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EXTENSIONS OF MODULES OVER POLYNOMIAL RINGS

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Abstract

For a commutative and unitary ring R, we characterize the classes of a classical equivalence relation defined on extensions of an $R[X]$-module, which is R-projective by another $R[X]$-module. The modules that are extensions of an $R[X]$-module M_2, which is R-projective by an $R[X]$-module M_1 have a fairly simple form which we denote by $M_1 \times_f M_2$, where f is an R-homomorphism from M_2 to M_1. For R, M_1, M_2 and f as in the foregoing, but M_2 is not necessarily R-projective, we search to classify the $R[X]$-modules $M_1 \times_f M_2$.

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1. Introduction

All rings considered in this paper are supposed to be with unit. Let R be a commutative ring. Let N and L be two R-modules. We recall (see [1, 7, 9]) that an extension of L by N is a triple (u, M, v), where M is an R-module and u, v are R-module homomorphisms such that the sequence

$$O \to N \xrightarrow{u} M \xrightarrow{v} L \to O$$

is exact. Two extensions (u, M, v) and (u', M', v') of L by N are equivalent if there exists an isomorphism of R-modules $\varphi : M \to M'$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "u"] \arrow[d, "1_N"] & M \arrow[r, "v"] \arrow[d, "\varphi"] & L \arrow[r] \arrow[d, "1_L"] & 0 \\
0 \arrow[r] & N \arrow[r, "u'"] & M' \arrow[r, "v'"] & L \arrow[r] & 0.
\end{tikzcd}$$

For this equivalence relation, the class of an extension (u, M, v) of L by N is denoted by $\overline{(u, M, v)}$.

On the group rings, one of the technics used to study the module extensions consists to show that a module, which is extension of a module L by another module N - under some conditions - can be written in a simple form that is denoted by $N \times_\alpha L$, where α is a cocycle or a derivation (see, for example, [2], [3], [8]). For the polynomial rings, this form is much simpler, indeed instead of cocycles or derivations we can take homomorphisms of R-modules.

The polynomial ring with coefficients in R will be denoted by $R[X]$.

In this paper, we study the extensions of $R[X]$-modules. We search essentially to characterize these extensions. In Section 2, we characterize the classes of this equivalence relation defined on extensions of an $R[X]$-modules M_2, which is R-projective by an $R[X]$-module M_1. As we already said, the modules that are extensions of an $R[X]$-module M_2, which is R-projective by an $R[X]$-modules M_1 have a fairly simple form which we denote by $M_1 \times_f M_2$, where f is an R-homomorphism from M_2 to M_1. In Section 3, for M_1, M_2 and f as in the foregoing, but M_2 is not necessarily R-projective, we search to classify the $R[X]$-modules $M_1 \times_f M_2$. In the same time, we give some properties of these modules.

2. Extension Equivalence of Modules Over Polynomial Rings

Let R be a commutative ring. Let M_1 and M_2 be two $R[X]$-modules. Let $f \in Hom_R(M_2, M_1)$, where $Hom_R(M_2, M_1)$ is the set of all R-module morphisms from the R-module M_2 to the R-module M_1. Then, we can define a structure of $R[X]$-module on $M_1 \times M_2$ by $\forall (m_1, m_2) \in M_1 \times M_2$, $X.(m_1, m_2) = (X.m_1 + f(m_2), X.m_2)$. The module $M_1 \times M_2$ equipped with this structure of $R[X]$-module will be denoted by $M_1 \times_f M_2$. In this case, the following sequence $O \to M_1 \xrightarrow{i_1} M_1 \times_f M_2 \xrightarrow{p_2} M_2 \to O$, is an exact sequence of $R[X]$-modules, where i_1 is the first injection and p_2 is the second projection.

