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ORDINAL QUASI POINT-SYMMETRY AND DECOMPOSITION OF POINT-SYMMETRY FOR CROSS-CLASSIFICATIONS

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Abstract

For cross-classifications with ordered categories, the present paper proposes the ordinal quasi point-symmetry model that indicates the structure of asymmetry for cell probabilities with respect to the center point or center cell in the table. It also gives the orthogonal decomposition of the point-symmetry model into the proposed model and the marginal mean point-symmetry model. An example is given.

1. Introduction

Consider an $R \times C$ contingency table with ordered categories. Let X and Y denote the row and column variables, respectively, and let p_{ij} denote the probability that an observation will fall in the cell in row i and column j $(i = 1, \ldots, R;\ j = 1, \ldots, C)$. When $R = C$, Wall and Lienert [15] considered the point-symmetry (PS) model defined by

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$$p_{ij} = p_{i^* j^*} \quad (i = 1, \ldots, R;\ j = 1, \ldots, R),$$

where $i^* = R + 1 - i$. For the $R \times C$ tables, Tomizawa [12] considered the PS model defined by

$$p_{ij} = p_{i^* j^{**}} \quad (i = 1, \ldots, R;\ j = 1, \ldots, C),$$

where $i^* = R + 1 - i$ and $j^{**} = C + 1 - j$. This indicates the structure of point-symmetry for cell probabilities with respect to the center point or center cell of the table.

Tomizawa [12] considered the quasi point-symmetry (QPS) and marginal point-symmetry (MPS) models and gave a theorem that the PS model holds if and only if both QPS and MPS models hold, although the detail is omitted. Also, Tahata and Tomizawa [11] showed that the test statistic for goodness-of-fit of the PS model is asymptotically equivalent to the sum of test statistics for the QPS and MPS models. It certifies that the incompatible situation that both QPS and MPS models are accepted with high probability for test of significance level α but the PS model is rejected with high probability would not arise. Thus, it is important to show the relationship between test statistics for the decomposition of model. Such decomposition of model as described above may be called the asymptotic *orthogonal decomposition*. For the orthogonal decompositions of models, see, e.g., Aitchison [4], Darroch and Silvey [6], Read [10], Lang and Agresti [9], and Tomizawa and Tahata [14]. Moreover, Tomizawa [13] showed the other orthogonal decomposition of the PS model. The purpose of the present paper is to show new decomposition of the PS model with the asymptotic orthogonality of test statistic.

Section 2 proposes a model which indicates the structure of asymmetry for cell probabilities with respect to the center point or center cell in the $R \times C$ table. Section 3 gives the decompositions of the PS model that is different from Tomizawa [12, 13]. Section 4 shows the orthogonality of test statistic for the model. Also an example is given.

2. Ordinal Quasi Point-Symmetry Model

The QPS model is defined by

$$p_{ij} = \mu\alpha_i\beta_j\psi_{ij} \quad (i = 1, \dots, R;\ j = 1, \dots, C),$$

where $\psi_{ij} = \psi_{i^* j^{**}}$. This model indicates the structure of point-symmetry for odds ratios with respect to the center point or center cell in the table although the detail is omitted. Note that the QPS model with $\{\alpha_i = \alpha_{i^*}\}$ and $\{\beta_j = \beta_{j^{**}}\}$ is identical with the PS model.

Consider a new model defined by

$$p_{ij} = \mu\alpha^i\beta^j\psi_{ij} \quad (i = 1, \dots, R;\ j = 1, \dots, C),$$

where $\psi_{ij} = \psi_{i^* j^{**}}$. We shall refer to this model as the ordinal quasi point-symmetry (OQPS) model. A special case of OQPS model obtained by putting $\alpha = \beta = 1$ is the PS model. Also, the OQPS model is a special case of the QPS model with $\{\alpha_i = \alpha^i\}$ and $\{\beta_j = \beta^j\}$.

The OQPS model is the analogy to the linear diagonals-parameter symmetry model (Agresti [1]) and the ordinal quasi-symmetry model (Agresti [3], p. 236). Also, we note that the OQPS model should be applied to the ordinal categorical data because the model is not invariant under arbitrary permutations of row and column classifications except the reverse order.

