EXTENSIONS OF MODULES OVER POLYNOMIAL RINGS

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Abstract

For a commutative and unitary ring $R$, we characterize the classes of a classical equivalence relation defined on extensions of an $R[X]$-module, which is $R$-projective by another $R[X]$-module. The modules that are extensions of an $R[X]$-module $M_2$, which is $R$-projective by an $R[X]$-module $M_1$ have a fairly simple form which we denote by $M_1 \times_f M_2$, where $f$ is an $R$-homomorphism from $M_2$ to $M_1$. For $R$, $M_1$, $M_2$ and $f$ as in the foregoing, but $M_2$ is not necessarily $R$-projective, we search to classify the $R[X]$-modules $M_1 \times_f M_2$. 

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1. Introduction

All rings considered in this paper are supposed to be with unit. Let $R$ be a commutative ring. Let $N$ and $L$ be two $R$-modules. We recall (see [1, 7, 9]) that an extension of $L$ by $N$ is a triple $(u, M, v)$, where $M$ is an $R$-module and $u, v$ are $R$-module homomorphisms such that the sequence

$$O \rightarrow N \xrightarrow{u} M \xrightarrow{v} L \rightarrow O$$

is exact. Two extensions $(u, M, v)$ and $(u', M', v')$ of $L$ by $N$ are equivalent if there exists an isomorphism of $R$-modules $\varphi : M \rightarrow M'$ such that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & N & \xrightarrow{u} & M & \xrightarrow{v} & L & \rightarrow & 0 \\
& & \downarrow{1_N} & & \downarrow{\varphi} & & \downarrow{1_L} & & \\
0 & \rightarrow & N & \xrightarrow{u'} & M' & \xrightarrow{v'} & L & \rightarrow & 0.
\end{array}
$$

For this equivalence relation, the class of an extension $(u, M, v)$ of $L$ by $N$ is denoted by $\overline{(u, M, v)}$.

On the group rings, one of the technics used to study the module extensions consists to show that a module, which is extension of a module $L$ by another module $N$ - under some conditions - can be written in a simple form that is denoted by $N \times_{\alpha} L$, where $\alpha$ is a cocycle or a derivation (see, for example, [2], [3], [8]). For the polynomial rings, this form is much simpler, indeed instead of cocycles or derivations we can take homomorphisms of $R$-modules.

The polynomial ring with coefficients in $R$ will be denoted by $R[X]$.

In this paper, we study the extensions of $R[X]$-modules. We search essentially to characterize these extensions. In Section 2, we characterize the classes of this equivalence relation defined on extensions of an $R[X]$-modules $M_2$, which is $R$-projective by an $R[X]$-module $M_1$. As we
already said, the modules that are extensions of an $R[X]$-module $M_2$, which is $R$-projective by an $R[X]$-modules $M_1$ have a fairly simple form which we denote by $M_1 \times_f M_2$, where $f$ is an $R$-homomorphism from $M_2$ to $M_1$. In Section 3, for $M_1$, $M_2$ and $f$ as in the foregoing, but $M_2$ is not necessarily $R$-projective, we search to classify the $R[X]$-modules $M_1 \times_f M_2$. In the same time, we give some properties of these modules.

### 2. Extension Equivalence of Modules

**Over Polynomial Rings**

Let $R$ be a commutative ring. Let $M_1$ and $M_2$ be two $R[X]$-modules.

Let $f \in \text{Hom}_R(M_2, M_1)$, where $\text{Hom}_R(M_2, M_1)$ is the set of all $R$-module morphisms from the $R$-module $M_2$ to the $R$-module $M_1$. Then, we can define a structure of $R[X]$-module on $M_1 \times M_2$ by $\forall (m_1, m_2) \in M_1 \times M_2$, $X(m_1, m_2) = (X.m_1 + f(m_2), X.m_2)$. The module $M_1 \times M_2$ equipped with this structure of $R[X]$-module will be denoted by $M_1 \times_f M_2$. In this case, the following sequence $O \rightarrow M_1 \overset{i_1}{\rightarrow} M_1 \times_f M_2 \overset{p_2}{\rightarrow} M_2 \rightarrow O$, is an exact sequence of $R[X]$-modules, where $i_1$ is the first injection and $p_2$ is the second projection.

**Theorem 2.1.** Let $O \rightarrow M_1 \overset{u}{\rightarrow} M \overset{v}{\rightarrow} M_2 \rightarrow O$ be an exact sequence of $R[X]$-modules. If $M_2$ is $R$-projective, then there exists $f \in \text{Hom}_R(M_2, M_1)$ such that $(u, M, v) = (i_1, M_1 \times_f M_2, p_2)$.

