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A NEW GENERALIZATION OF KUMARASWAMY LINDLEY DISTRIBUTION

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Abstract

In this paper, a new class of generalized distributions called the Exponentiated Kumaraswamy Lindley (EKL) distribution is introduced. The new distribution is a quite flexible model in analyzing positive data. We provide a comprehensive mathematical treatment of the statistical properties of this distribution. Some structural properties of the proposed new distribution are discussed including probability density function and explicit forms for its survival function, hazard function, and quantile function. The method of maximum likelihood is used to estimate the model parameters and the observed and expected information matrices are derived. A real data set is used to compare the new model with widely known distributions. A simulation study is conducted and the mean, bias and mean squared error of estimates are presented for different sample sizes.

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1. Introduction

In many fields of applied sciences such as medicine, engineering, and finance amongst others, modelling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data for instance the exponential, Weibull, gamma, Rayleigh distributions and their generalizations (see, e.g., Gupta and Kundu [1, 2]). Each distribution has its own characteristics due specifically to the shape of the failure rate function which may be only monotonically decreasing or increasing or constant in its behaviour, as well as non-monotone, being bathtub shaped or even unimodal. Lindley distribution is important for studying stress-strength reliability modelling. Ghitany et al. [3] discussed various properties of this distribution and showed that the Lindley distribution provides a better model than the exponential distribution in many ways. Sankaran [4] introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Mazucheli and Achcar [5] discussed the applications of Lindley distribution to competing risks lifetime data. Ghitany and Al-Mutairi [6], and Ghitany et al. [7] obtained size-biased and zero-truncated version of Poisson-Lindley distribution and discussed their various properties and applications. Bakouch et al. [8] obtained an extended Lindley distribution and discussed its various properties and applications. Ghitany et al. [9] developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Ghitany and Al-Mutairi [10] discussed various estimation methods for the discrete Poisson-Lindley distribution. Rama and Mishra [11] studied quasi Lindley distribution. Ghitany et al. [12] studied power Lindley distribution and associated inference. Zakerzadah and Dolati [13] and Elbatal et al. [14] have obtained the generalized Lindley distribution and the Kumaraswamy quasi Lindley distribution, respectively, and discussed various properties and applications. Cakmakyapan and Kadilar [15] have obtained the Kumaraswamy Lindley distribution.

The Kumaraswamy distribution as defined by Kumaraswamy (1980) in [16] is identified as an alternative to beta distribution because they both have the same basic shape properties (unimodal, uniantimodal, increasing, decreasing, monotone or constant). The probability density function (pdf) of the Kumaraswamy distribution is given as follows:

$$f(x) = abx^{a-1}(1 - x^{a})^{b-1}; \quad 0 < x < 1. \tag{1}$$

The corresponding cumulative density function (cdf) is given as

$$F(x) = 1 - (1 - x^{a})^{b}; \quad 0 < x < 1, \tag{2}$$

where $a, b > 0$ are shape parameters.

The pdf given in Equation (1) does not involve any incomplete beta function ratio and it is regarded as being tractable because of its mild algebraic properties. Jones [17] explored the background of this distribution and discussed the advantages and disadvantages of this distribution compared with the beta distribution. Cordeiro and Castro [18] have proposed a generalization of the probability distribution, by using (1) and (2). The density and distribution function of generalized class given by Cordeiro and Castro have following forms:

$$f(x) = abg(x)G(x)^{a-1}(1 - G(x)^{a})^{b-1}, \tag{3}$$

and

$$F(x) = 1 - (1 - G(x)^{a})^{b}, \tag{4}$$

where $g(x) = \dfrac{dG(x)}{dx}$ and parameters $a, b > 0$ are shape parameters.

Cordeiro et al. [19] introduced the Kumaraswamy Weibull distribution. Also, Bourguignon et al. [20] introduced the Kumaraswamy Pareto distribution. Later, Paranaiba et al. [21] and Gomes et al. [22] introduced the Kumaraswamy generalized Burr and Rayleigh distributions, followed by the paper of de Pascoa et al. [23] where they introduced the Kumaraswamy generalized gamma distribution.

In this article, we present a new generalization of Lindley distribution called the exponentiated Kumaraswamy Lindley (EKL) distribution. The study examines various properties of the new model. The rest of the paper is organized as follows: In Section 2, we define the generalization of the EKL distribution. The probability density function and cumulative distribution are given. Quantile function, moments, and moment generating function are discussed in Section 3. Renyi and Shannon entropies are calculated in Section 4. In Section 5, we discuss maximum likelihood estimation and determine the elements of the observed information matrix and expected Fisher information matrix. Section 6 provides an application to a real data set and a simulation study is presented. Section 7 ends the paper with some conclusions.

2. Exponentiated Kumaraswamy Lindley (EKL) Distribution

In this section, we introduce the exponentiated Kumaraswamy Lindley (EKL) distribution. The cumulative distribution function (cdf) of the Lindley distribution, as introduced by Lindley [24], is given by

$$G(x) = 1 - e^{-\theta x}\left[1 + \frac{\theta x}{\theta + 1}\right], \quad x > 0,\ \theta > 0. \tag{5}$$

Using (5) in (4), we obtain the cdf of the Kumaraswamy Lindley (KL) distribution. The study of exponentiated distributions is useful in statistics, as indicated by Mudholkar and Srivastava [25] for a very important reason: It provides methods of extending distributions for added flexibility in fitting data. For a baseline cdf $G(x)$, the exponentiated distribution $F(x) = G(x)^{\delta}$, where the exponentiation parameter $\delta > 0$ gives the flexibility to accommodate both monotone and non-monotone hazard rate functions. Nassar and Eissa [26, 27] and many authors studied the various properties of the exponentiated distributions. In this paper, we consider the EKL with probability density function (pdf) given by

$$f_{EKL}(x) = abc\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a-1}$$

$$\times\left\{1-\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a}\right\}^{b-1}$$

$$\times\left[1-\left\{1-\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a}\right\}^{b}\right]^{c-1}. \tag{6}$$

Using the generalized binomial theorem, if $\beta = n$ is a natural number, the series is given by

$$(1-z)^{\beta} = \sum_{i=0}^{\beta}\binom{\beta}{i}(-1)^{i}z^{i}. \tag{7.a}$$

If β is neither a natural number nor zero, the series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$, (see Gradshteyn and Ryzhik [28]). In this case, the series is written using the following infinite sum:

$$(1-z)^{\beta} = \sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^{i}z^{i}. \tag{7.b}$$

The EKL pdf can be rewritten, for a, b, c non-integers, as follows:

$$f_{EKL}(x) = \sum_{j=0}^{\infty}(-1)^{j}\binom{c-1}{j}abc\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a-1}$$

$$\times\left\{1-\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a}\right\}^{b(j+1)-1}.$$

Since

$$\left\{1-\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a}\right\}^{b(j+1)-1} = \sum_{k=0}^{\infty}(-1)^{k}\binom{b(j+1)-1}{k} \times\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{ak},$$

we get

$$f_{EKL}(x) = abc\sum_{j,k,l,m=0}^{\infty}(-1)^{j+k+l}\binom{c-1}{j}\binom{b(j+1)-1}{k}\binom{a(k+1)-1}{l}\binom{l}{m} \times\frac{\theta^{(m+2)}}{(\theta+1)^{(m+1)}}(1+x)e^{-\theta(l+1)x}x^{m}.$$

This EKL pdf can be written in the form

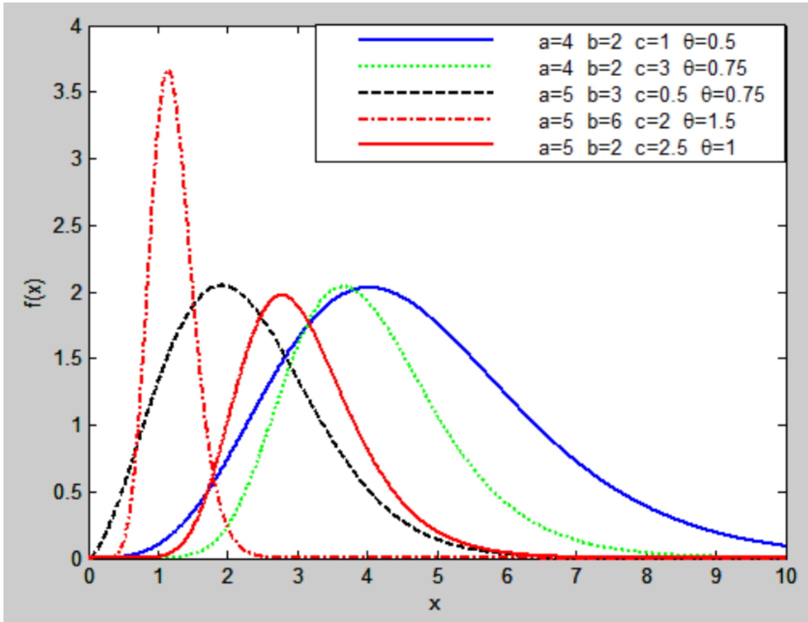
$$f_{EKL}(x) = w_{jklm}(1+x)e^{-\theta(l+1)x}x^{m}, \tag{8}$$

where

$$w_{jklm} = abc\sum_{j,k,l,m=0}^{\infty}(-1)^{j+k+l}\binom{c-1}{j}\binom{b(j+1)-1}{k}\binom{a(k+1)-1}{l}\binom{l}{m} \times\frac{\theta^{(m+2)}}{(\theta+1)^{(m+1)}}. \tag{9}$$