The OQPS model may be expressed as

$$\frac{p_{ij}}{p_{i^* j^{**}}} = \theta_1^{\frac{R+1}{2}-i}\theta_2^{\frac{C+1}{2}-j} \quad (i = 1, \dots, R;\ j = 1, \dots, C).$$

Thus, the OQPS model indicates the structure of asymmetry for cell probabilities with respect to the center point or center cell in the $R \times C$ table. When the column value *j* is fixed, the log-odds, $\log(p_{ij}/p_{i^* j^{**}})$, is a linear function of row value i with slope $-\log \theta_1$. Also, when the row value i is fixed, the log-odds is a linear function of column value j with slope $-\log \theta_2$.

Under the OQPS model, we see

$$\left(\frac{p_{ij}}{p_{i+k,\,j}}\right) \Big/ \left(\frac{p_{i^* j^{**}}}{p_{(i+k)^*,\,j^{**}}}\right) = \theta_1^k \quad (1 \le i + k \le R;\ 1 \le j \le C).$$

This indicates that the odds that an observation will fall in row i instead of in row $i + k$ with column j is θ_1^k times higher than the odds that the observation falls in point-symmetric row i^* instead of in row $(i + k)^*$ with point-symmetric column j^{**}. The analogous interpretation follows for θ_2 by interchanging the role of rows and columns.

Next, consider a model defined by

$$p_{ij} = \mu\alpha^i \psi_{ij} \quad (i = 1, \dots, R;\ j = 1, \dots, C),$$

where $\psi_{ij} = \psi_{i^* j^{**}}$. We shall refer to this model as the row ordinal quasi point-symmetry (ROQPS) model. This is a special case of the OQPS model with $\beta = 1$. The ROQPS model indicates that (1) the log-odds, $\log(p_{ij} / p_{i^* j^{**}})$, is constant for any j, and (2) it is a linear function of row i. The ROQPS model replaced $\{\alpha^i\}$ by $\{\alpha_i\}$ is the row conditional point-symmetry model proposed by Tomizawa [13]. The column ordinal quasi point-symmetry (COQPS) model can be defined analogously.

3. Decompositions of Point-Symmetry

Consider a model defined by

$$E(X) = E(X^*) \quad \text{and} \quad E(Y) = E(Y^{**}).$$

Since $E(X^*) = R + 1 - E(X)$ and $E(Y^{**}) = C + 1 - E(Y)$, this model is expressed as

$$E(X) = \frac{R+1}{2} \quad \text{and} \quad E(Y) = \frac{C+1}{2}.$$

These indicate that $E(X)$ and $E(Y)$ equal the mid-value of row and column categories, respectively. Also, $E(X) = E(X^*)$ implies that both means are the same when the order of categories (as $1, 2, \ldots, R$) is rearranged in reverse order (as $R, R-1, \ldots, 1$). We shall refer to this model as the marginal mean point-symmetry (MMPS) model.

We obtain the following theorem:

Theorem 1. *The PS model holds if and only if both OQPS and MMPS models hold.*

Proof. If the PS model holds, then the OQPS and MMPS models obviously hold. Assuming that both OQPS and MMPS models hold, then we shall show the PS model holds. Let $\{q_{ij}\}$ denote the cell probabilities which satisfy both OQPS and MMPS models. Since the OQPS model holds, we see

$$\log\left(\frac{q_{ij}}{\pi_{ij}}\right) = \log c\mu + i \log \alpha + j \log \beta,$$

where $\pi_{ij} = \psi_{ij} / c$ with $c = \sum\sum \psi_{ij}$. Note that $\sum\sum \pi_{ij} = 1$, and $\pi_{ij} = \pi_{i^* j^{**}}$ because $\psi_{ij} = \psi_{i^* j^{**}}$. Let

$$\mu_X(\{q_{ij}\}) = \sum_{s=1}^{R} \sum_{t=1}^{C} s q_{st} \quad \text{and} \quad \mu_Y(\{q_{ij}\}) = \sum_{s=1}^{R} \sum_{t=1}^{C} t q_{st}.$$