**Proof.** $O \rightarrow M_1 \overset{u}{\rightarrow} M \overset{v}{\rightarrow} M_2 \rightarrow O$ is exact and $M_2$ is projective as $R$-module. Then, there exist $v' \in \text{Hom}_R(M_2, M)$ and $u' \in \text{Hom}_R(M, M_1)$ such that $u'v = id_{M_1}$ and $v'u = id_{M_2}$. It is well-known that $0 : M \rightarrow M_1 \oplus M_2$ defined by $m \mapsto (u'(m), v(m))$ is an isomorphism of
$R$-modules, and $\theta^{-1} : M_1 \oplus M_2 \to M$ is defined by $(m_1, m_2) \mapsto m + u(x)$, where $m$ is such that $v(m) = m_2$ and $x \in M_1$ is such that $x = m_1 - u'(m)$.

We define a structure of $R[X]$-module on $M_1 \oplus M_2$ by

$$\forall (m_1, m_2) \in M_1 \oplus M_2, \ X(m_1, m_2) = \theta(X.\theta^{-1}((m_1, m_2))).$$

Then,

$$X(m_1, m_2) = \theta(X.\theta^{-1}((m_1, m_2))) = \theta(X(m + u(x))) = \theta(X.m + X.u(x)) = (u'(X.m + X.u(x)), v(X.m + X.u(x))) = (u'(X.m) + X.x, X.m_2) = (X.m_1 + u'(X.m) - X.u'(m), X.m_2).$$

We have $v(m) = m_2$, so there exists $m' \in \ker(v)$ such that $v'(m_2) = m + m'$. As $\ker(v) = \Im(u)$, then there exists $m'_2 \in M_1$ such that $v'(m_2) = m + u(m'_2)$. So,

$$u'(X.m) - X.u'(m) = u'(X.(v'(m_2) - u(m'_2))) - X.u'(v'(m_2) - u(m'_2)) = u'(X.v'(m_2)) - u'(X.u(m'_2)) - X.(u'o(u'(m'_2))) + X.(u'o(u'(m'_2))) = u'(X.v'(m_2)) - X.(u'o(u'(m'_2))).$$

Therefore,

$$X(m_1, m_2) = (X.m_1 + u'(X.v'(m_2)) - X.(u'o(u'(m'_2))), X.m_2) = (X.m_1 + f(m_2), X.m_2),$$

where $f \in \text{Hom}_R(M_2, M_1)$ is defined by

$$\forall m_2 \in M_2, \ f(m_2) = u'(X.v'(m_2)) - X.(u'o(u'(m'_2))).$$

$\theta : M \to M_1 \times_f M_2$ is an isomorphism of $R[X]$-modules and we have $\theta u = i_1$ and $p_2 \theta = v$.
Theorem 2.2. Let \( f, g \in \text{Hom}_R(M_2, M_1) \), where \( M_1 \) and \( M_2 \) are two \( R[X] \)-modules. Then, the following conditions are equivalent:

1. \((i_1, M_1 \times_f M_2, p_2) = (i_1, M_1 \times_g M_2, p_2)\).

2. There exists \( h \in \text{Hom}_R(M_2, M_1) \) such that, for all \( m_2 \in M_2 \),
   \[
   f(m_2) + h(Xm_2) = Xh(m_2) + g(m_2).
   \]

Proof. (1) \(\Rightarrow\) (2): If \((i_1, M_1 \times_f M_2, p_2) = (i_1, M_1 \times_g M_2, p_2)\), then there exists a homomorphism of \( R[X] \)-modules \( \varphi : M_1 \times_f M_2 \to M_1 \times_g M_2 \) such that \( \varphi i_1 = i_1 \) and \( p_2 \varphi = p_2 \). Let \((m_1, m_2) \in M_1 \times M_2 \). We have

\[
\varphi((m_1, 0)) = \varphi i_1(m_1) = i_1(m_1) = (m_1, 0).
\]

We put \( \varphi((0, m_2)) = (m_1', m_2') \). As \( p_2 \varphi = p_2 \), then \( m_2' = m_2 \). We define an application \( h : M_2 \to M_1 \) by \( h(m_2) = m_1' \). Then, we have \( \varphi((m_1, m_2)) = (m_1 + h(m_2), m_2) \). We first show that \( h \) is a homomorphism of \( R \)-modules. Let \( m_2, m_2' \in M_2 \) and let \( r \in R \). \( \varphi((0, m_2 + m_2')) = (h(m_2 + m_2'), m_2 + m_2') \). But,

\[
\varphi((0, m_2 + m_2')) = \varphi((0, m_2)) + \varphi((0, m_2'))
= (h(m_2), m_2) + (h(m_2'), m_2')
= (h(m_2) + h(m_2'), m_2 + m_2').
\]