Also, the cumulative distribution function (cdf) of the EKL is given by

$$F_{EKL}(x) = \left[1-\left\{1-\left(1-e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]\right)^{a}\right\}^{b}\right]^{c},\ a>0,\ b>0,\ c>0,\ \theta>0. \tag{10}$$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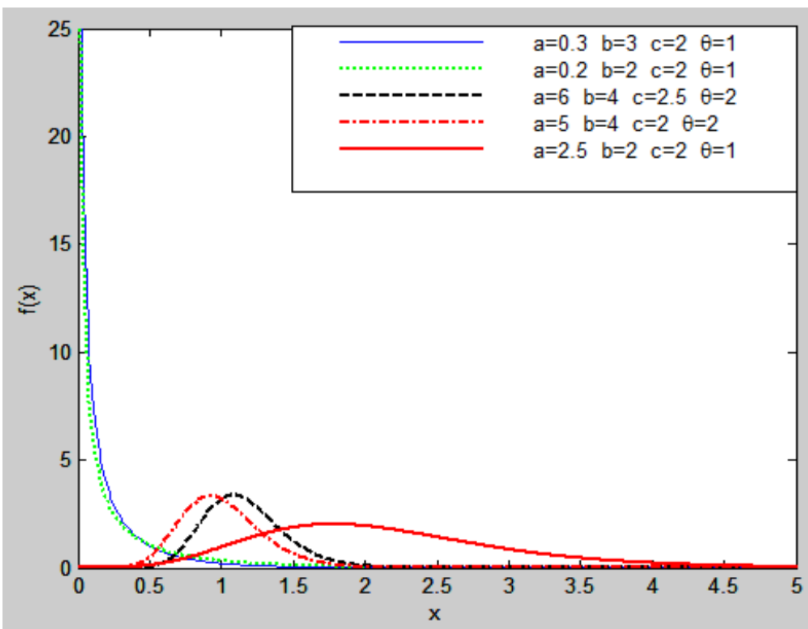
a=4 b=2 c=3 θ=0.75
a=5 b=3 c=0.5 θ=0.75
a=5 b=6 c=2 θ=1.5
a=5 b=2 c=2.5 θ=1
f(x)
x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a=0.2 b=2 c=2 θ=1
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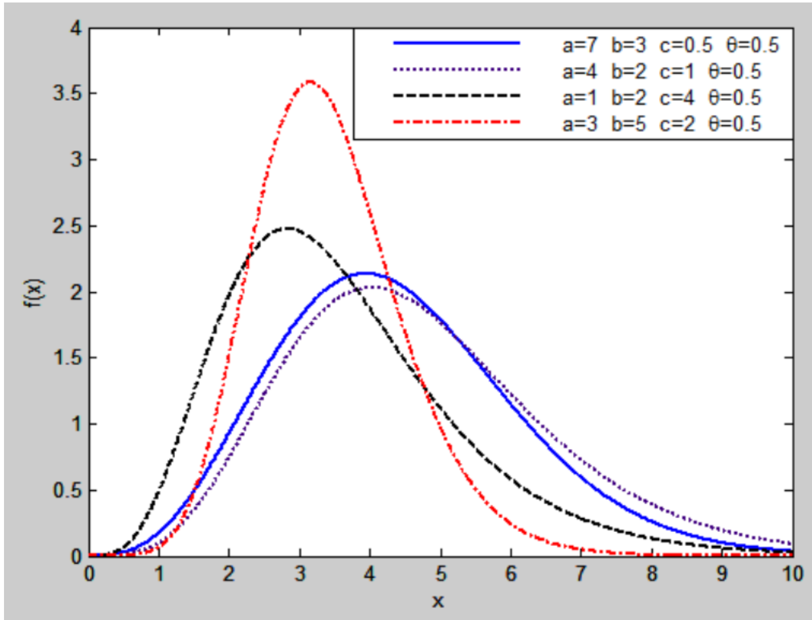
f(x)
x


a=7 b=3 c=0.5 θ=0.5
a=4 b=2 c=1 θ=0.5
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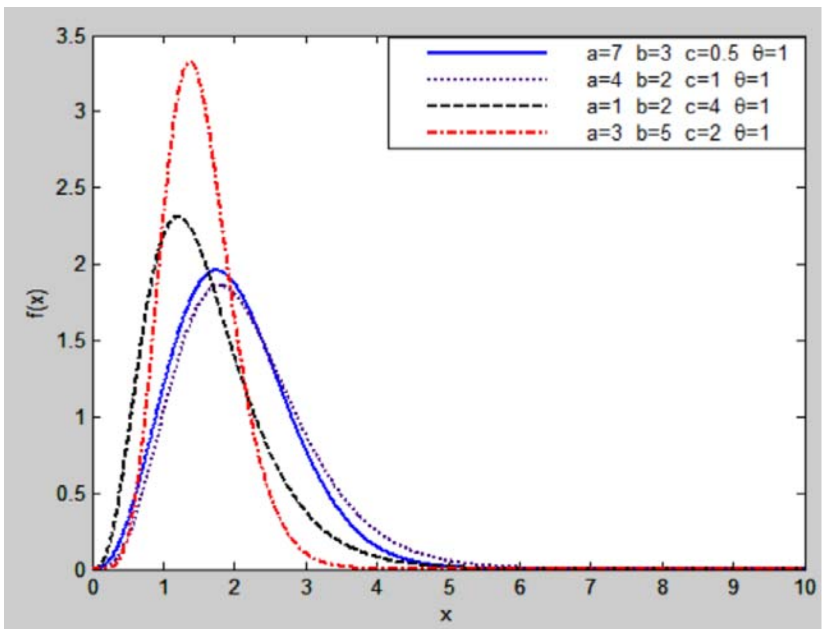
a=3 b=5 c=2 θ=0.5
f(x)
x