The MMPS model is expressed as

$$\mu_X(\{q_{ij}\}) = \frac{R+1}{2} \quad \text{and} \quad \mu_Y(\{q_{ij}\}) = \frac{C+1}{2}.$$

Consider the arbitrary cell probabilities $\{p_{ij}\}$ satisfying

$$\mu_X(\{p_{ij}\}) = \frac{R+1}{2} \quad \text{and} \quad \mu_Y(\{p_{ij}\}) = \frac{C+1}{2}.$$

Let $K(\{a_{ij}\}; \{b_{ij}\})$ be the Kullback-Leibler information between $\{a_{ij}\}$ and $\{b_{ij}\}$, where

$$K(\{a_{ij}\}; \{b_{ij}\}) = \sum_{i=1}^{R} \sum_{j=1}^{C} a_{ij} \log\left(\frac{a_{ij}}{b_{ij}}\right).$$

From above equations, we see

$$\sum_{i=1}^{R} \sum_{j=1}^{C} (p_{ij} - q_{ij}) \log\left(\frac{q_{ij}}{\pi_{ij}}\right) = 0.$$

Thus, we can obtain $K(\{p_{ij}\}; \{\pi_{ij}\}) = K(\{q_{ij}\}; \{\pi_{ij}\}) + K(\{p_{ij}\}; \{q_{ij}\})$. Since $\{\pi_{ij}\}$ is fixed, we see

$$\min_{\{p_{ij}\}} K(\{p_{ij}\}; \{\pi_{ij}\}) = K(\{q_{ij}\}; \{\pi_{ij}\}).$$

Then $\{q_{ij}\}$ uniquely minimize $K(\{p_{ij}\}; \{\pi_{ij}\})$ as described in Darroch and Ratcliff [5].

Let $r_{ij} = q_{i^* j^{**}}$ for $i = 1, \ldots, R$ and $j = 1, \ldots, C$. Then noting that $\{\pi_{ij} = \pi_{i^* j^{**}}\}$, we similarly obtain

$$\min_{\{p_{ij}\}} K(\{p_{ij}\}; \{\pi_{ij}\}) = K(\{r_{ij}\}; \{\pi_{ij}\}).$$

Then $\{r_{ij}\}$ uniquely minimize $K(\{p_{ij}\}; \{\pi_{ij}\})$. Therefore, we see $q_{ij} = r_{ij} = q_{i^* j^{**}}$ for $i = 1, \ldots, R$ and $j = 1, \ldots, C$. Namely, the probabilities $\{q_{ij}\}$ satisfy point-symmetry. The proof is completed.

Consider a model defined by

$$E(X) = E(X^*).$$

We shall refer to this model as the row marginal mean point-symmetry (RMMPS) model. Also, consider a model defined by

$$E(Y) = E(Y^{**}).$$

We shall refer to this model as the column marginal mean point-symmetry (CMMPS) model. We can obtain the following corollary from Theorem 1.

Corollary 1. *The PS model holds if and only if both ROQPS (COQPS) and RMMPS (CMMPS) models hold.*

4. Orthogonality of Test Statistic

Let n_{ij} denote the observed frequency in the (i, j)-th cell of the $R \times C$ table $(i = 1, \ldots, R;\ j = 1, \ldots, C)$ with $n = \sum\sum n_{ij}$. Assume that the observed frequencies have a multinomial distribution. Let $G^2(M)$ denote the likelihood ratio statistic for testing goodness-of-fit of model *M*. The number of degrees of freedom for testing goodness-of-fit of the OQPS model is $(RC - 5)/2$ (when each R and C is odd) and $(RC - 4)/2$ (otherwise), which is two less than that for the PS model. The MMPS model has two degrees of freedom. Thus, the number of degrees of freedom for the PS model is equal to the sum of those for the OQPS model and the MMPS model. The number of degrees of freedom for testing goodness-of-fit of the ROQPS (COQPS) model is one less than that for the PS model. Also, the RMMPS (CMMPS) model has one degree of freedom.

