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THE SAMPLING DISTRIBUTION OF THE MAXIMUM LIKELIHOOD ESTIMATORS FROM WEIBULL-UNIFORM DISTRIBUTION BASED ON UPPER RECORD VALUES

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Abstract

This paper devoted with the sampling distribution of the maximum likelihood estimators for the Weibull-uniform distribution based on upper record values. The numerical study will used to obtain the value of the maximum likelihood estimators for different distribution parameters. Using EasyFit 5.6 Professional, the sampling distribution for the unknown parameters will be obtained.

1. Introduction

The Weibull-uniform (WU) distribution is defined in the following way (Bourguignon et al. [1]). It has the distribution function given by

$$F(x) = 1 - e^{-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}}, \qquad 0 < x < \theta, \tag{1.1}$$

and therefore its probability density function (pdf) of the form

$$f(x) = \frac{\alpha\beta\theta}{(\theta - x)^2}\left(\frac{x}{\theta - x}\right)^{\beta - 1} e^{-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}}, \qquad 0 < x < \theta. \tag{1.2}$$

The corresponding survival function is

$$\overline{F}(x) = S(x) = 1 - F(x) = e^{-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}}, \qquad 0 < x < \theta, \tag{1.3}$$

and the hazard function is

$$h(x) = \frac{\alpha\beta\theta}{(\theta - x)^2}\left(\frac{x}{\theta - x}\right)^{\beta - 1}, \tag{1.4}$$

where $\alpha, \beta, \theta > 0$. Here α denotes the scale parameter, β denotes the shape parameter, and θ is the location parameter.

Record values and associated statistics are of great importance in several real-life applications involving weather, economic and sports data "Olympic records or world records in sport". The statistical study of record values started with Chandler [2], he formulated the theory of record values as a model for successive extremes in a sequence of independently and identically random variables. Feller [3] gave some examples of record values with respect to gambling problems. Resnick [4] discussed the asymptotic theory of records. Theory of record values and its distributional properties have been extensively studied in the literature, for example, see Ahsanullah [5], Nagaraja [6], and Arnold et al. [7].

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d) random variables with cumulative function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Set $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence, if $Y_{j+1} > Y_j$. By definition, X_1 is an upper record values. The indices at which the upper record values occur are called upper record times $\{U(m), m \geq 0\}$, where $U(0) = 1$ and $U(m) = \min\{j : j > U(m-1), X_j > X_{U(m-1)}\}$. Then $R_m = X_{U(m)}, m \geq 0$ are called the upper record values (see Ahsanullah [5]).

Let $R_0, R_1, R_2, \ldots, R_m$ represent the first $(m+1)$ upper record values arising from a sequence $\{X_i\}$ of iid Weibull-uniform variable. Then the pdf of R_m is given by

$$f_{R_m}(r_m) = \frac{1}{\Gamma(m+1)}[-\ln \overline{F}(r_m)]^m f(r_m), \quad 0 < r_m < \theta, \quad m = 0, 1, 2, \ldots, \tag{1.5}$$

while the likelihood function based on the $(m+1)$ upper record values $R_0, R_1, R_2, \ldots, R_m$ is given by

$$f(r_0, r_1, \ldots, r_m) = \prod_{i=0}^{m-1} \frac{f(r_i)}{\overline{F}(r_i)} f(r_m), \tag{1.6}$$

where $0 < r_0 < r_1 < \ldots < r_{m-1} < r_m < \theta$.

Ashour and Amin [8] are obtained the sampling distribution for the maximum likelihood estimator for the Weibull distribution based on upper record values. Amin [9] obtained the sampling distribution for the maximum likelihood estimator for the generalized logistic type I distribution based on lower record values.

Present work is devoted to obtain, numerically, the sampling distribution of the maximum likelihood estimators for the unknown parameters of the Weibull-uniform distribution (1.1) based on upper record values. MathCad (2001) package and EasyFit 5.6 Professional package are used to obtain such distribution.

