *Let* Ω *be an orientable noncompact* C^{∞} *Riemannian manifold and let* $f \in C^{\infty}(\Omega)$. *Let F be a closed subset of* Ω, *and let u be a solution of* $\Delta u = f$ *on an open set containing F. If* $\varepsilon > 0$, *then there exists a singular solution* v *of* $\Delta v = f$ *on* Ω *such that* $\sup_F |u - v| < \varepsilon$.

Proof. Let u be a solution of $\Delta u = f$ on an open set $N \supset F$. By Lemma 2, there exists $u_1 \in C^\infty(\Omega)$ such that $\Delta u_1 = f$ on Ω. Since $\Delta(u - u_1) = \Delta u - \Delta u_1 = f - f = 0$ on N we see that $u - u_1$ is harmonic on N. By [2, Theorem 5.1], there exists a Newtonian function h on Ω such that $\sup_F |u - u_1 - h| < \varepsilon$. We set $v = h + u_1$. Thus v is a singular solution of $\Delta v = f$ on Ω which has the required property.

Theorem 4. *Let Ω be an orientable noncompact C^∞ Riemannian manifold and let $f \in C^\infty(\Omega)$. Let F be a closed subset of Ω, and let u be a singular solution of $\Delta u = f$ on an open set containing F. If $\varepsilon > 0$, then there exists a singular solution h of $\Delta h = f$ on Ω such that $\sup_F |u - h| < \varepsilon$.*

Proof. Let u be a singular solution of $\Delta u = f$ on an open set $N \supset F$. We may write $u = v + w$, where v is Newtonian on N and where w is a regular C^∞ solution of $\Delta w = f$ on N. By [2, Theorem 9.4], there exists a Newtonian function v_1 on Ω such that $v - v_1$ is harmonic on N. By Lemma 2, there exists $w_1 \in C^\infty(\Omega)$ such that $\Delta w_1 = f$ on Ω. Since $\Delta(w - w_1) = f - f = 0$ on N we see that $w - w_1$ is harmonic on N. By [2, Theorem 5.1], there exists a Newtonian function h_1 on Ω such that $\sup_F |v - v_1 + w - w_1 - h_1| < \varepsilon$. We set $h = h_1 + v_1 + w_1$. Since the sum of two Newtonian functions is Newtonian we see that h is a singular solution of $\Delta h = f$ on Ω such that $\sup_F |u - h| < \varepsilon$. This completes the proof of Theorem 4.

Theorem 5. *Let Ω be an orientable noncompact C^∞ Riemannian manifold and let $f \in C^\infty(\Omega)$. Let F be a closed subset of Ω such that $\Omega^* \setminus F$ is connected and locally connected. If u is a solution of $\Delta u = f$ on an open set containing F and if ε is a positive constant, then there exists $v \in C^\infty(\Omega)$ solution of $\Delta v = f$ on Ω such that $\sup_F |u - v| < \varepsilon$.*

Proof. Let u be a solution of $\Delta u = f$ on an open set $N \supset F$. By Lemma 2, there exists $u_1 \in C^\infty(\Omega)$ such that $\Delta u_1 = f$ on Ω. Since $\Delta(u - u_1) = f - f = 0$ on N we see that $u - u_1$ is harmonic on N. By [2, Theorem 9.3], there exists a harmonic function h on Ω such that $\sup_F |u - u_1 - h| < \varepsilon$. We set $v = h + u_1$. Then $v \in C^\infty(\Omega)$ and $\Delta v = \Delta h + \Delta u_1 = 0 + f = f$ on Ω. In addition $\sup_F |u - v| < \varepsilon$. This completes the proof of Theorem 5.

The definition of singular solution of $\Delta u = f$ should be compared with our definition of subharmonic singular function presented in [4] and [5].

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