a=7 b=3 c=0.5 θ=1
a=4 b=2 c=1 θ=1
a=1 b=2 c=4 θ=1
a=3 b=5 c=2 θ=1
f(x)
x

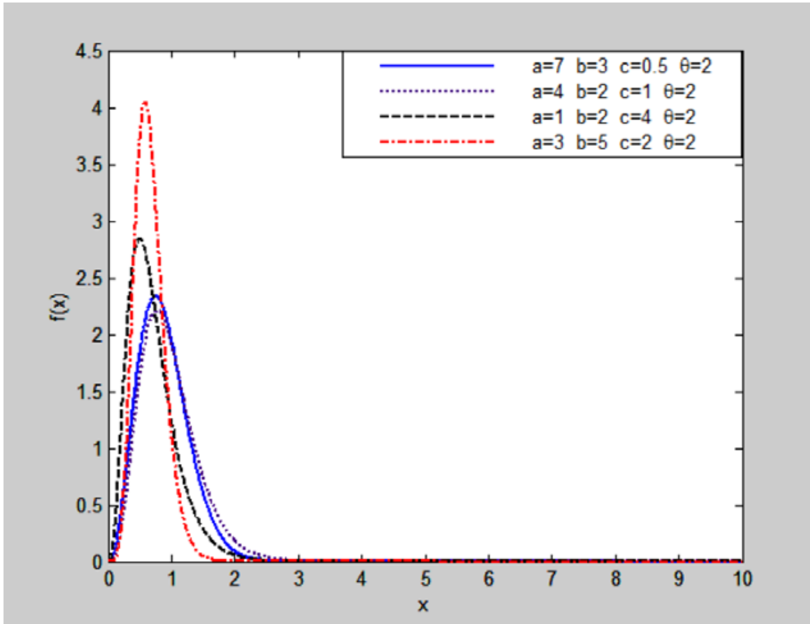
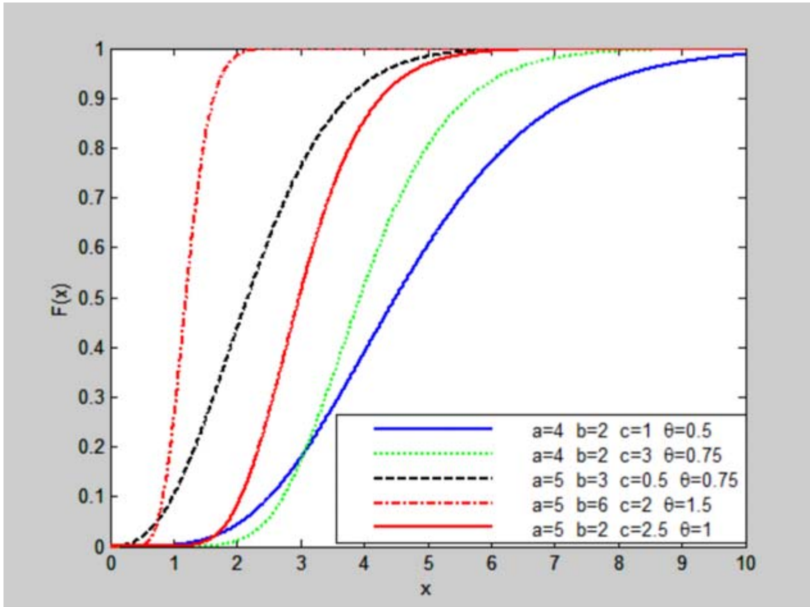
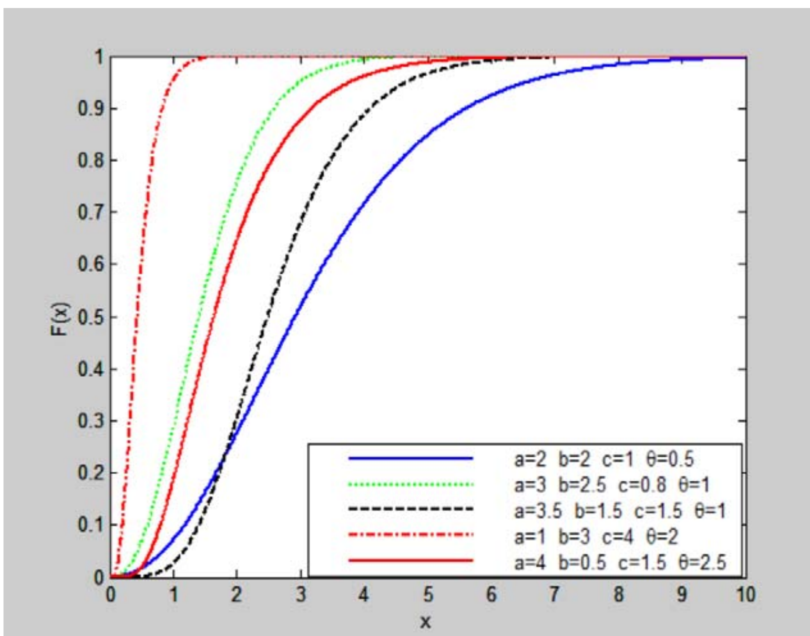
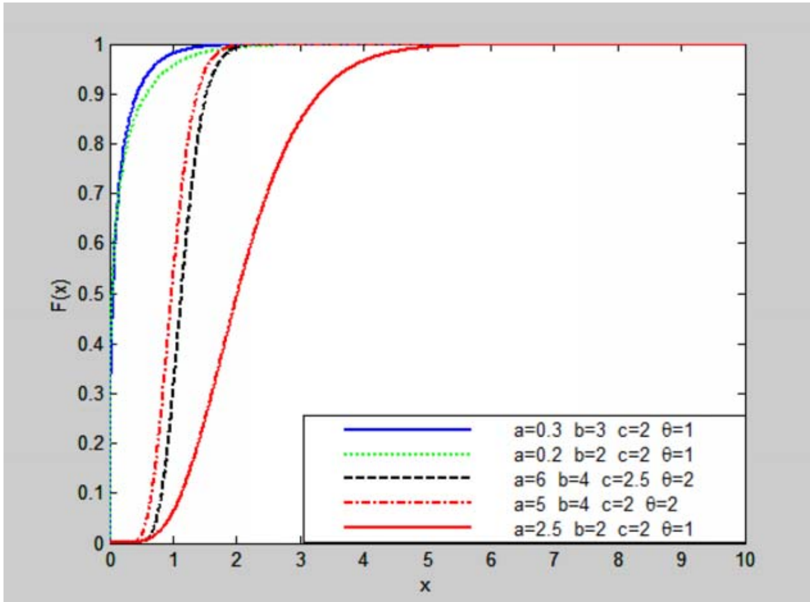


Figure 1. Plots of the pdf of the EKL for selected values of the parameters.

Figures 1 and 2 are the plots of the pdf and cdf of the EKL for different values of the parameters. It is obvious from the displayed plots in Figure 1 that pdf of the EKL distribution is unimodal for all values of the parameters, skewed to the right, and for $\theta < 1$ the graph expands while for $\theta > 1$ the graph diminishes. Also, for increasing values of *b,* the graph has a higher mode peak.


F(x)
x
a=4 b=2 c=1 θ=0.5
a=4 b=2 c=3 θ=0.75
a=5 b=3 c=0.5 θ=0.75
a=5 b=6 c=2 θ=1.5
a=5 b=2 c=2.5 θ=1


F(x)
x
a=2 b=2 c=1 θ=0.5
a=3 b=2.5 c=0.8 θ=1
a=3.5 b=1.5 c=1.5 θ=1
a=1 b=3 c=4 θ=2
a=4 b=0.5 c=1.5 θ=2.5



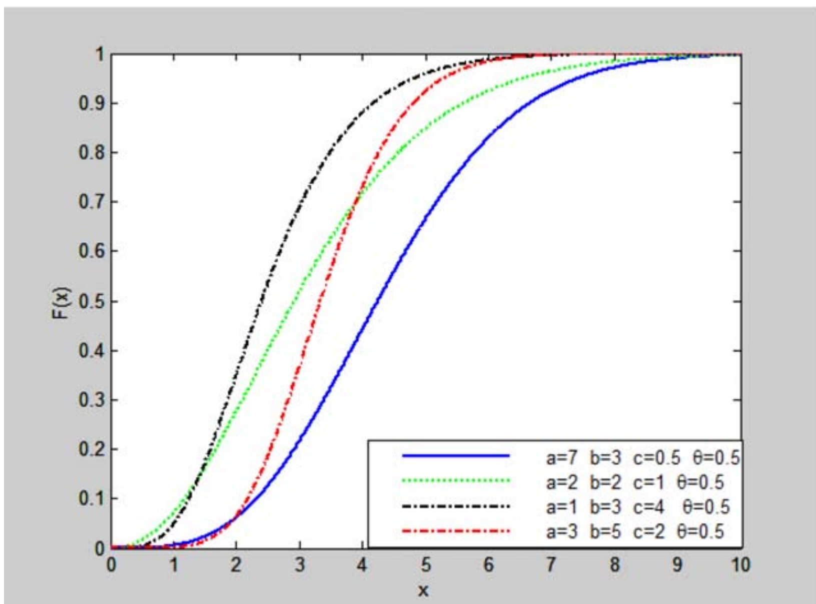


Figure 2. Plots of the cdf of EKL for selected values of the parameters.

The survival (reliability) function of the EKL, is defined as

$$S_{EKL}(x) = 1 - F_{EKL}(x) = 1 - \left[1 - \left\{1 - \left(1 - e^{-\theta x}\left[1 + \frac{\theta x}{\theta + 1}\right]\right)^{a}\right\}^{b}\right]^{c}. \quad (11)$$

Now, the hazard rate function of the EKL is given by

$$h_{EKL}(x) = \frac{f(x)}{1 - F(x)}$$

$$= w_{jklm}(1 + x)e^{-\theta(l+1)x}x^{m} \times S_{EKL}^{-1}(x). \quad (12)$$

In Figure 3, the shape of the hazard rate function can be observed from the following various plots of the hazard rate function for different values of the parameters.