2. Sampling Distribution of the Maximum Likelihood Estimators for the Scale Parameter when Shape Parameter is Known

Let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed random variables having the Weibull-uniform distribution (1.1). Let $R_0, R_1, R_2, \ldots, R_m$ be the first $(m+1)$ upper record values arising from a sequence $\{X_i\}$ of iid Weibull-uniform variables. The maximum likelihood estimators for the parameter scale α when the shape parameter β and the location parameter θ are known is given by (see Amin and Abdel-Aziz [10])

$$\hat{\alpha} = (m+1)\left(\frac{\theta - r_m}{r_m}\right)^{\beta}. \tag{2.1}$$

Now consider the random variable $Y = (m+1)\left(\frac{\theta - R_m}{R_m}\right)^{\beta}$, where R_m is a random variable has the probability density function is given by

$$f_{R_m}(r_m) = \frac{\alpha^{(m+1)}\beta\theta}{\Gamma(m+1)(\theta - r_m)^2}\left(\frac{r_m}{\theta - r_m}\right)^{m\beta+\beta-1} e^{-\alpha\left(\frac{r_m}{\theta - r_m}\right)^{\beta}},$$

$$0 < r_m < \theta, \quad m = 0, 1, 2, \ldots. \tag{2.2}$$

After some calculation, one can prove that the probability density function of Y is given by

$$f(y) = \frac{1}{\Gamma(m+1)y}\left(\frac{\alpha(m+1)}{y}\right)^{m+1} e^{\frac{\alpha(m+1)}{y}}, \qquad y > 0,$$

which means that the distribution of $\hat{\alpha}$ has the inverse gamma distribution with parameters $(m+1)$ and $\alpha(m+1)$. Also, the mean and the variance of $\hat{\alpha}$ is $\frac{\alpha(m+1)}{m}$ and $\left(\frac{1}{m-1}\right)\left(\frac{\alpha(m+1)}{m}\right)^2$, respectively.

3. Sampling Distribution of Maximum Likelihood Estimators for Scale and Shape Parameters when the Two Parameters are Unknowns

Let $X_1, X_2, \cdots$ be an infinite sequence of independent and identically distributed random variables having the Weibull-uniform distribution (1.1). Let $R_0, R_1, R_2, \ldots, R_m$ be the first $(m+1)$ upper record values arising from a sequence $\{X_i\}$ of iid Weibull-uniform variables. The likelihood function is given by (see Amin and Abdel-Aziz [10])

$$L(\Theta) = (\beta\theta\alpha)^{m+1}\prod_{i=0}^{m}\left(r_i^{\beta-1}\left(\frac{1}{\theta - r_m}\right)^{\beta+1}\right) e^{-\alpha\left(\frac{r_m}{\theta - r_m}\right)^{\beta}},$$

$$0 < r_0 < r_1 < \cdots < r_m < \theta. \tag{3.1}$$

Amin and Abdel-Aziz [10] used (3.1) to obtain the maximum likelihood estimators by using the following equations:

$$\hat{\alpha} = (m+1)\left(\frac{\theta - r_m}{r_m}\right)^{\hat{\beta}}, \tag{3.2}$$

and

$$\hat{\beta} = \frac{(m+1)}{(m+1)\ln\left(\frac{r_m}{\theta - r_m}\right) - \sum_{i=0}^{m}\ln(\theta - r_i) + \sum_{i=0}^{m}\ln(r_i)}. \tag{3.3}$$

One can solve (3.3) to obtain $\hat{\beta}$ and then back substitution in (3.2) to find $\hat{\alpha}$. The following steps will be used to obtain the sampling distribution for the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ for α and β.

(1) Generate $n = 100$ independent and identically distributed random variates $X_1, X_2, \ldots, X_n$ Weibull-uniform distribution (1.1) with $\alpha, \beta = 0.5$, 1 and 1.5, and known location parameter $\theta = 1$.

(2) Do step (1) 5000 times, therefore there are 5000 vectors of iid random variates.

(3) Choose from each vector the first $(m + 1)$ upper record values where $m = 3$, then there are k vectors containing four upper record values, where $k < 5000$.

(4) Compute the values of $\hat{\alpha}$ and $\hat{\beta}$ in each vector, then there exist two vectors of length k containing the random variables from the distribution of $\hat{\alpha}$ and $\hat{\beta}$.

(5) Do steps (1) to (5), for $m = 5$ and $m = 7$.

(6) The distribution of each vector was fitted using EasyFit 5.6 Professional package and the results recorded in Tables 1 and 2.

Tables 1 and 2 contains the results which show

(i) The values of the parameters α and β and the number of record m.

(ii) The fitted distribution for the estimator $\hat{\alpha}$ in Table 1 and the fitted distribution for the estimator $\hat{\beta}$ in Table 2.

(iii) The parameters of the fitted distribution.