So, \( h(m_2 + m_2') = h(m_2) + h(m_2') \). We have also \( \varphi((0, r.m_2)) = (h(r.m_2), r.m_2) \). But,

\[
\varphi((0, r.m_2)) = \varphi(r(0, m_2)) = r\varphi((0, m_2)) = r(h(m_2), m_2) = (r.h(m_2), r.m_2).
\]
So, $h(r.m_2) = r.h(m_2)$. Therefore, $h \in \text{Hom}_R(M_2, M_1)$. Let $m_2 \in M_2$.

\[ \varphi(X(0, m_2)) = X.\varphi((0, m_2)). \]

But,

\[ \varphi(X(0, m_2)) = \varphi(f(m_2), X.m_2) \]

\[ = (f(m_2) + h(X.m_2), X.m_2), \]

and

\[ X.\varphi((0, m_2)) = X.(h(m_2), m_2) \]

\[ = (X.h(m_2), g(m_2), X.m_2). \]

Therefore, $\forall m_2 \in M_2$, $f(m_2) + h(X.m_2) = X.h(m_2) + g(m_2)$.

(2) $\Rightarrow$ (1): Assume that there exists $h \in \text{Hom}_R(M_2, M_1)$ such that, $\forall m_2 \in M_2$, $f(m_2) + h(X.m_2) = X.h(m_2) + g(m_2)$. We define an application $\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2$ by $\forall (m_1, m_2) \in M_1 \times_f M_2$, $\varphi((m_1, m_2)) = (m_1 + h(m_2), m_2)$. We easily see that $\varphi$ is a homomorphism of $R$-modules.

Let $(m_1, m_2) \in M_1 \times_f M_2$.

\[ \varphi(X.(m_1, m_2)) = \varphi((X.m_1 + f(m_2), X.m_2)) \]

\[ = (X.m_1 + f(m_2) + h(X.m_2), X.m_2) \]

\[ = (X.m_1 + X.h(m_2) + g(m_2), X.m_2) \]

\[ = (X.(m_1 + h(m_2)) + g(m_2), X.m_2) \]

\[ = X.(m_1 + h(m_2), m_2) \]

\[ = X.\varphi((m_1, m_2)). \]

Therefore, $\varphi$ is a homomorphism of $R[X]$-modules, and it is easy to see that $\varphi i_1 = i_1$ and $p_2 \circ \varphi = p_2$. \hfill \Box
Remark 2.3. If we assume that the conditions of Theorem 2.2 are satisfied, then we have seen in the proof of this theorem, that the homomorphism $\varphi$ such that the following diagram of $R[X]$-modules is commutative:

$$
\begin{array}{ccc}
0 & \longrightarrow & M_1 \\
\bigg\downarrow 1_{M_1} & & \bigg\downarrow \varphi \\
0 & \longrightarrow & M_1 \\
\end{array}
\begin{array}{ccc}
i_1 & \longrightarrow & M_1 \times_f M_2 \\
\bigg\downarrow 1_{M_1} & & \bigg\downarrow 1_{M_2} \\
p_2 & \longrightarrow & M_2 \\
\end{array}
\begin{array}{ccc}
i_1 & \longrightarrow & M_1 \times_g M_2 \\
p_2 & \longrightarrow & M_2 \\
\longrightarrow & \longrightarrow & 0,
\end{array}
$$

is defined by $\forall (m_1, m_2) \in M_1 \times_f M_2$, $\varphi((m_1, m_2)) = (m_1 + h(m_2), m_2)$. (Here, we kept the notation of Theorem 2.2.) $\varphi$ is necessarily an isomorphism of $R[X]$-modules and $\varphi^{-1}$ is defined by $\forall (m_1, m_2) \in M_1 \times_g M_2$, $\varphi^{-1}((m_1, m_2)) = (m_1 - h(m_2), m_2)$.