Since the OQPS model has a multiplicative form, the maximum likelihood estimates of expected frequencies under the model can be obtained using the iterative procedure, for example, the general iterative procedure for log-linear models of Darroch and Ratcliff [5]. Those under the MMPS model can be obtained by using the Newton-Raphson method in the log-likelihood equation. Also, we note that Kateri and Papaioannou [8] connected the point symmetric models for two-way tables to the standard log-linear models for three-way tables and provided ways to fit them more easily.

We obtain the following theorem:

Theorem 2. *The test statistic* $G^2(PS)$ *is asymptotically equivalent to the sum of* $G^2(OQPS)$ *and* $G^2(MMPS)$.

Proof. We shall show the proof when each of R and C is even. The OQPS model may be expressed as

$$\log p_{ij} = i\lambda_1 + j\lambda_2 + \lambda_{ij} \quad (i = 1, \dots, R;\ j = 1, \dots, C),$$

where $\lambda_{ij} = \lambda_{i^* j^{**}}$. Let

$$\boldsymbol{p} = (p_{11}, \dots, p_{1C}, p_{21}, \dots, p_{2C}, \dots, p_{R1}, \dots, p_{RC})^t,$$

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \boldsymbol{\lambda}_3)^t,$$

where "t" denotes the transpose, and where

$$\boldsymbol{\lambda}_3 = (\lambda_{11}, \dots, \lambda_{1C}, \lambda_{21}, \dots, \lambda_{2C}, \dots, \lambda_{\frac{R}{2},1}, \dots, \lambda_{\frac{R}{2},C}),$$

is a $1 \times RC/2$ vector. Then the OQPS model is expressed as

$$\log \boldsymbol{p} = X\boldsymbol{\lambda} = (\boldsymbol{x}_1, \boldsymbol{x}_2, X_3)\boldsymbol{\lambda},$$

where X is an $RC \times K$ matrix with $K = (RC + 4)/2$ and

$$\boldsymbol{x}_1 = \boldsymbol{j}_R \otimes \boldsymbol{1}_C;\ \text{an}\ RC \times 1\ \text{vector},$$

$$\boldsymbol{x}_2 = \boldsymbol{1}_R \otimes \boldsymbol{j}_C;\ \text{an}\ RC \times 1\ \text{vector},$$

$$X_3 = \begin{bmatrix} I_{\frac{RC}{2}} \\ L_{\frac{RC}{2}} \end{bmatrix};\ \text{an}\ RC \times \frac{RC}{2}\ \text{matrix},$$

and $I_{\frac{RC}{2}}$ is an $\frac{RC}{2} \times \frac{RC}{2}$ identity matrix, $L_{\frac{RC}{2}}$ is an $\frac{RC}{2} \times \frac{RC}{2}$ matrix with 1 in the $(i, \frac{RC}{2} + 1 - i)$-th elements $(i = 1, \dots, \frac{RC}{2})$ and 0 in the others, and where $\mathbf{1}_s$ is a $s \times 1$ vector of 1 elements and $\boldsymbol{j}_s = (1, 2, \dots, s)^t$ and $\otimes$ denotes the Kronecker product. Note that $X_3 \mathbf{1}_{RC/2} = \mathbf{1}_{RC}$. Note that the matrix X is full column rank which is K. We denote the liner space spanned by the columns of the matrix X by $S(X)$ with the dimension K. Let U be an $RC \times d_1$, where $d_1 = (RC - 4)/2$, and full column rank matrix such that the linear space spanned by the columns of U, i.e., $S(U)$, is the orthogonal complement of the space $S(X)$. Thus, $U^t X = O_{d_1, K}$, where $O_{d_1, K}$ is a $d_1 \times K$ zero matrix. Therefore, the OQPS model is expressed as $\boldsymbol{h}_1(\boldsymbol{p}) = \mathbf{0}_{d_1}$, where $\mathbf{0}_{d_1}$ is a $d_1 \times 1$ zero vector and $\boldsymbol{h}_1(\boldsymbol{p}) = U^t \log \boldsymbol{p}$.