(iv) The P-value for the Kolmogorov-Smirnov (K-S) and Chi-square (Chi-sq.) test.

4. Conclusion

Based on the set of upper record values, this paper developed the sampling distributions of the maximum likelihood estimators of the two unknown parameters of the Weibull-uniform distribution. MathCad (2001) package and EasyFit 5.6 Professional package are used to obtain such distribution numerically and the following were observed:

(1) When the shape parameter β is known, the sampling distribution for the estimator $\hat{\alpha}$ is the inverted gamma distribution with parameters $(m+1)$ and $\alpha(m+1)$.

(2) Table 1 shows the fitted sampling distribution for the maximum likelihood estimators $\hat{\alpha}$ is the Johnson SB distribution with the following pdf:

$$f(x) = \frac{\delta}{\lambda\sqrt{2\pi}z(1-z)} e^{-\left\{\frac{1}{2}\left(\gamma+\delta\ln\left(\frac{z}{1-z}\right)\right)^2\right\}}, \qquad 0 < z < 1,$$

where both γ and $\delta > 0$ are the shape parameter, $\lambda > 0$ is the scale parameter, $z = \frac{x-\xi}{\lambda}$, and $\xi < x < \xi + \lambda$, where ξ is the location parameter.

(3) Table 2 shows the fitted sampling distribution for the maximum likelihood estimator $\hat{\beta}$ is fatigue life with 3 parameters with the following pdf:

$$f(x) = \frac{\sqrt{\frac{x-\mu}{\upsilon}} + \sqrt{\frac{\upsilon}{x-\mu}}}{2\omega(x-z)} \phi\left(\frac{1}{\omega}\left(\sqrt{\frac{x-\mu}{\upsilon}} - \sqrt{\frac{\upsilon}{x-\mu}}\right)\right), \qquad x < \mu,$$

where $\upsilon > 0$ is the scale parameter, $\omega > 0$ is the shape parameter, and μ is the location parameter.

(4) Some cases in Table 2 the sampling fitted distribution is Johnson SB.

Table 1. Fitted sampling distribution for the MLE's $\hat{\alpha}$ for different values of α, β, and m

# of record values m	Parameters							P-Value	
	α	β	Fitted					K.S	Chi. Sq.
3	0.5	0.5	Johnson SB	$\gamma = 1.12926$	$\delta = 0.74849$	$\lambda = 2.2007$	$\xi = -0.0510$	0.058	0.874
		1	Johnson SB	$\gamma = 1.1674$	$\delta = 0.81123$	$\lambda = 2.3014$	$\xi = -0.01624$	0.839	0.262
		1.5	Johnson SB	$\gamma = 0.96465$	$\delta = 1.0189$	$\lambda = 2.4079$	$\xi = -0.089$	0.956	0.876
	1	0.5	Johnson SB	$\gamma = 0.36239$	$\delta = 0.692$	$\lambda = 2.1041$	$\xi = -0.04956$	0.683	0.184
		1	Johnson SB	$\gamma = 0.33798$	$\delta = 0.82029$	$\lambda = 2.0571$	$\xi = 0.11244$	0.950	0.823
		1.5	Johnson SB	$\gamma = 0.11446$	$\delta = 0.994$	$\lambda = 1.9476$	$\xi = 0.21849$	0.825	0.732
	1.5	0.5	Johnson SB	$\gamma = 0.19275$	$\delta = 0.78398$	$\lambda = 3.2632$	$\xi = -0.11181$	0.979	0.821
		1	Johnson SB	$\gamma = 0.27202$	$\delta = 0.97139$	$\lambda = 2.8747$	$\xi = 0.4872$	0.984	0.933
		1.5	Johnson SB	$\gamma = 0.11651$	$\delta = 0.876$	$\lambda = 3.0441$	$\xi = 0.15484$	0.847	0.663
5	0.5	0.5	Johnson SB	$\gamma = 0.58316$	$\delta = 0.84312$	$\lambda = 2.1062$	$\xi = -0.10402$	0.88	0.38
		1	Johnson SB	$\gamma = 0.7472$	$\delta = 0.91277$	$\lambda = 2.2559$	$\xi = -0.01262$	0.922	0.988
		1.5	Johnson SB	$\gamma = 0.95329$	$\delta = 1.0118$	$\lambda = 2.2709$	$\xi = 0.14926$	0.973	0.887
	1	0.5	Johnson SB	$\gamma = -0.0597$	$\delta = 0.77219$	$\lambda = 2.1185$	$\xi = -0.00266$	0.946	0.607
		1	Johnson SB	$\gamma = -0.0495$	$\delta = 0.80203$	$\lambda = 1.8918$	$\xi = 0.20754$	0.942	0.866
		1.5	Johnson SB	$\gamma = -0.2004$	$\delta = 0.74802$	$\lambda = 1.589$	$\xi = 0.45268$	0.994	0.873
	1.5	0.5	Johnson SB	$\gamma = -0.2268$	$\delta = 1.1125$	$\lambda = 3.5952$	$\xi = -0.25194$	0.821	0.221
		1	Johnson SB	$\gamma = -0.2716$	$\delta = 0.99002$	$\lambda = 3.2084$	$\xi = 0.04916$	0.994	0.954
		1.5	Johnson SB	$\gamma = 0.38563$	$\delta = 0.92523$	$\lambda = 2.7096$	$\xi = 0.62582$	0.8662	0.443