Let $M_1$ and $M_2$ be two $R[X]$-modules that are $R$-free of finite rank. Let $B_1$ and $B_2$ be two bases of $M_1$ and $M_2$, respectively. Let $C_f = \text{Mat}_{B_2, B_1}(f)$ and $C_g = \text{Mat}_{B_2, B_1}(g)$ be the matrices of $f$ and $g$, respectively, with respect to the bases $B_2$ and $B_1$, and let $A$ (respectively, $B$) be the matrix representing the action of $X$ on $M_1$ with respect to the base $B_1$ (respectively, $M_2$ with respect to the base $B_2$).

**Corollary 2.4.** The following conditions are equivalent:

1. $(i_1, M_1 \times_f M_2, p_2) = (i_1, M_1 \times_g M_2, p_2)$.

2. There exists $H \in M_{n_1, n_2}(R)$ such that, $AH - HB = C_f - C_g$.

**Proof.** Just take $H = \text{Mat}_{B_2, B_1}(h)$. \qed

In the rest of this section, we give some interesting results on the $R[X]$-modules $M_1 \times_f M_2$, where we use the following notations:

$$\text{Ker}(f) = \{x \in M_1 \mid f(x) = 0\} \text{ and } \text{Im}(f) = \{f(x) \mid x \in M_1\}.$$
Proposition 2.5. The following assertions are true:

1. Each direct summand of $M_2$ included in $\text{Ker}(f)$ is a direct summand of $M_1 \times_f M_2$.

2. If $N_1$ is a direct summand of $M_1$ and $\text{Im}(f) \subseteq N_1$, then $N_1 \times_f M_2$ is a direct summand of $M_1 \times_f M_2$.

Proof. (1) If $M_2 = M'_2 \oplus M''_2$, where $M'_2$ and $M''_2$ are two $R[X]$-submodules of $M_2$ such that $M'_2 \subseteq \text{Ker}(f)$, then $M_1 \times_f M_2 = M_1 \times_f M'_2 \oplus \{0\} \times_f M''_2$.

(2) If $N_1$ is an $R[X]$-submodule of $M_1$ such that $\text{Im}(f) \subseteq N_1$, and there exists an $R[X]$-submodule of $M_1$ such that $M_1 = N_1 \oplus N'_1$, then $M_1 \times_f M_2 = N'_1 \times_f \{0\} \oplus N_1 \times_f M_2$.

Corollary 2.6. (1) If $f$ is zero, then $M_1 \times_f M_2$ is a decomposable $R[X]$-module.

(2) If $\text{Im}(f)$ is a direct summand of $M_1$, then $\text{Im}(f) \times_f M_2$ is a direct summand of $M_1 \times_f M_2$.

Proof. Obvious.

3. Isomorphism Classes

Let $R$ be a commutative ring. Let $M_{n,m}(R)$, $M_n(R)$, and $\text{Gl}_n(R)$ denote, respectively, the set of the $n \times m$ matrices, the set of the $n \times n$ matrices, and the set of the $n \times n$ invertible matrices, with entries in $R$, where $m$ and $n$ are two nonzero natural numbers.

Let $M_1$ and $M_2$ be two $R[X]$-modules that are $R$-free of respective finite rank $n$ and $m$. We assume that $M_1 = R^n$ and $M_2 = R^m$ (as $R$-modules). Let $B_1$ and $B_2$ be the respective canonical bases of $M_1$ and $M_2$. Let $f, g \in \text{Hom}_R(M_2, M_1)$, let $C_f = \text{Mat}_{B_2, B_1}(f)$ and
\[ C_g = \text{Mat}_{B_2, B_1}(g), \] and let \( A \) (respectively, \( B \)) be the matrix representing the action of \( X \) on \( M_1 \) with respect to the base \( B_1 \) (respectively, \( M_2 \) with respect to the base \( B_2 \)).

**Remark 3.1.** (1) Let \( \tilde{B} \) be the canonical base of \( M_1 \times_f M_2 \) seen as \( R \)-module. Then, the matrix representing the action of \( X \) on \( M_1 \times_f M_2 \) with respect to the base \( \tilde{B} \) is
\[
\begin{pmatrix}
A & C_f \\
0 & B \\
\end{pmatrix}.
\]

(2) The modules \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are isomorphic if and only if
\[
\begin{pmatrix}
A & C_f \\
0 & B \\
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
A & C_g \\
0 & B \\
\end{pmatrix}
\]
are similar.

The study of isomorphism classes of modules \( M_1 \times_f M_2 \) led us to introduce an equivalence relation on the matrices that is stronger than that of the similarity, and that is the object of the following definition.