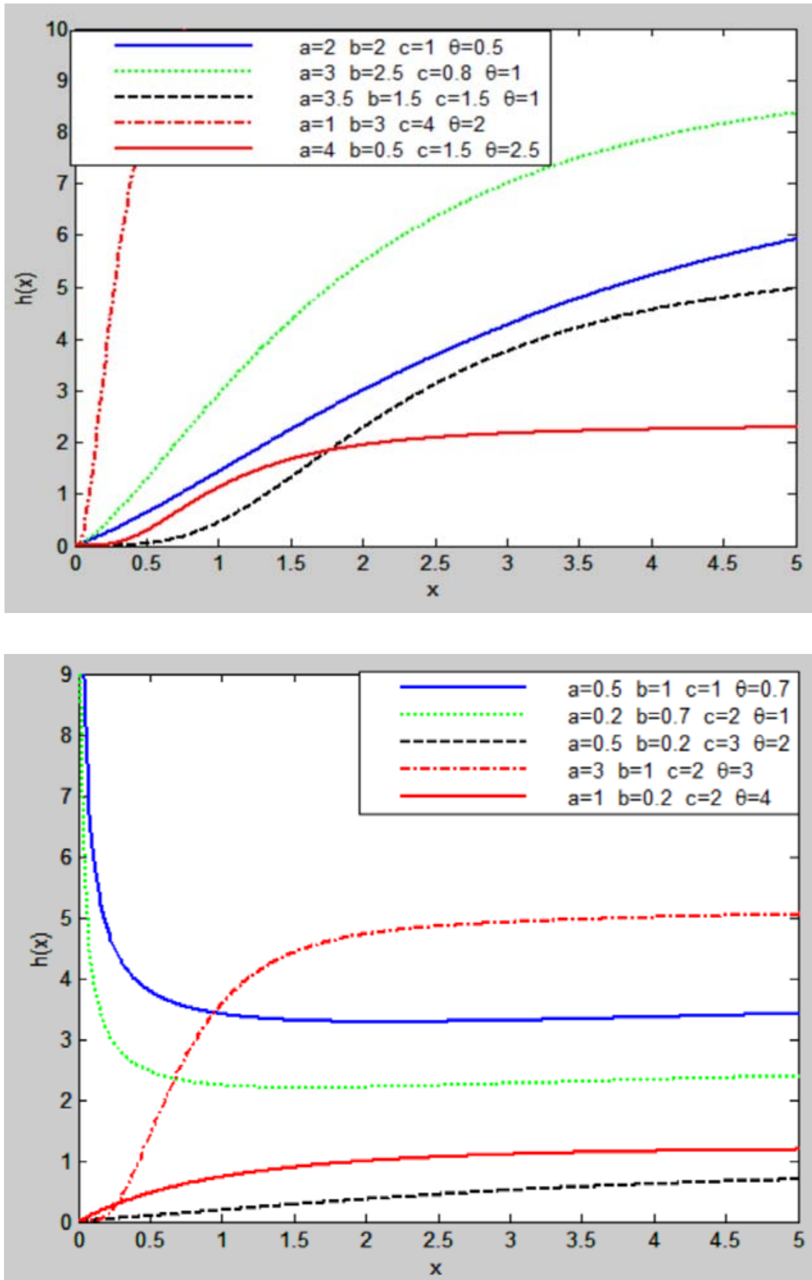


Figure 3. Plots of the hazard rate of EKL for selected values of the parameters.

Note that the EKL distribution has several well-known models as special cases, which make it of distinguishable scientific importance from other distributions.

- If $a = b = c = 1$, then Equation (6) reduces to the Lindley distribution.

- If $a = b = c = 1$, then Equation (10) reduces to the gamma distribution with parameters $(2, \theta)$.

- If $c = 1$, we get the Kumaraswamy Lindley distribution.

The graphs of the hazard function for several combinations of the parameters represent various shapes including monotonically increasing, monotonically decreasing, bathtub and up-side down bathtub shapes. This attractive flexibility makes EKL hazard rate function useful and suitable for nonmonotone empirical hazard behaviours, which are more likely to be encountered or observed in real life situations.

3. Basic Properties of the Distribution

In this section, we study some basic properties of EKL distribution, specifically the quantile function, moments, and moment generating function.

3.1. Quantile function

Theorem 1. *The quantile function of EKL distribution is at the following approximate point*:

$$X \approx \frac{\theta + 1}{\theta^2}\left(1 - \left(1 - (u)^{1/c}\right)^{1/b}\right)^{1/a}. \tag{13}$$

Proof. If $F(x)$ is continuous and strictly increasing, then the quantile function $Q(x) = F^{-1}(u)$, $u \in (0, 1)$ can be straightforward computed by inverting (10) to obtain

$$e^{-\theta x}\left[1+\frac{\theta x}{\theta+1}\right]=1-\left(1-\left(1-(u)^{1/c}\right)^{1/b}\right)^{1/a}. \tag{14}$$

Using the Taylor series expansion given by $e^{tx}=\sum_{j=0}^{\infty}\frac{(tx)^j}{j!}$, an approximate quantile function of the EKL distribution is given by (13).

In particular, the median of the EKL distribution is given by

$$median(x)\approx\frac{\theta+1}{\theta^2}\left(1-\left(1-(1/2)^{1/c}\right)^{1/b}\right)^{1/a}. \tag{15}$$

The random sample can also be easily generated from (14) by taking U as a uniform random variable in $(0, 1)$.

3.2. Moments

Many of the interesting characteristics and features of a distribution can be studied through its moments (e.g., tendency, dispersion, skewness, and kurtosis). Moments are necessary and important in any statistical analysis, especially in applications. The following theorem gives the moments of the EKL distribution.

Theorem 2. *If X~EKL, then the r-th non-central moment of the EKL distribution is given by the following*:

$$\mu_r'(x)=E(X^r)=w_{jklm}\left[\frac{\Gamma(r+m+1)}{(\theta(l+1))^{(r+m+1)}}+\frac{\Gamma(r+m+2)}{(\theta(l+1))^{(r+m+2)}}\right], \tag{16}$$

where w_{jklm} *is given by Equation* (9).

Proof. Using the pdf given in Equation (8),

$$\mu_r'(x)=E(X^r)=\int_0^{\infty}x^r f(x)dx=\int_0^{\infty}x^r w_{jklm}e^{-\theta(l+1)x}x^m(1+x)dx.$$

Therefore,

$$\mu'_r(x) = w_{jklm} \int_0^\infty x^{r+m}(1 + x)e^{-\theta(l+1)x} dx$$

$$= w_{jklm}\left[\int_0^\infty x^{r+m} e^{-\theta(l+1)x} dx + \int_0^\infty x^{r+m+1} e^{-\theta(l+1)x} dx\right].$$

Hence, we obtain the result given by Equation (16).

3.3. Moment generating function

In this subsection, we derive the moment generating function of EKL distribution.

Theorem 3. *If $X \sim EKL$, the moment generating function of X is then given as*

$$M_x(t) = w_{jklm}\left[\frac{\Gamma(m+1)}{(\theta(l+1)-t)^{(m+1)}} + \frac{\Gamma(m+2)}{(\theta(l+1)-t)^{(m+2)}}\right], \tag{17}$$

where w_{jklm} is given by Equation (9).

Proof. We start with the well-known definition of the moment generating function and using the pdf form in Equation (8)

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{EKL}(x) dx$$

$$= w_{jklm} \int_0^\infty x^m (1 + x)e^{-x[\theta(l+1)-t]} dx$$

$$= w_{jklm}\left[\frac{\Gamma(m+1)}{(\theta(l+1)-t)^{(m+1)}} + \frac{\Gamma(m+2)}{(\theta(l+1)-t)^{(m+2)}}\right],$$

which completes the proof.

4. Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is Rényi entropy [29]. If X has the pdf $f(.)$, then Rényi entropy is defined by

$$\Im_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\right\}, \tag{18}$$

here $\gamma > 0$ and $\gamma \neq 1$. Suppose $X \sim EKL$. Setting

$$A = \left(1 - e^{-\theta x}\left[1 + \frac{\theta x}{\theta + 1}\right]\right), \tag{19}$$

then one can calculate

$$\int f^{\gamma}(x) = \left(\frac{abc\theta^2}{\theta+1}\right)^{\gamma}\int_0^{\infty}(1+x)^{\gamma}e^{-\theta\gamma x}A^{a\gamma-\gamma}(1-A^a)^{b\gamma-\gamma}[1-(1-A^a)^b]^{c\gamma-\gamma}dx$$

$$= \left(\frac{abc\theta^2}{\theta+1}\right)^{\gamma}\sum_{i,j,k=0}^{\infty}\binom{b\gamma-\gamma}{i}\binom{c\gamma-\gamma}{j}\binom{bj}{k}$$

$$\times(-1)^{i+j+k}\int_0^{\infty}(1+x)^{\gamma}e^{-\theta\gamma x}A^{a\gamma-\gamma+ai+ak}dx$$

$$= \left(\frac{abc\theta^2}{\theta+1}\right)^{\gamma}\sum_{i,j,k,l=0}^{\infty}\binom{b\gamma-\gamma}{i}\binom{c\gamma-\gamma}{j}\binom{bj}{k}\binom{a(\gamma+i+k)-\gamma}{l}$$

$$\times(-1)^{i+j+k+l}\int_0^{\infty}(1+x)^{\gamma}e^{-\theta\gamma x}e^{-\theta lx}\left(1+\frac{\theta x}{\theta+1}\right)^{l}dx$$

$$= \left(\frac{abc\theta^2}{\theta+1}\right)^{\gamma}\sum_{i,j,k,l,m,n=0}^{\infty}\binom{b\gamma-\gamma}{i}\binom{c\gamma-\gamma}{j}\binom{bj}{k}\binom{a(\gamma+i+k)-\gamma}{l}$$

$$\times\binom{\gamma}{m}\binom{l}{n}(-1)^{i+j+k+l}\left(\frac{\theta}{\theta+1}\right)^{n}\int_0^{\infty}x^{m+n}e^{-\theta(\gamma+l)x}dx$$

$$= \left(\frac{abc\theta^2}{\theta+1}\right)^{\gamma} \sum_{i,j,k,l,m,n=0}^{\infty} \binom{b\gamma-\gamma}{i}\binom{c\gamma-\gamma}{j}\binom{bj}{k}\binom{a(\gamma+i+k)-\gamma}{l}$$

$$\times \binom{\gamma}{m}\binom{l}{n}(-1)^{i+j+k+l}\left(\frac{\theta}{\theta+1}\right)^{n}\left[\frac{\Gamma_{(m+n+1)}}{(\theta(\gamma+l))^{(m+n+1)}}\right].$$