Table 1. (Continued)

7	0.5	0.5	Johnson SB	$\gamma = 0.62545$	$\delta = 2.2091$	$\lambda = 4.1052$	$\xi = -0.87759$	0.674	0.451
		1	Johnson SB	$\gamma = 0.62545$	δ=2.2091	$\lambda = 4.1052$	$\xi = -0.87759$	0.674	0.451
		1.5	Johnson SB	$\gamma = 0.44439$	$\delta = 0.8959$	$\lambda = 1.8264$	$\xi = 0.31362$	0.934	0.952
	1	0.5	Johnson SB	$\gamma = -0.2997$	$\delta = 0.81925$	$\lambda = 2.0353$	$\xi = 0.004$	0.832	0.747
		1	Johnson SB	$\gamma = 0.0796$	$\delta = 0.84962$	$\lambda = 1.5802$	$\xi = 0.58141$	0.943	0.545
		1.5	Johnson SB	$\gamma = -0.9718$	$\delta = 1.2975$	$\lambda = 1.9301$	$\xi = 0.23924$	0.778	0.370
	1.5	0.5	Johnson SB	$\gamma = -1.5294$	$\delta = 1.3315$	$\lambda = 4.6395$	$\xi = -1.2506$	0.967	0.708
		1	Johnson SB	$\gamma = -0.6924$	$\delta = 0.59479$	$\lambda = 2.2062$	$\xi = 0.6796$	0.743	0.457
		1.5	Johnson SB	$\gamma = -0.41895$	$\delta = 1.2392$	$\lambda = 2.9989$	$\xi = 0.43718$	0.962	0.973