The MMPS model may be expressed as $\boldsymbol{h}_2(\boldsymbol{p}) = \mathbf{0}_{d_2}$, where $\boldsymbol{h}_2(\boldsymbol{p}) = W\boldsymbol{p}$ with

$$W = \begin{pmatrix} (\boldsymbol{j}_R \otimes \mathbf{1}_C)^t - \frac{1}{2}(R+1)\mathbf{1}_{RC}^t \\ (\mathbf{1}_R \otimes \boldsymbol{j}_C)^t - \frac{1}{2}(C+1)\mathbf{1}_{RC}^t \end{pmatrix}; \text{ a } 2 \times RC \text{ matrix,}$$

and $d_2 = 2$. Thus W^t belongs to the space $S(X)$, i.e., $S(W^t) \subset S(X)$. Hence $WU = O_{d_2, d_1}$. From Theorem 1, the PS model may be expressed as $\boldsymbol{h}_3(\boldsymbol{p}) = \mathbf{0}_{d_3}$, where $d_3 = d_1 + d_2 = RC/2$ and $\boldsymbol{h}_3 = (\boldsymbol{h}_1^t, \boldsymbol{h}_2^t)^t$.

Let $H_s(\boldsymbol{p})$, $s = 1, 2, 3$, denote the $d_s \times RC$ matrix of partial derivatives of $\boldsymbol{h}_s(\boldsymbol{p})$ with respect to $\boldsymbol{p}$, i.e., $H_s(\boldsymbol{p}) = \partial \boldsymbol{h}_s(\boldsymbol{p}) / \partial \boldsymbol{p}^t$, and let $\Sigma(\boldsymbol{p}) = \mathrm{diag}(\boldsymbol{p}) - \boldsymbol{p}\boldsymbol{p}^t$, where $\mathrm{diag}(\boldsymbol{p})$ denotes a diagonal matrix with i-th component of $\boldsymbol{p}$ as i-th diagonal component.

We see that $H_1(\boldsymbol{p})\boldsymbol{p} = U^t \mathbf{1}_{RC} = \mathbf{0}_{d_1}$, $H_1(\boldsymbol{p})\mathrm{diag}(\boldsymbol{p}) = U^t$ and $H_2(\boldsymbol{p}) = W$. Therefore, we obtain

$$H_1(\boldsymbol{p})\Sigma(\boldsymbol{p})H_2(\boldsymbol{p})^t = U^t W^t = O_{d_1, d_2}.$$

Thus we obtain $\Delta_3(\boldsymbol{p}) = \Delta_1(\boldsymbol{p}) + \Delta_2(\boldsymbol{p})$, where

$$\Delta_s(\boldsymbol{p}) = \boldsymbol{h}_s(\boldsymbol{p})^t [H_s(\boldsymbol{p})\Sigma(\boldsymbol{p})H_s(\boldsymbol{p})^t]^{-1}\boldsymbol{h}_s(\boldsymbol{p}).$$

From the asymptotic equivalence of the Wald statistic and likelihood ratio statistic, we obtain Theorem 2 when each of R and C is even. The proofs of Theorem 2 for the other cases are omitted because it is obtained in a similar way. The proof is completed.

From Theorem 2, we obtain the following corollary:

Corollary 2. *The test statistic* $G^2(PS)$ *is asymptotically* (*i*) *equivalent to the sum of* $G^2(ROQPS)$ *and* $G^2(RMMPS)$, *and* (*ii*) *equivalent to the sum of* $G^2(COQPS)$ *and* $G^2(CMMPS)$.

5. An Example

Consider the data in Table 1 taken directly from Agresti ([2], p.12), which originally was presented by Grizzle et al. [7]. These data describe a comparison of four different operations for treating duodenal ulcer patients. The operations correspond to removal of various amounts of the stomach. Operation A is drainage and vagotomy, B is 25 percent resection (antrectomy) and vagotomy, C is 50 percent resection (hemigastrectomy) and vagotomy, and D is 75 percent resection.