**Definition 3.2.** Let \( A, A' \in M_{n}(R), B, B' \in M_{m}(R), \) and \( C, C' \in M_{n,m}(R), \) where \( n \) and \( m \) are two nonzero natural numbers. The matrices
\[
\begin{pmatrix}
A & C \\
0 & B \\
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
A' & C' \\
0 & B' \\
\end{pmatrix}
\]
are called strongly similar if there exists \( (U, V) \in \text{Gl}_n(R) \times \text{Gl}_m(R) \) and \( T \in M_{n,m}(R) \) such that
\[
\begin{pmatrix}
U & T \\
0 & V \\
\end{pmatrix}
\begin{pmatrix}
A & C \\
0 & B \\
\end{pmatrix}
= \begin{pmatrix}
A' & C' \\
0 & B' \\
\end{pmatrix}
\begin{pmatrix}
U & T \\
0 & V \\
\end{pmatrix}.
\]

For two matrices \( M \) and \( M' \), we put \( M \tilde{s} M' \) (respectively, \( M \tilde{s} M' \)) to mean that \( M \) and \( M' \) are similar (respectively, strongly similar).

**Remark 3.3.** For \( U, V \in M_{n}(R) \times M_{m}(R) \) and \( T \in M_{n,m}(R) \), it is easy to see that
\[
\begin{pmatrix}
U & T \\
0 & V \\
\end{pmatrix}
\]
is invertible if and only if \( U \) and \( V \) are invertible.
**Definition 3.4.** Let $f, g \in \text{Hom}_R(M_2, M_1)$. $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are called globally isomorphic, if there exists an isomorphism of $R[X]$-modules $\varphi: M_1 \times_f M_2 \rightarrow M_1 \times_g M_2$, such that $\varphi(M_1 \times \{0\}) \subset M_1 \times \{0\}$.

We have the following interesting result:

**Proposition 3.5.** The modules $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic if and only if 

$$
\begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}
$$

are strongly similar.

**Proof.** We denote by $\tilde{X}_f$ (respectively, $\tilde{X}_g$) the endomorphism which represents the action of $X$ on $M_1 \times_f M_2$ (respectively, $M_1 \times_g M_2$). The matrix of $\tilde{X}_f$ (respectively, $\tilde{X}_g$) in the canonical base of $M_1 \times_f M_2$ (respectively, $M_1 \times_g M_2$) is

$$
\begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix}
$$

(respectively,

$$
\begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}
$$

). Assume that $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic by an isomorphism $\varphi$. In the canonical bases of $M_1 \times_f M_2$ and $M_1 \times_g M_2$, $\varphi$ admits a matrix of the form

$$
\begin{pmatrix}
U & T \\
0 & V
\end{pmatrix},
$$

where $(U, V) \in \text{Gl}_n(R) \times \text{Gl}_m(R)$ and $T \in M_{n,m}(R)$. As for all $m \in M_1 \times_f M_2$, $\varphi(X.m) = X.\varphi(m)$, then $\varphi \circ \tilde{X}_f = \tilde{X}_g \circ \varphi$. So,

$$
\begin{pmatrix}
U & T \\
0 & V
\end{pmatrix} \begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix} = \begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix} \begin{pmatrix}
U & T \\
0 & V
\end{pmatrix}.
$$

Therefore, 

$$
\begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}
$$

are strongly similar.
Reciprocally, assume that \( \begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} \) and \( \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix} \) are strongly similar. So, there exist \( U, V \in GL_n(R) \times GL_m(R) \) and \( T \in M_{n,m}(R) \), such that \( \begin{pmatrix} U & T \\ 0 & V \end{pmatrix} \begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix} \begin{pmatrix} U & T \\ 0 & V \end{pmatrix} \). Let \( \varphi : M_1 \times_f M_2 \to M_1 \times_g M_2 \) be the morphism of \( R \)-modules, which is represented by the matrix \( \begin{pmatrix} U & T \\ 0 & V \end{pmatrix} \). Then, we have \( \varphi \circ X_f = X_g \circ \varphi \). So, for all \( m \in M_1 \times_f M_2 \), \( \varphi(X,m) = X \varphi(m) \). As \( \varphi \) is obviously bijective and \( \varphi(M_1 \times \{0\}) \subset M_1 \times \{0\} \), then \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic. \( \square \)

It is clear that if \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic, then they are isomorphic. In the following, we show the equivalence in special cases.