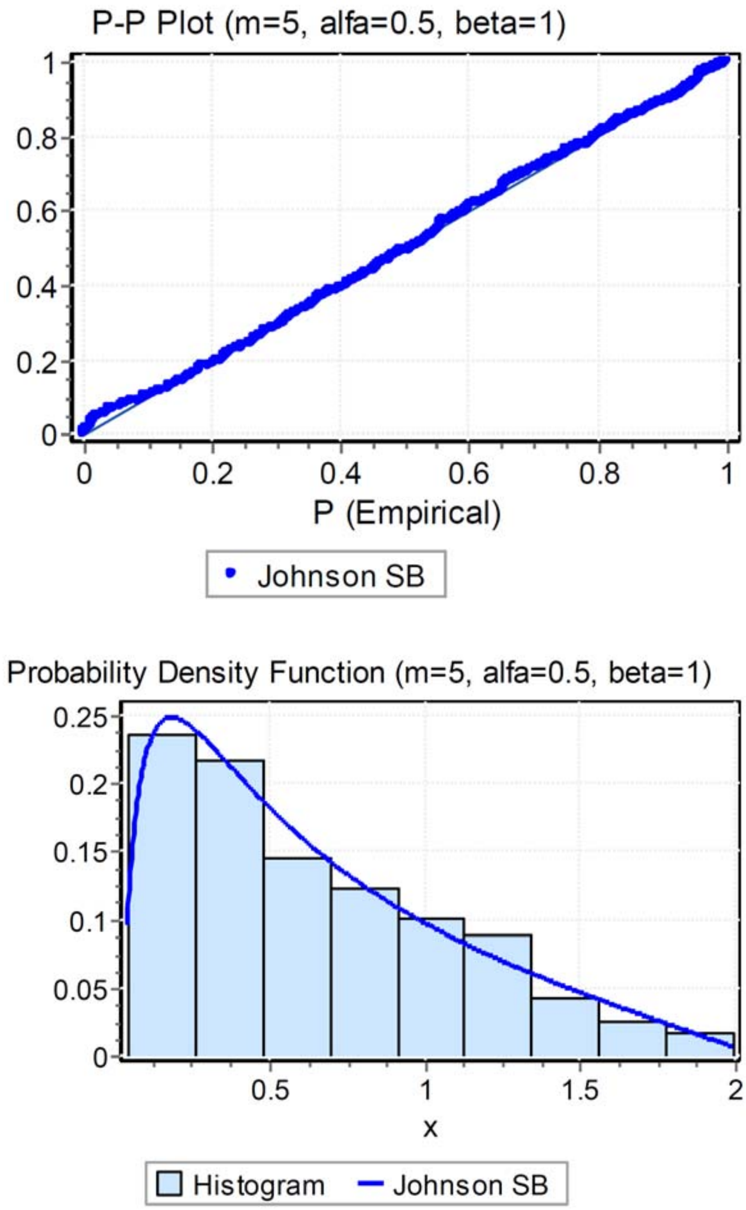

Figure 1. The histogram and the *P-P* plot for the fitted distribution for $\hat{\alpha}$ when $m = 5$, alfa = 0.5, and beta = 1.

Table 2. Fitted sampling distribution for the MLE's $\hat{\beta}$ for different values of α, β, and m

# of record values m	Parameters							P-Value	
	α	β	Fitted					K.S	Chi. Sq.
3	0.5	0.5	Fatigue Life	$\omega = 0.55518$	$\upsilon = 0.61279$	$\mu = 0.0596$		0.539	0.82
		1	Fatigue Life	$\omega = 0.15493$	$\upsilon = 2.7807$	$\mu = -1.694$		0.0942	0.653
		1.5	Johnson SB	$\gamma = -0.2688$	$\delta = 0.87193$	$\lambda = 1.9831$	$\xi = 0.13009$	0.978	0.616
	1	0.5	Fatigue Life	$\omega = 0.54018$	$\upsilon = 0.67156$	$\mu = 0.0587$		0.465	0.391
		1	Fatigue Life	$\omega = 0.06432$	$\upsilon = 5.9452$	$\mu = -4.722$		0.198	0.157
		1.5	Johnson SB	$\gamma = -0.4555$	$\delta = 0.94549$	$\lambda = 1.5049$	$\xi = 0.60011$	0.948	0.984
	1.5	0.5	Fatigue Life	$\omega = 0.66273$	$\upsilon = 0.62166$	$\mu = 0.0823$		0.262	0.346
		1	Fatigue Life	$\omega = 0.42921$	$\upsilon = 1.3664$	$\mu = -0.065$		0.521	0.602
		1.5	Fatigue Life	$\omega = 0.15101$	$\upsilon = 3.9836$	$\mu = -2.269$		0.421	0.352
5	0.5	0.5	Fatigue Life	$\omega = 0.62598$	$\upsilon = 0.37847$	$\mu = 0.1847$		0.945	0.526
		1	Fatigue Life	$\omega = 0.35989$	$\upsilon = 0.91926$	$\mu = 0.1269$		0.724	0.937
		1.5	Fatigue Life	$\omega = 0.00542$	$\upsilon = 57.16$	$\mu = -55.73$		0.541	0.21
	1	0.5	Fatigue Life	$\omega = 0.54282$	$\upsilon = 0.44172$	$\mu = 0.1791$		0.794	0.689
		1	Fatigue Life	$\omega = 0.33388$	$\upsilon = 1.0213$	$\mu = 0.1023$		0.811	0.696
		1.5	Johnson SB	$\gamma = -0.1791$	$\delta = 0.78814$	$\lambda = 1.1857$	$\xi = 0.87662$	0.974	0.656
	1.5	0.5	Fatigue Life	$\omega = 0.52581$	$\upsilon = 0.47802$	$\mu = 0.1285$		0.421	0.0959
		1	Fatigue Life	$\omega = 0.00646$	$\upsilon = 92.482$	$\mu = -90.52$		0.171	0.285
		1.5	Fatigue Life	$\omega = 0.18106$	$\upsilon = 2.8438$	$\mu = -1.118$		0.112	0.418

Table 2. (Continued)