Table 1. Cross-classification of duodenal ulcer patients according to operation and dumping severity. The parenthesized values are the maximum likelihood estimates of expected frequencies under the OQPS model

	Dumping Severity			
Operation	None	Slight	Moderate	Total
A	61	28	7	96
	(60.689)	(27.456)	(7.200)	
B	68	23	13	104
	(65.878)	(29.721)	(10.367)	
C	58	40	12	110
	(60.633)	(33.279)	(14.122)	
D	53	38	16	107
	(52.800)	(38.544)	(16.311)	
Total	240	129	48	417

Source: Grizzle et al. [7].

The PS model fits the data poorly (see Table 2). Using Theorem 1, we shall explore the reason for the poor fit of the PS model. We see from Table 2 that the OQPS model fits the data in Table 1 very well, however, the MMPS model fits these data very poorly. Therefore, we can infer that the poor fit of the PS model is caused by the influence of the lack of structure of the MMPS model rather than the OQPS model. Also, we can explore the reason for the poor fit of the PS model by using Corollary 1, although the details are omitted.

Both of the COQPS and the OQPS models fit these data well. The COQPS model is a special case of the OQPS model. Therefore, we shall test the hypothesis that $\alpha = 1$ (i.e., the COQPS model holds) assuming that the OQPS model holds. The, difference between the likelihood ratio Chi-squared values of the COQPS and the OQPS models is 4.44 with one degree of freedom. Thus, this hypothesis is rejected at the 0.05 significance level. Therefore, the OQPS model would be preferable to the COQPS model for these data.

Table 2. Likelihood ratio Chi-squared values for models applied to the data in Table 1

Applied model	Degrees of freedom	Likelihood ratio Chi-square
PS	6	148.26†
ROQPS	5	147.51†
COQPS	5	8.53
OQPS	4	4.09
MMPS	2	143.27†
RMMPS	1	0.75
CMMPS	1	139.73†

The symbol "†" means significant at 5% level.

The maximum likelihood estimates of parameters θ_1 and θ_2 under the OQPS model are $\hat{\theta}_1 = 0.798$ and $\hat{\theta}_2 = 5.223$. Thus, (i) the odds that Operation is i rather than $i + k$ when Dumping Severity is j is $\hat{\theta}_1^k$ times higher than the odds that Operation is i^* rather than $(i + k)^*$ when Dumping Severity is j^{**}, and (ii) the odds that Dumping Severity is j rather than $j + m$ when Operation is i is $\hat{\theta}_2^m$ times higher than the odds that Dumping Severity is j^{**} rather than $(j + m)^{**}$ when Operation is i^*. For example, when $m = 1$, (a) the odds that Dumping Severity is None rather than Slight when Operation is A is estimated to be 5.223 times higher than the odds that it is Moderate rather than Slight when Operation is D, and (b) the odds that Dumping Severity is None rather than Slight when Operation is B is estimated to be 5.223 times higher than the odds that it is Moderate rather than Slight when Operation is C; and when $m = 2$, (c) the odds that Dumping Severity is None rather than Moderate when Operation is A is estimated to be $(5.223)^2 = 27.280$ times higher than the odds that it is Moderate rather than None when

Operation is D, and (d) the odds that Dumping Severity is None rather than Moderate when Operation is B is estimated to be $(5.223)^2 = 27.280$ times higher than the odds that it is Moderate rather than None when Operation is C.

6. Concluding Remarks

We have proposed the OQPS model. Also, we have shown the orthogonal decomposition of the PS model into the OQPS and MMPS models. When the PS model fits the data poorly, the orthogonal decomposition of the PS model would be useful for seeing the reason for its poor fit. Indeed, as seen in Example, we can see that for the data in Table 1 the poor fit of the PS model is caused by the poor fit of the MMPS model rather than the OQPS model.

The orthogonal decomposition of the PS model into the OQPS and MMPS models would guarantee that an incompatible situation that both OQPS and MMPS models are accepted with high probability but the PS model is rejected with high probability, would not arise.

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