Therefore, one obtain the Rényi entropy as

$$\Im_R(\gamma) = \frac{1}{1-\gamma}\log\left(\frac{abc\theta^2}{\theta+1}\right)$$

$$+ \frac{1}{1-\gamma}\log\left\{\sum_{i,j,k,l,m,n=0}^{\infty} \binom{b\gamma-\gamma}{i}\binom{c\gamma-\gamma}{j}\binom{bj}{k}\binom{a(\gamma+i+k)-\gamma}{l}\right.$$

$$\left.\times \binom{\gamma}{m}\binom{l}{n}(-1)^{i+j+k+l}\left[\frac{\Gamma_{(m+n+1)}}{(\theta(\gamma+l))^{(m+n+1)}}\right]\right\}. \tag{20}$$

Now, Shannon entropy [30] defined by $E[-\log f(x)]$ is the particular case of Equation (18) as $\gamma \uparrow 1$. From Equation (6) and (19),

$$E[-\log f(x)] = -\log\left(\frac{abc\theta^2}{\theta+1}\right) - E[\log(1+X)] + \theta E(X) - (a-1)E[\log(A)]$$

$$-(b-1)E[\log(1-A^{a})] - (c-1)E[\log(1-(1-A^{a})^{b})].$$

Using the series expansion defined in Equation (7.b) and

$$\log(1-z) = -\sum_{i=1}^{\infty}\frac{z^i}{i},\ -1 < z < 1, \tag{21}$$

$$E[\log(A)] = -\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{1}{r}\left(\frac{\theta}{\theta+1}\right)^{s}\binom{r}{s}E(X^{s}e^{-\theta r x}),$$

$$E[\log(1-A^{a})] = E\left[-\sum_{t=1}^{\infty}\frac{(A^{a})^{t}}{t}\right]$$

$$= -\sum_{t=1}^{\infty}\sum_{r,s=0}^{\infty}\binom{at}{r}\frac{(-1)^{r}}{t}\binom{r}{s}\left(\frac{\theta}{\theta+1}\right)^{s}E(X^{s}e^{-\theta rx}),$$

$$E[\log(1-(1-A^{a})^{b})] = E\left[-\sum_{u=1}^{\infty}\frac{(1-A^{a})^{bu}}{u}\right]$$

$$= -\sum_{u=1}^{\infty}\sum_{t,r,s=0}^{\infty}\binom{bu}{t}\binom{at}{r}\frac{(-1)^{t+r}}{u}\binom{r}{s}\left(\frac{\theta}{\theta+1}\right)^{s}E(X^{s}e^{-\theta rx}),$$

$$E[-\log f(x)] = -\log\left(\frac{abc\theta^{2}}{\theta+1}\right) - E[\log(1+X)] + \theta E(X) + (a-1)\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{1}{r}\left(\frac{\theta}{\theta+1}\right)^{s}$$

$$\times\binom{r}{s}E(X^{s}e^{-\theta rx}) + (b-1)\sum_{t=1}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{at}{r}\frac{(-1)^{r}}{t}\binom{r}{s}\left(\frac{\theta}{\theta+1}\right)^{s}E(X^{s}e^{-\theta rx})$$

$$+ (c-1)\sum\nolimits_{u=1}^{\infty}\sum\nolimits_{t,r,s=0}^{\infty}\binom{bu}{t}\binom{at}{r}\frac{(-1)^{t+r}}{u}\binom{r}{s}\left(\frac{\theta}{\theta+1}\right)^{s}E(X^{s}e^{-\theta rx}). \quad (22)$$

Now,

$$E(X) = w_{jklm}\left[\frac{\Gamma(m+2)}{(\theta(l+1))^{(m+2)}} + \frac{\Gamma(m+3)}{(\theta(l+1))^{(m+3)}}\right],$$

$$E[\log(1+X)] = E\left(\sum_{n=1}^{\infty}\frac{X^{n}}{n}\right) = -w_{jklm}\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n}$$

$$\times\left[\frac{\Gamma(m+n+1)}{(\theta(l+1))^{(m+n+1)}} + \frac{\Gamma(m+n+2)}{(\theta(l+1))^{(m+n+2)}}\right],$$

$$E(X^r e^{-\theta r x}) = w_{jklm}\left[\frac{\Gamma(m+s+1)}{(\theta(l+r+1))^{(m+s+1)}} + \frac{\Gamma(m+s+2)}{(\theta(l+r+1))^{(m+s+2)}}\right],$$

where w_{jklm} is given by Equation (9).

5. Estimation of the Parameters

Here, we consider estimating the parameters of the EKL distribution by the method of maximum likelihood and provide expressions for the associated expected information matrix. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $EKL(\xi)$, where $\xi = (a, b, c, \theta)$. The log likelihood function of the parameters is given by

$$\ell = \log L = n\log a + n\log b + n\log c + 2n\log\theta - n\log(\theta+1)$$

$$+\sum_{i=1}^{n}\log(1+x_i) - \theta\sum_{i=1}^{n}x_i + (a-1)\sum_{i=1}^{n}\log(A) + (b-1)\sum_{i=1}^{n}\log(1-A^a)$$

$$+(c-1)\sum_{i=1}^{n}\log[1-(1-A^a)^b], \tag{23}$$

where A is defined in Equation (19).

The components of the score vector $(\xi) = (\partial\ell/\partial a, \partial\ell/\partial b, \partial\ell/\partial c, \partial\ell/\partial\theta)^T$ are obtained by differentiating Equation (23) with respect to the different parameters. Thus, we have

$$\frac{\partial\log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log(A) - (b-1)\sum_{i=1}^{n}\frac{A^a\log A}{1-A^a} + (c-1)\sum_{i=1}^{n}\frac{(1-A^a)^{b-1}A^a\log A}{[1-(1-A^a)^b]};$$

$$\frac{\partial\log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log(1-A^a) - (c-1)\sum_{i=1}^{n}\frac{(1-A^a)^b\log(1-A^a)}{[1-(1-A^a)^b]};$$

$$\frac{\partial\log L}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\log[1-(1-A^a)^b];$$

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta + 1} - \sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n} \frac{A'}{A} - a(b-1)\sum_{i=1}^{n} \frac{A^{a-1}A'}{1 - A^a}$$

$$+ ab(c-1)\sum_{i=1}^{n} \frac{(1 - A^a)^{b-1} A^{a-1} A'}{[1 - (1 - A^a)^b]},$$

where $A = \partial A / \partial \theta = x_i e^{-\theta x_i}\left[1 + \frac{\theta x_i}{\theta + 1} - \frac{1}{(\theta + 1)^2}\right].$ (24)