7	0.5	0.5	Fatigue Life	ω = 0.71053	υ = 0.20922	μ = 0.2912		0.987	0.804
		1	Fatigue Life	ω = 0.37732	υ = 0.82058	μ = 0.2226		0.338	0.504
		1.5	Fatigue Life	ω = 0.23827	υ = 1.0848	μ = 0.3028		0.945	0.556
	1	0.5	Fatigue Life	ω = 0.50365	υ = 0.3916	μ = 0.2286		0.994	0.977
		1	Fatigue Life	ω = 0.41006	υ = 0.6301	μ = 0.5094		0.818	0.578
		1.5	Fatigue Life	ω = 0.13426	υ = 1.5051	μ = 0.0418		0.965	0.851
	1.5	0.5	Fatigue Life	ω = 0.46295	υ = 0.42791	μ = 0.1357		0.833	0.583
		1	Fatigue Life	ω = 0.53835	υ = 0.66447	μ = 0.4436		0.946	0.677
		1.5	Fatigue Life	ω = 0.64233	υ = 0.62348	μ = 0.9209		0.676	0.561

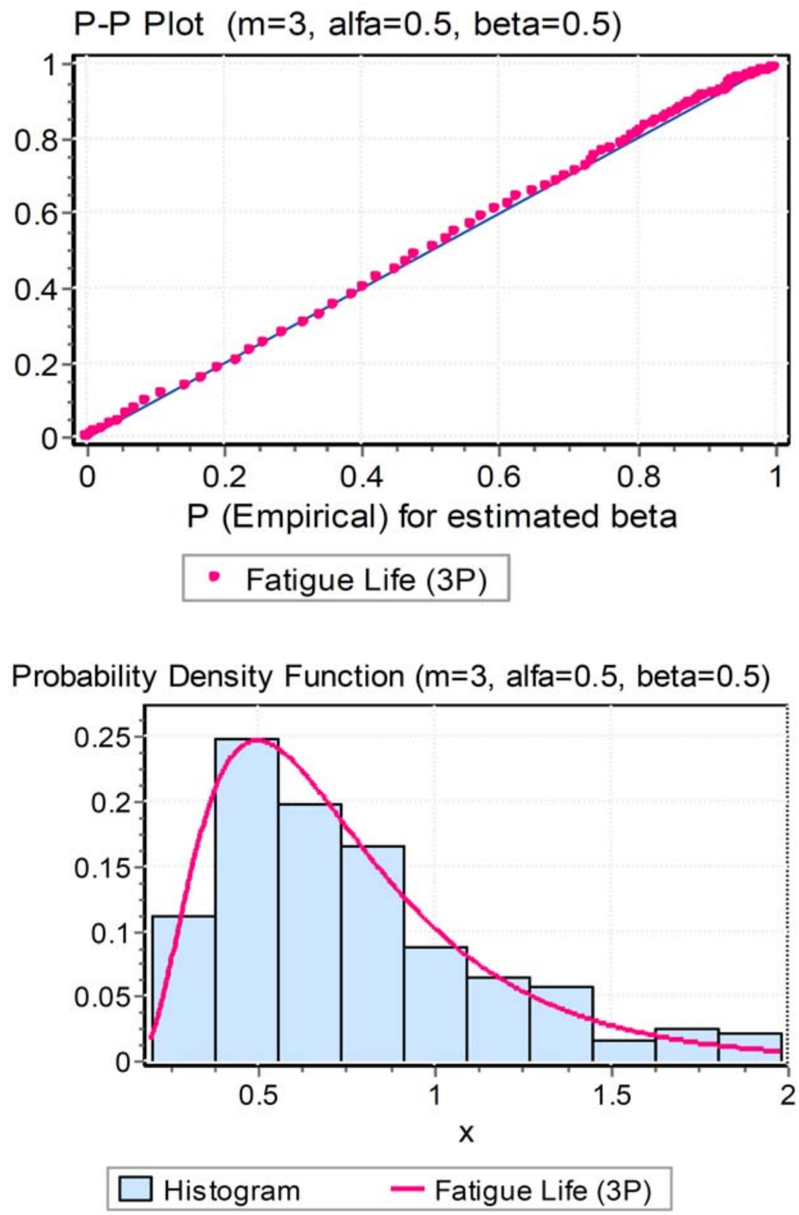

Figure 2. The histogram and the *P-P* plot for the fitted distribution for $\widehat{\beta}$ when $m = 3$, alfa = 0.5, and beta = 0.5.

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