**Proposition 3.6.** The following conditions are equivalent:

1. The modules \( M_1 \times_f M_2 \) and \( M_1 \oplus M_2 \) are globally isomorphic.
2. The modules \( M_1 \times_f M_2 \) and \( M_1 \oplus M_2 \) are isomorphic.
3. There exists \( T \in M_{n,m}(R) \) such that \( AT - TB = C_f \).
4. \( (i_1, M_1 \times_f M_2, p_2) = (i_1, M_1 \oplus M_2, p_2) \).

**Proof.** (3) \( \Leftrightarrow \) (4): Comes from Corollary 2.4. Then, it suffices to show that (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3).

(1) \( \Rightarrow \) (2): Evident.

(2) \( \Rightarrow \) (3): Let \( \varphi : M_1 \times_f M_2 \to M_1 \oplus M_2 \) be an isomorphism of \( R[X] \)-modules. Then, we have \( \varphi \circ X_f = X_g \circ \varphi \). This implies that \( \begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} \) and \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) are similar. By [5] or [6], the equation \( AX - XB = C_f \) has a solution in \( M_{n,m}(R) \).
(3) \( \Rightarrow \) (1): By Corollary 2.4 and Remark 2.3,

\[
\varphi : M_1 \times_f M_2 \to M_1 \oplus M_2
\]

\[
(m_1, m_2) \mapsto (m_1 + Tm_2, m_2)
\]

is an isomorphism of \( R[X] \)-modules leaving \( M_1 \) globally invariant. \( \square \)

**Remark 3.7.** If we assume that the conditions of Proposition 3.6 are satisfied, then

\[
\varphi : M_1 \times_f M_2 \to M_1 \oplus M_2
\]

\[
(m_1, m_2) \mapsto (m_1 + Tm_2, m_2)
\]

is an isomorphism of \( R[X] \)-modules leaving \( M_1 \) globally invariant. Its reciprocal isomorphism is defined by

\[
\varphi^{-1} : M_1 \oplus M_2 \to M_1 \times_f M_2
\]

\[
(m_1, m_2) \mapsto (m_1 - Tm_2, m_2).
\]

**Proposition 3.8.** If one of the following conditions is true:

(i) The equation \( BX -XA = 0 \) admits \( 0 \) as a unique solution.

(ii) There existe \( \alpha \in R \) such that \( A = \alpha I_n \) and \( C_g \) is invertible.

(iii) There existe \( \beta \in R \) such that \( B = \beta I_m \) and \( C_f \) is invertible,

then the following assertions are equivalent:

(1) \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are isomorphic.

(2) \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic.
Proof. (1) ⇒ (2): Assume that $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are isomorphic by an isomorphism $\varphi$. Then, by (2) of Remark 3.1, there exist $(U, V) \in M_n(R) \times M_n(R)$, $T \in M_{n,n}(R)$, and $R \in M_{m,n}(R)$ such that
\[
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix}
\]
is invertible, and
\[
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix}
\begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix}
= \begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix}.
\]
But,
\[
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix}
\begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix}
= \begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix}
\Leftrightarrow
\begin{cases}
TA = BT; \\
UA = AU + C_g T; \\
TC_f + VB = BV; \\
UC_f + RB = AR + C_g V.
\end{cases}
\]
If one of the conditions (i), (ii) or (iii) is true, then $T = 0$. Therefore,
\[
\begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix}
\cong_s
\begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}.
\]
At last, by Proposition 3.5, $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic.

(2) ⇒ (1): Obvious. \qed

Remark 3.9. (1) If $m = n$ and there exists $a \in R$ such that $A = aI_n$ and $B - aI_n$ is invertible or there exists $\beta \in R$ such that $B = \beta I_n$ and $A - \beta I_n$ is invertible, then the condition (i) in Proposition 3.8 is satisfied.

(2) In a field the equation $BX - XA = 0$ admits 0 as a unique solution means that $A$ and $B$ have no eigenvalue in common.
Lemma 3.10. Let \((m_1, m_2) \in M_1 \times f M_2\) and \(n\) be a nonzero natural number. Then,

\[
X^n (m_1, m_2) = (X^n m_1 + \sum_{k=0}^{n-1} X^{n-1-k} f(X^k m_2), X^n m_2).
\]

**Proof.** We use an induction on \(n\). Let \((m_1, m_2) \in M_1 \times f M_2\). For \(n = 1\), we have

\[
X(m_1, m_2) = (Xm_1 + f(m_2), Xm_2)
\]

\[
= (X^1 m_1 + \sum_{k=0}^{1-1} X^{1-1-k} f(X^k m_2), X^1 m_2).
\]