By equating each of the four equations to zero, the maximum likelihood estimates (MLEs) of the parameters are the solution of this system of nonlinear equations, which are solved iteratively. The observed information matrix given by

$$J_n(\xi) = -\begin{bmatrix} J_{aa} & J_{ab} & J_{ac} & J_{a\theta} \\ J_{ba} & J_{bb} & J_{bc} & J_{b\theta} \\ J_{ca} & J_{cb} & J_{cc} & J_{c\theta} \\ J_{\theta a} & J_{\theta b} & J_{\theta c} & J_{\theta\theta} \end{bmatrix},$$

where

$$J_{aa} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n} \frac{A^a \log^2 A}{(1 - A^a)^2} + b(c-1)\sum_{i=1}^{n} \frac{(1 - A^a)^{b-1} A^a \log^2 A}{[1 - (1 - A^a)^b]^2}$$

$$\times \left[1 - (1 - A^a)^b - (b-1)A^a(1 - A^a)^{-1} - (1 - A^a)^{b-1} A^a\right];$$

$$J_{ab} = -\sum_{i=1}^{n} \frac{A^a \log A}{1 - A^a} + (c-1)$$

$$\times \sum_{i=1}^{n} \frac{(1 - A^a)^{b-1} A^a \log A[1 - (1 - A^a)^b + b\log(1 - A^a)]}{[1 - (1 - A^a)^b]^2};$$

$$J_{ac} = b\sum_{i=1}^{n} \frac{(1 - A^a)^{b-1} A^a \log A}{[1 - (1 - A^a)^b]};$$

$$J_{a\theta} = \sum_{i=1}^{n} \frac{A'}{\theta} - (b-1)\sum_{i=1}^{n} \frac{A^{a-1}A'[1 - A^{a} + a\log A]}{(1-A^{a})^2}$$

$$+ b(c-1)\sum_{i=1}^{n} \frac{(1-A^{a})^{b-1}A^{a-1}A'}{[1-(1-A^{a})^{b}]^2}$$

$$\times [1 + a\log A - a(b-1)(1-A^{a})^{-1}A^{a}\log A - (1-A^{a})^{b} - a(1-A^{a})^{b}$$

$$\times \log A - a(1-A^{a})^{b-1}A^{a}\log A];$$

$$J_{bb} = -\frac{n}{b^2} - (c-1)\sum_{i=1}^{n} \frac{(1-A^{a})^{b}\log^2(1-A^{a})}{[1-(1-A^{a})^{b}]^2};$$

$$J_{bc} = -\sum_{i=1}^{n} \frac{(1-A^{a})^{b}\log(1-A^{a})}{[1-(1-A^{a})^{b}]};$$

$$J_{b\theta} = -a\sum_{i=1}^{n} \frac{A^{a-1}A'}{1-A^{a}} + a(c-1)\sum_{i=1}^{n} \frac{(1-A^{a})^{b-1}A^{a-1}A'[1-(1-A^{a})^{b} + b\log(1-A^{a})]}{[1-(1-A^{a})^{b}]^2};$$

$$J_{cc} = -\frac{n}{c^2};$$

$$J_{c\theta} = ab\sum_{i=1}^{n} \frac{(1-A^{a})^{b-1}A^{a-1}A'}{[1-(1-A^{a})^{b}]};$$

$$J_{\theta\theta} = -\frac{2n}{\theta^2} + \frac{n}{(\theta+1)^2} + (a-1)\sum_{i=1}^{n} \frac{AA'' - A'^2}{A^2} - a(b-1)\sum_{i=1}^{n} \frac{1}{[1-A^{a}]^2}$$

$$\times \left[A^{a-1}A'' - A^{2a-1}A'' + (a-1)A^{a-2}A'^2 + A^{2a-2}A'^2\right] + ab(c-1)$$

$$\sum_{i=1}^{n} \frac{A^{a-1}(1-A^{a})^{b-1}}{[1-(1-A^{a})^{b}]^2}[A'' - (1-A^{a})^{b}A'' + (a-1)A^{-1}A'^2 - (a-1)$$

$$(1-A^{a})^{b}A^{a-2}A'^2 - a(b-1)A^{a-1}(1-A^{a})^{-1}A'^2 - aA^{a-1}(1-A^{a})^{b-1}A'^2],$$

where $A'' = \frac{\partial A'}{\partial \theta}$.

Applying the usual large sample approximation, the ML estimates $\hat{\xi} = (\hat{a}, \hat{b}, \hat{c}, \hat{\theta})$ can be treated as being approximately $N_4(\xi, I_n^{-1}(\xi))$, where $I_n(\xi) = E[J_n(\xi)]$ under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary. The asymptotic distribution $\sqrt{n}(\hat{\xi} - \xi) \to N_4(0, I_n^{-1}(\xi))$.

Approximate two sided $(1 - \alpha)\,\%$ confidence intervals for a, b, c, and θ, respectively, given by

$$\hat{a} \pm z_{\alpha/2}\sqrt{I_{aa}^{-1}(\xi)}, \quad \hat{b} \pm z_{\alpha/2}\sqrt{I_{bb}^{-1}(\xi)}, \quad \hat{c} \pm z_{\alpha/2}\sqrt{I_{cc}^{-1}(\xi)},$$

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{I_{\theta\theta}^{-1}(\xi)},$$

where z_α is the upper $100\,\alpha$-th percentile of the standard normal distribution.

The elements of the expected Fisher information matrix are

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = \frac{n}{a^2} + \left[\frac{1}{a} bc(b-1)\sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^j \binom{c-1}{j} \beta(2,\, b + bj + k + l - 2)\right]$$

$$- \left[\frac{1}{a} b^2 c(c-1)\sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^j \binom{c-3}{j}\left(\frac{1}{k+l}\right)\right.$$

$$\{\beta(2,\, 2b + bj + k + l + 1) - \beta(2,\, 3b + bj + k + l - 1)$$

$$\left. - (b-1)\beta(3,\, 2b + bj + k + l - 2) - \beta(3,\, 3b + bj + k + l - 2)\}\right],$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) = -\frac{1}{a} bc\sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(-1)^j \binom{c-1}{j}\left(\frac{1}{k}\right)\beta(2,\, b + bj + n - 1)$$

$$-\left[\frac{1}{a} bc(c-1)\sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(-1)^j \binom{c-3}{j}\left\{-\left(\frac{1}{k}\right)\beta(2,\, 2b + bj + k - 1)\right.\right.$$

$$\left.\left. + \left(\frac{1}{k}\right)\beta(2,\, 3b + bj + k - 2) + \sum_{l=1}^{\infty}\left(\frac{1}{k+l}\right)\beta(k + 2,\, 2b + bj + l - 1)\right\}\right],$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial c}\right) = -\frac{1}{a} b^2 c \sum_{i=1}^{n} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} (-1)^j \binom{c-2}{j} \left(\frac{1}{k}\right) \beta(2,\ 2b + bj + k - 1),$$

$$\left(-\frac{\partial^2 \log L}{\partial a \partial \theta}\right) = abc \sum_{i=1}^{n} \sum_{j,k,l,m=0}^{\infty} \binom{l}{m} \left(\frac{\theta}{\theta+1}\right)^m (-1)^{j+k+l}$$

$$\times \left[\left\{\left(\frac{\Gamma_{(m+2)}}{(\theta(l+2))^{(m+2)}} + \frac{\Gamma_{(m+3)}}{(\theta(l+2))^{(m+3)}}\right)\right.\right.$$

$$\left.+ \frac{\theta^3}{(\theta+1)^2}\left(\frac{\Gamma_{(m+3)}}{(\theta(l+2))^{(m+3)}} + \frac{\Gamma_{(m+4)}}{(\theta(l+2))^{(m+4)}}\right)\right\}$$

$$\left.- \left\{\frac{\theta^2}{(\theta+1)^3}\left(\frac{\Gamma_{(m+2)}}{(\theta(l+2))^{(m+2)}} + \frac{\Gamma_{(m+3)}}{(\theta(l+2))^{(m+3)}}\right)\right\}\right]$$

$$\times \left[\left\{\binom{c-1}{j}\binom{b(j+1)-1}{k}\binom{a(k+1)-1}{l}\right\}\right.$$

$$+ \left\{(b-1)\binom{c-1}{j}\left(\left\{\binom{b(j+1)-3}{k}\binom{a(k+2)-2}{l}\right\}\right.\right.$$

$$- \left\{\binom{b(j+1)-3}{k}\binom{a(k+3)-2}{l}\right\}$$

$$\left.\left.- \left\{\sum_{i=1}^{n}\binom{b(j+1)+n-3}{k}\binom{a(k+2)-2}{l}\left(\frac{1}{n}\right)\right\}\right)\right\}$$

$$- \left\{b(c-1)\binom{c-3}{j}\left(\left\{\binom{b(j+2)-2}{k}\binom{a(k+2)-2}{l}\right\}\right.\right.$$

$$- \left\{\sum_{n=1}^{\infty}\binom{b(j+2)+n-2}{k}\binom{a(k+2)-2}{l}\binom{l}{m}\left(\frac{1}{n}\right)\right\}$$

$$+\left\{(b-1)\sum_{n=1}^{\infty}\binom{b(j+2)+n-3}{k}\binom{a(k+3)-2}{l}\left(\frac{1}{n}\right)\right\}$$

$$-\left\{\binom{b(j+3)-2}{k}\binom{a(k+2)-2}{l}\right\}$$

$$+\left\{\sum_{n=1}^{\infty}\binom{b(j+3)+n-2}{k}\binom{a(k+2)-2}{l}\left(\frac{1}{n}\right)\right\}$$

$$+\left\{\sum_{n=1}^{\infty}\binom{b(j+3)+n-3}{k}\binom{a(k+3)-2}{l}\left(\frac{1}{n}\right)\right\}\Bigg\}\Bigg\}\Bigg];$$