Theorem 2.1. *Let $O \to M_1 \xrightarrow{u} M \xrightarrow{v} M_2 \to O$ be an exact sequence of $R[X]$-modules. If M_2 is R-projective, then there exists $f \in Hom_R(M_2, M_1)$ such that $\overline{(u, M, v)} = \overline{(i_1, M_1 \times_f M_2, p_2)}$.*

Proof. $O \to M_1 \xrightarrow{u} M \xrightarrow{v} M_2 \to O$ is exact and M_2 is projective as R-module. Then, there exist $v' \in Hom_R(M_2, M)$ and $u' \in Hom_R(M, M_1)$ such that $u'ou = id_{M_1}$ and $vov' = id_{M_2}$. It is well-known that $\theta : M \to M_1 \oplus M_2$ defined by $m \mapsto (u'(m), v(m))$ is an isomorphism of

R-modules, and $\theta^{-1} : M_1 \oplus M_2 \to M$ is defined by $(m_1, m_2) \mapsto m + u(x)$, where m is such that $v(m) = m_2$ and $x \in M_1$ is such that $x = m_1 - u'(m)$. We define a structure of $R[X]$-module on $M_1 \oplus M_2$ by

$$\forall (m_1, m_2) \in M_1 \oplus M_2,\ X.(m_1, m_2) = \theta(X.\theta^{-1}((m_1, m_2))).$$

Then,

$$\begin{aligned} X.(m_1, m_2) &= \theta(X.\theta^{-1}((m_1, m_2))) \\ &= \theta(X.(m + u(x))) \\ &= \theta(X.m + X.u(x)) \\ &= (u'(X.m + X.u(x)),\ v(X.m + X.u(x))) \\ &= (u'(X.m) + X.x,\ X.m_2) \\ &= (X.m_1 + u'(X.m) - X.u'(m),\ X.m_2). \end{aligned}$$

We have $v(m) = m_2$, so there exists $m' \in \ker(v)$ such that $v'(m_2) = m + m'$. As $\ker(v) = \operatorname{Im}(u)$, then there exists $m_2' \in M_1$ such that $v'(m_2) = m + u(m_2')$. So,

$$\begin{aligned} u'(X.m) - X.u'(m) &= u'(X.(v'(m_2) - u(m_2'))) - X.u'(v'(m_2) - u(m_2')) \\ &= u'(X.v'(m_2)) - u'(X.u(m_2')) - X.(u'ov'(m_2)) + X.(u'ou(m_2')) \\ &= u'(X.v'(m_2)) - X.(u'ov'(m_2)). \end{aligned}$$

Therefore,

$$\begin{aligned} X.(m_1, m_2) &= (X.m_1 + u'(X.v'(m_2)) - X.(u'ov'(m_2)),\ X.m_2) \\ &= (X.m_1 + f(m_2),\ X.m_2), \end{aligned}$$

where $f \in Hom_R(M_2, M_1)$ is defined by

$$\forall m_2 \in M_2, \quad f(m_2) = u'(X.v'(m_2)) - X.(u'ov'(m_2)).$$

$\theta : M \to M_1 \times_f M_2$ is an isomorphism of $R[X]$-modules and we have $\theta o u = i_1$ and $p_2 o \theta = v$. □

Theorem 2.2. *Let* $f, g \in Hom_R(M_2, M_1)$, *where* M_1 *and* M_2 *are two* $R[X]$*-modules. Then, the following conditions are equivalent*:

(1) $\overline{(i_1, M_1 \times_f M_2, p_2)} = \overline{(i_1, M_1 \times_g M_2, p_2)}$.

(2) *There exists* $h \in Hom_R(M_2, M_1)$ *such that, for all* $m_2 \in M_2$,

$$f(m_2) + h(X.m_2) = X.h(m_2) + g(m_2).$$

Proof. (1) $\Rightarrow$ (2): If $\overline{(i_1, M_1 \times_f M_2, p_2)} = \overline{(i_1, M_1 \times_g M_2, p_2)}$, then there exists a homomorphism of $R[X]$-modules $\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2$ such that $\varphi o i_1 = i_1$ and $p_2 o \varphi = p_2$. Let $(m_1, m_2) \in M_1 \times M_2$. We have

$$\begin{aligned} \varphi((m_1, 0)) &= \varphi o i_1(m_1) \\ &= i_1(m_1) \\ &= (m_1, 0). \end{aligned}$$