Assume that

\[
X^n (m_1, m_2) = (X^n m_1 + \sum_{k=0}^{n-1} X^{n-1-k} f(X^k m_2), X^n m_2).
\]

Then,

\[
X^{n+1} (m_1, m_2) = X(X^n m_1 + \sum_{k=0}^{n-1} X^{n-1-k} f(X^k m_2), X^n m_2)
\]

\[
= (X^{n+1} m_1 + \sum_{k=0}^{n-1} X^{n-k} f(X^k m_2) + f(X^n m_2), X^{n+1} m_2)
\]

\[
= (X^{n+1} m_1 + \sum_{k=0}^{n} X^{n-k} f(X^k m_2), X^{n+1} m_2).
\]

\[\square\]

**Proposition 3.11.** Let \(\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2\) be a homomorphism of \(R[X]-\)modules. If \(g\) is an injective homomorphism of \(R[X]-\)modules, and there exists a nonzero natural number \(l\) such that \(l\) is invertible in \(R\) and both \(M_1\) and \(M_2\) are annihilated by \(X^l-1\), then 

\(\varphi(M_1 \times \{0\}) \subset M_1 \times \{0\}\).
In particular, if furthermore \( \varphi \) is an isomorphism, then \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic by the isomorphism \( \varphi \).

**Proof.** Let \((m_1, 0) \in M_1 \times \{0\}\). We put \( \varphi(m_1, 0) = (m'_1, m'_2) \). Then, we have

\[
\varphi(m_1, 0) = \varphi(X^l.m_1, 0)
\]

\[
= (X^l.m'_1 + \sum_{k=0}^{l-1} X^{l-1-k} g(X^k.m'_2), X^l.m'_2) \quad \text{(By Lemma 3.10)}
\]

\[
= (m'_1, m'_2).
\]

So,

\[
\sum_{k=0}^{l-1} X^{l-1-k} g(X^k.m'_2) = 0.
\]

\[
\Rightarrow \sum_{k=0}^{l-1} g(X^{l-1}.m'_2) = 0. \quad \text{(Since \( g \) is a homomorphism of \( R[X] \)-modules)}
\]

\[
\Rightarrow l g(X^{l-1}.m'_2) = 0.
\]

\[
\Rightarrow g(X^{l-1}.m'_2) = 0. \quad \text{(Since \( l \) is invertible in \( R \)}
\]

\[
\Rightarrow X^{l-1}.m'_2 = 0. \quad \text{(Since \( g \) is injective)}
\]

\[
\Rightarrow X.X^{l-1}.m'_2 = X^l.m'_2 = m'_2 = 0.
\]

So, \( \varphi(m_1, 0) = (m'_1, 0) \). Therefore, \( \varphi(M_1 \times \{0\}) \subset M_1 \times \{0\} \). \( \square \)

**Corollary 3.12.** Let \( A \in M_n(R) \), \( B \in M_m(R) \) and \( C, C' \in M_{n,m}(R) \), where \( n \) and \( m \) are two nonzero natural numbers. Let \((U, V) \in M_n(R) \times M_m(R) \) and \( T \in M_{m,n}(R) \) such that

\[
\begin{bmatrix}
U & R \\
T & V
\end{bmatrix}
\]
\[
\begin{pmatrix}
A & C \\
0 & B
\end{pmatrix} = \begin{pmatrix}
A & C' \\
0 & B
\end{pmatrix} \begin{pmatrix}
U & R \\
T & V
\end{pmatrix}. \text{ If } C' \text{ is invertible, } AC = CB \text{ and there exists a nonzero natural number } l \text{ such that } l \text{ is invertible in } R,
\]
\[A^l = I_n \text{ and } B^l = I_m, \text{ then } T = 0.\]

In particular, if
\[
\begin{pmatrix}
A & C \\
0 & B
\end{pmatrix} \overset{\text{is}}{=} \begin{pmatrix}
A & C' \\
0 & B
\end{pmatrix}, \text{ then } \begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix} \overset{l}{=} \begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix}.
\]

**Proof.** Just take \(M_1 = R^n, M_2 = R^m, B_1\) and \(B_2\) the respective canonical bases of \(M_1\) and \(M_2, C = C_f = Mat_{B_2,B_1}(f), C' = C_g = Mat_{B_2,B_1}(g)\) and \(A\) (respectively, \(B\)) the matrix representing the action of \(X\) on \(M_1\) with respect to the base \(B_1\) (respectively, \(M_2\) with respect to the base \(B_2\)). \(\Box\)