$$E\left(-\frac{\partial^2 \log L}{\partial b^2}\right)=\frac{n}{b^2}+bc(c-1)\sum_{n=1}^{\infty}\sum_{j,k,l=0}^{\infty}\binom{c-3}{j}\left(\frac{1}{k+l}\right)(-1)^j\beta(k+l+1,\, b(j+2)),$$

$$E\left(-\frac{\partial^2 \log L}{\partial b\partial c}\right)=-abc\sum_{n=1}^{n}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\binom{c-2}{j}\left(\frac{1}{n}\right)(-1)^j\beta(n+1,\, b(j+2)),$$

$$E\left(-\frac{\partial^2 \log L}{\partial b\partial\theta}\right)=abc\sum_{n=1}^{n}\sum_{j,k,l,m=0}^{\infty}\binom{l}{m}\left(\frac{\theta}{\theta+1}\right)^m(-1)^{j+k+l}$$

$$\times\Bigg[\Bigg\{\frac{\theta^2}{\theta+1}\left(\frac{\Gamma(m+2)}{(\theta(l+2))^{(m+2)}}+\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}\right)$$

$$+\frac{\theta^3}{(\theta+1)^2}\left(\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}+\frac{\Gamma(m+4)}{(\theta(l+2))^{(m+4)}}\right)\Bigg\}$$

$$-\left\{\frac{\theta^2}{(\theta+1)^3}\left(\frac{\Gamma(m+2)}{(\theta(l+2))^{(m+2)}}+\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}\right)\right\}\Bigg]$$

$$\times\Bigg[\Bigg\{\binom{c-1}{j}\binom{b(j+1)-2}{k}\binom{a(k+2-2)}{l}\Bigg\}$$

$$-\left\{a(c-1)\left(\binom{c-3}{j}\binom{b(j+2)-2}{k}\binom{a(k+2)-2}{l}\right.\right.$$

$$-\binom{b(j+3)-2}{k}\binom{a(k+2-2)}{l}$$

$$\left.\left.-b\sum_{n=1}^{n}\binom{b(j+2)-2}{k}\binom{a(k+2)+n-2}{l}\right)\right\}\Bigg];$$

$$E\left(-\frac{\partial^2 \log L}{\partial c^2}\right)=\frac{n}{c^2},$$

$$E\left(-\frac{\partial^2 \log L}{\partial c \partial \theta}\right)=a^2b^2c\sum_{l=1}^{n}\sum_{j,k,l,m=0}^{\infty}\binom{c-2}{j}\binom{b(j+2)-2}{k}\binom{a(k+2)-2}{l}\binom{l}{m}$$

$$\times\left(\frac{\theta}{\theta+1}\right)^m(-1)^{j+k+l}\left[\left\{\frac{\theta^2}{\theta+1}\left(\frac{\Gamma(m+2)}{(\theta(l+2))^{(m+2)}}+\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}\right)\right.\right.$$

$$\left.+\frac{\theta^3}{(\theta+1)^2}\left(\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}+\frac{\Gamma(m+4)}{(\theta(l+2))^{(m+4)}}\right)\right\}$$

$$\left.-\left\{\frac{\theta^2}{(\theta+1)^3}\left(\frac{\Gamma(m+2)}{(\theta(l+2))^{(m+2)}}+\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}\right)\right\}\right];$$

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)=\frac{2n}{\theta^2}-\frac{n}{(\theta+1)^2}+abc\sum_{i=1}^{n}\sum_{j,k,l,m=0}^{\infty}\binom{l}{m}\left(\frac{\theta}{\theta+1}\right)^m(-1)^{j+k+l}$$

$$\times\left[-\left\{(a-1)\binom{c-1}{j}\binom{b(j+1)-1}{k}\left(\binom{a(k+1)-2}{l}\tau-\binom{a(k+1)-3}{l}\omega\right)\right\}\right.$$

$$+\left\{a(b-1)\binom{c-1}{j}\binom{b(j+1)-3}{k}\left(\binom{a(k+2)-1}{l}\tau-\binom{a(k+3)-2}{l}\tau\right.\right.$$

$$+ (a-1)\binom{a(k+2)-3}{l}\omega + \binom{a(k+3)-3}{l}\omega\Bigg)\Bigg\}$$

$$- \left\{ab(c-1)\binom{c-3}{j}\left(\binom{b(j+2)-2}{k}\binom{a(k+2)-2}{l}\tau\right.\right.$$

$$-\binom{b(j+3)-2}{k}\binom{a(k+2)-2}{l}\tau + (a-1)\binom{b(j+2)-2}{k}\binom{a(k+2)-3}{l}\omega$$

$$-(a-1)\binom{b(j+3)-2}{k}\binom{a(k+2)-3}{l}\omega$$

$$-a(b-1)\binom{b(j+2)-3}{k}\binom{a(k+3)-3}{l}\omega$$

$$\left.\left.-a\binom{b(j+3)-3}{k}\binom{a(k+3)-3}{l}\omega\right)\right\}\Bigg],$$

where

$$\tau = \left\{\frac{2\theta^2}{(\theta+1)^4}\left(\frac{\Gamma(m+2)}{(\theta(l+2))^{(m+2)}} + \frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}\right)\right\}$$

$$+\left\{\frac{2}{(\theta+1)^2}\left(\frac{\theta^2}{\theta+1}\left\{\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}}\right\} + \frac{\Gamma(m+4)}{(\theta(l+2))^{(m+4)}}\right)\right\}$$

$$-\left\{\frac{\theta}{\theta+1}\left(\frac{\theta^2}{\theta+1}\left\{\frac{\Gamma(m+4)}{(\theta(l+2))^{(m+4)}}\right\} + \frac{\Gamma(m+5)}{(\theta(l+2))^{(m+5)}}\right)\right\}$$

$$-\left\{\frac{\theta^2}{\theta+1}\left(\frac{\Gamma(m+3)}{(\theta(l+2))^{(m+3)}} + \frac{\Gamma(m+4)}{(\theta(l+2))^{(m+4)}}\right)\right\};$$

$$\omega = \left\{\frac{\theta^2}{\theta+1}\left(\frac{\Gamma(m+3)}{(\theta(l+3))^{(m+3)}} + \frac{\Gamma(m+4)}{(\theta(l+3))^{(m+4)}}\right)\right\}$$

$$+ \left\{\frac{\theta^2}{(\theta+1)^5}\left(\frac{\Gamma(m+3)}{(\theta(l+3))^{(m+3)}} + \frac{\Gamma(m+4)}{(\theta(l+3))^{(m+4)}}\right)\right\}$$

$$- \left\{\frac{2\theta^3}{(\theta+1)^4}\left(\frac{\Gamma(m+4)}{(\theta(l+3))^{(m+4)}} + \frac{\Gamma(m+5)}{(\theta(l+3))^{(m+5)}}\right)\right\}$$

$$+ \left\{\frac{\theta^4}{(\theta+1)^3}\left(\frac{\Gamma(m+5)}{(\theta(l+3))^{(m+5)}} + \frac{\Gamma(m+6)}{(\theta(l+3))^{(m+6)}}\right)\right\}$$

$$- \left\{\frac{2\theta^2}{(\theta+1)^3}\left(\frac{\Gamma(m+3)}{(\theta(l+3))^{(m+3)}} + \frac{\Gamma(m+4)}{(\theta(l+3))^{(m+4)}}\right)\right\}$$

$$+ \left\{\frac{2\theta^3}{(\theta+1)^2}\left(\frac{\Gamma(m+4)}{(\theta(l+3))^{(m+4)}} + \frac{\Gamma(m+5)}{(\theta(l+3))^{(m+5)}}\right)\right\}.$$

6. Numerical Application

In this section, we illustrate the usefulness of the EKL distribution.

6.1. Numerical example

A real data set is used to show that the EKL distribution can be a better model than the Lindley distribution.

The data set given in Table 1 represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [31].

Table 1. The remission times (in months) of bladder cancer patients

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
0.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
0.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
0.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
0.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
0.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
0.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
0.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
0.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
0.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85
0.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02
0.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
0.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49		

The maximum likelihood method is applied to estimate the parameters of the three models Lindley (L), Kumaraswamy Lindley (KL), and exponentiated Kumaraswamy Lindley (EKL) distributions. The resulting estimates with the negative of the likelihood function $(-\ell)$.