**Lemma 3.13.** If \((U,V) \in M_n(R) \times M_m(R), R \in M_{n,m}(R), T \in M_{m,n}(R)\)
\[
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix} \begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix} = \begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix} \begin{pmatrix}
U & R \\
T & V
\end{pmatrix}, \text{ then } \begin{pmatrix}
A & UC_f \\
0 & B
\end{pmatrix} \overset{\text{is}}{=} \begin{pmatrix}
A & C_g V \\
0 & B
\end{pmatrix}.
\]

**Proof.**
\[
\begin{pmatrix}
U & R \\
T & V
\end{pmatrix} \begin{pmatrix}
A & C_f \\
0 & B
\end{pmatrix} = \begin{pmatrix}
A & C_g \\
0 & B
\end{pmatrix} \begin{pmatrix}
U & R \\
T & V
\end{pmatrix}
\]
\[
\Rightarrow UC_f + RB = AR + C_g V.
\]
Then, we have

\[
\begin{pmatrix}
I_n & R \\
0 & I_m
\end{pmatrix}
\begin{pmatrix}
A & UC_f \\
0 & B
\end{pmatrix}
\begin{pmatrix}
I_n & -R \\
0 & I_m
\end{pmatrix}
= 
\begin{pmatrix}
A & C_g V \\
0 & B
\end{pmatrix}.
\]

As

\[
\begin{pmatrix}
I_n & R \\
0 & I_m
\end{pmatrix}^{-1} = 
\begin{pmatrix}
I_n & -R \\
0 & I_m
\end{pmatrix},
\]

then

\[
\begin{pmatrix}
A & UC_f \\
0 & B
\end{pmatrix} \stackrel{s}{\sim} 
\begin{pmatrix}
A & C_g V \\
0 & B
\end{pmatrix}.
\]

\[\square\]

**Proposition 3.14.** Assume that \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are isomorphic by an isomorphism \( \varphi \). Let \( f' = p_1 \circ \varphi \circ p_1 \) and \( g' = g \circ p_2 \circ \varphi \circ p_2 \). Then, \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic.

**Proof.** Comes from Lemma 3.13. \(\square\)

**Remark 3.15.** In general, \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic does not necessarily imply that \((i_1, M_1 \times_f M_2, p_2) = (i_1, M_1 \times_g M_2, p_2)\). In particular, \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are isomorphic does not necessarily imply that \( (i_1, M_1 \times_f M_2, p_2) = (i_1, M_1 \times_g M_2, p_2) \). Indeed, let \( R = \mathbb{R} \) and let \( \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \) (respectively, \( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \)) be the matrix representing the action of \( X \) on \( M_1 \times_f M_2 \) (respectively, \( M_1 \times_g M_2 \)) with respect to its canonical base (as \( R \)-module). We have \( \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \stackrel{s}{\sim} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \). So, by Proposition 3.5, \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic. But, the equation \( 2 \times X - X \times 2 = 3 - 1 \) has not any solution. By Corollary 2.4, \( (i_1, M_1 \times_f M_2, p_2) \neq (i_1, M_1 \times_g M_2, p_2) \).
Lemma 3.16. If \( \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \in M_2(R) \) and \( R \) is an integral domain, then

\[
\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \overset{s}{\sim} \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \Rightarrow \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \overset{s}{\sim} \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix}.
\]

Proof. If \( \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \overset{s}{\sim} \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \), then there exists \( \begin{pmatrix} u & r \\ t & v \end{pmatrix} \in GL_2(R) \) such that

\[
\begin{pmatrix} u & r \\ t & v \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \begin{pmatrix} u & r \\ t & v \end{pmatrix}.
\]

Or

\[
\begin{pmatrix} u & r \\ t & v \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \begin{pmatrix} u & r \\ t & v \end{pmatrix} \Leftrightarrow \begin{cases}
ta = bt; \\
u a = u a + c' t; \\
t c + v b = b v; \\
u c + r b = a r + c' v.
\end{cases}
\]

\[
\Rightarrow t = 0 \text{ or } c = c' = 0.
\]

It is clear that if \( t = 0 \) or \( c = c' = 0 \), then \( \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \overset{s}{\sim} \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \).

If \( R \) is an integral domain and \( n = m = 1 \), then we have the following proposition:

**Proposition 3.17.** The following conditions are equivalent:

1. \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are isomorphic.
2. \( M_1 \times_f M_2 \) and \( M_1 \times_g M_2 \) are globally isomorphic.
Proof. Clear by (2) of Remark 3.1, Lemma 3.16 and Proposition 3.5.

References