Table 2. The ML estimated and Log-likelihood for data set

Model	ML estimates	$-\ell$
L	$\hat{\theta} = 0.196$	419.529
KL	$\hat{\theta} = 0.503$	414.229
	$\hat{a} = 0.978$	
	$\hat{b} = 0.280$	
EKL	$\hat{\theta} = 0.120$	411.015
	$\hat{a} = 4.479$	
	$\hat{b} = 1.768$	
	$\hat{c} = 0.121$	

The variance covariance matrix $I(\hat{\xi})^{-1}$ of the MLEs under the EKL distribution for the data set is computed as

$$\begin{pmatrix} 1.0928 & 0.0307 & -0.0329 & -0.0002 \\ 0.0307 & 0.5087 & -0.0093 & -0.0163 \\ -0.0329 & -0.0093 & 0.0009 & 0.0002 \\ -0.0002 & -0.0163 & 0.0002 & 0.0004 \end{pmatrix}.$$

Thus $\text{var}(\hat{a}) = 1.0928, \text{var}(\hat{b}) = 0.5087, \text{var}(\hat{c}) = 0.0009,$ and $\text{var}(\hat{\theta}) = 0.0004.$ Therefore, 95% confidence intervals for a, b, c, and θ are [2.4300, 6.5280], [0.3701, 3.1659], [0.0818, 0.1602], and [0.0808, 0.1592], respectively.

The LR test statistic to test the hypotheses $H_0 : a = b = c = 1$ versus $H_1 = a \neq 1 \vee b \neq 1 \vee c \neq 1$ is $\omega = 17.028 > 7.815 = \chi^2_{3, 0.05}$, so we reject the null hypothesis.

The values of the parameters' estimates are used to plot the pdf and the cdf for the three distributions L, KL, and EKL in Figure 4 and Figure 5.

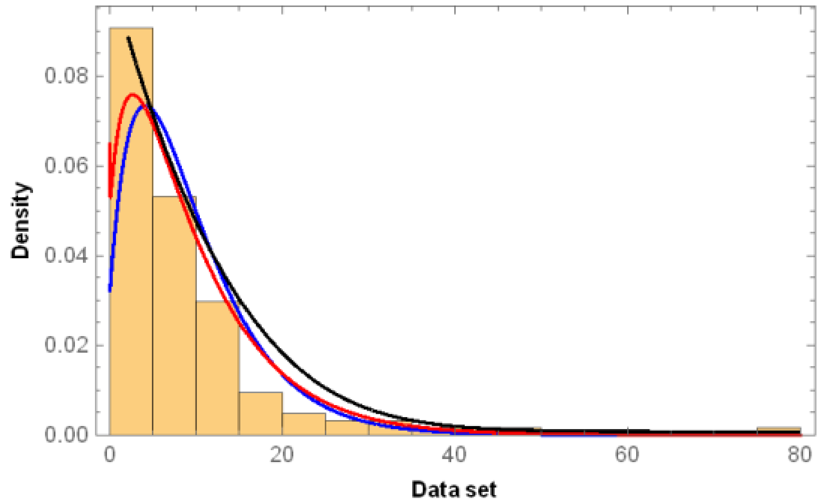


Figure 4. Estimated densities of the models for data set.

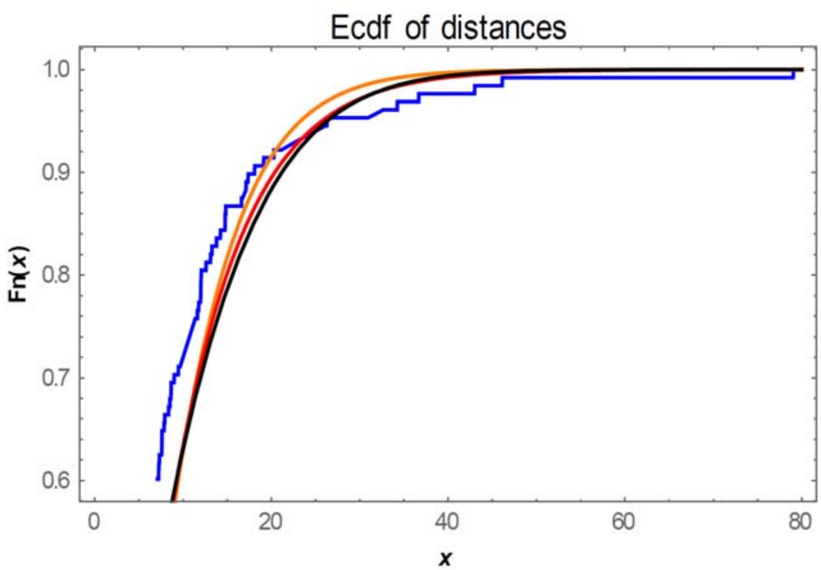


Figure 5. Estimated cumulative densities of the models for data set.

Table 3. Criteria comparison

Model	-2ℓ	AIC	AICC	BIC
Lindley	839.04	841.06	841.091	843.892
Kum-Lindley	828.458	834.458	834.651	843.014
ExpKum-Lindley	822.03	830.03	830.355	841.438

In order to compare the three distributions, we consider criteria like -2ℓ, AIC (Akaike information criterion), AICC (corrected Akaike information criteria), and BIC (Bayesian information criterion), for the data set. As shown in Table 3 the better distribution corresponds to smaller -2ℓ, AIC, AICC, and BIC values, where

$$AIC = 2K - 2\ell; \quad AICC = AIC + \frac{2k(k+1)}{n-k-1};$$

$$BIC = k^* \log(n) - 2\ell.$$

Here k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of log-likelihood function under the considered model. Table 2 shows the MLEs under the three distribution, Table 3 shows the values of -2ℓ, AIC, AICC, and BIC, values. Table 3 indicates that the EKL distribution leads to a better fit than the Lindley and KL distribution.

6.2. Simulation study

We conducted Monte Carlo simulation studies to assess on the finite sample behavior of the MLEs of a, b, c, and θ. All results were obtained from 1000 Monte Carlo replications simulations. The EKL random number generation was performed using the inversion method. In each replication, a random sample of size n is drawn from the $EKL(a, b, c, \theta)$ distribution and the maximum likelihood estimates of the parameters were obtained. The mean, bias and mean squared error (MSE) for each parameter was computed under different sample sizes $n = 25, 50, 75, 100,$ and 200.

Table 4. Mean estimates, biases and mean square errors of the MLEs of $a = 0.2$, $b = 2$, $c = 2$, $\theta = 1$

n	Parameter	Mean	Bias	MSE
25	a	0.1774	– 0.0226	0.0050
	b	2.6447	0.6447	1.0902
	c	4.5051	2.5051	7.4921
	θ	0.9342	– 0.0658	0.0257
50	a	0.1566	– 0.0434	0.0033
	b	1.5234	– 0.4766	0.2317
	c	2.5142	0.5142	0.6204
	θ	1.2577	0.2577	0.0684
75	a	0.1394	– 0.0606	0.0039
	a	1.5260	– 0.4740	0.2278
	c	3.1250	1.1250	1.3893
	θ	1.4350	0.4350	0.1930
100	a	0.1941	– 0.0059	0.0001
	b	1.1362	– 0.8639	0.7476
	c	1.5537	– 0.4463	0.2288
	θ	2.2233	1.2233	1.5049
200	a	0.1630	– 0.0370	0.0022
	b	1.9876	– 0.0124	0.0125
	c	3.1817	1.1817	1.7684
	θ	1.2877	0.2877	0.0868

Table 5. Mean estimates, biases and mean square errors of the MLEs of $a = 3$, $b = 5$, $c = 2$, $\theta = 0.5$

n	Parameter	Mean	Bias	MSE
25	a	2.5531	– 0.4469	0.3169
	b	5.6413	0.6413	0.6218
	c	4.2663	2.2663	5.4789
	θ	0.4968	– 0.0032	0.0015
50	a	3.6644	0.6644	0.4419
	b	4.8606	– 0.1394	0.0203
	c	3.7244	1.7244	2.9783
	θ	0.6071	0.1071	0.0115
75	a	3.7179	0.7179	0.5429
	b	5.2223	0.2223	0.0515
	c	3.8310	1.8310	3.4088
	θ	0.5958	0.0958	0.0092
100	a	4.9931	1.9931	7.2825
	b	2.6961	– 2.3039	5.3230
	c	3.4341	1.4341	3.5870
	θ	0.8422	0.3422	0.1208
200	a	2.7023	– 0.2977	2.7909
	b	6.8421	1.8421	10.2354
	c	3.9669	1.9669	7.8353
	θ	0.4542	– 0.0458	0.4563

7. Conclusion

We proposed a new distribution, named the EKL distribution which extends the KL distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modelling real data. We have derived various properties of the new distribution, including the moment, moment generating function, and quantile function. Rényi and Shannon entropies have been obtained. The estimation of the model parameters is approached by maximum likelihood and the observed information matrix is derived. Finally, an application to real data set shows that the fit of the new model is superior to the fits of its main sub-models. We hope that the proposed model may attract wider applications in statistics.

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