We put $\varphi((0, m_2)) = (m_1', m_2')$. As $p_2 o \varphi = p_2$, then $m_2' = m_2$. We define an application $h : M_2 \to M_1$ by $h(m_2) = m_1'$. Then, we have $\varphi((m_1, m_2)) = (m_1 + h(m_2), m_2)$. We first show that h is a homomorphism of R-modules. Let $m_2, m_2' \in M_2$ and let $r \in R$. $\varphi((0, m_2 + m_2'))$ $= (h(m_2 + m_2'), m_2 + m_2')$. But,

$$\begin{aligned} \varphi((0, m_2 + m_2')) &= \varphi((0, m_2)) + \varphi((0, m_2')) \\ &= (h(m_2), m_2) + (h(m_2'), m_2') \\ &= (h(m_2) + h(m_2'), m_2 + m_2'). \end{aligned}$$

So, $h(m_2 + m_2') = h(m_2) + h(m_2')$. We have also $\varphi((0, r.m_2)) = (h(r.m_2), r.m_2)$. But,

$$\begin{aligned} \varphi((0, r.m_2)) &= \varphi(r(0, m_2)) \\ &= r\varphi((0, m_2)) \\ &= r(h(m_2), m_2) \\ &= (r.h(m_2), r.m_2). \end{aligned}$$

So, $h(r.m_2) = r.h(m_2)$. Therefore, $h \in Hom_R(M_2, M_1)$. Let $m_2 \in M_2$. $\varphi(X.(0, m_2)) = X.\varphi((0, m_2))$. But,

$$\begin{aligned}\varphi(X.(0, m_2)) &= \varphi(f(m_2), X.m_2)) \\ &= (f(m_2) + h(X.m_2), X.m_2),\end{aligned}$$

and

$$\begin{aligned}X.\varphi((0, m_2)) &= X.(h(m_2), m_2) \\ &= (X.h(m_2). g(m_2), X.m_2).\end{aligned}$$

Therefore, $\forall m_2 \in M_2$, $f(m_2) + h(X.m_2) = X.h(m_2) + g(m_2)$.

(2) $\Rightarrow$ (1): Assume that there exists $h \in Hom_R(M_2, M_1)$ such that, $\forall m_2 \in M_2$, $f(m_2) + h(X.m_2) = X.h(m_2) + g(m_2)$. We define an application $\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2$ by $\forall (m_1, m_2) \in M_1 \times_f M_2$, $\varphi((m_1, m_2)) = (m_1 + h(m_2), m_2)$. We easily see that φ is a homomorphism of R-modules. Let $(m_1, m_2) \in M_1 \times_f M_2$.

$$\begin{aligned}\varphi(X.(m_1, m_2)) &= \varphi((X.m_1 + f(m_2), X.m_2)) \\ &= (X.m_1 + f(m_2) + h(X.m_2), X.m_2) \\ &= (X.m_1 + X.h(m_2) + g(m_2), X.m_2) \\ &= (X.(m_1 + h(m_2)) + g(m_2), X.m_2) \\ &= X.(m_1 + h(m_2), m_2) \\ &= X.\varphi((m_1, m_2)).\end{aligned}$$

Therefore, φ is a homomorphism of $R[X]$-modules, and it is easy to see that $\varphi o i_1 = i_1$ and $p_2 o \varphi = p_2$. $\square$

Remark 2.3. If we assume that the conditions of Theorem 2.2 are satisfied, then we have seen in the proof of this theorem, that the homomorphism φ such that the following diagram of $R[X]$-modules is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{i_1} & M_1 \times_f M_2 & \xrightarrow{p_2} & M_2 & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle 1_{M_1}} & & \downarrow {\scriptstyle \varphi} & & \downarrow {\scriptstyle 1_{M_2}} & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{i_1} & M_1 \times_g M_2 & \xrightarrow{p_2} & M_2 & \longrightarrow & 0, \end{array}$$

is defined by $\forall (m_1, m_2) \in M_1 \times_f M_2$, $\varphi((m_1, m_2)) = (m_1 + h(m_2), m_2)$. (Here, we kept the notation of Theorem 2.2.) φ is necessarily an isomorphism of $R[X]$-modules and φ^{-1} is defined by $\forall (m_1, m_2) \in M_1 \times_g M_2$, $\varphi^{-1}((m_1, m_2)) = (m_1 - h(m_2), m_2)$.

Let M_1 and M_2 be two $R[X]$-modules that are R-free of finite rank. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two bases of M_1 and M_2, respectively. Let $C_f = Mat_{\mathbf{B}_2, \mathbf{B}_1}(f)$ and $C_g = Mat_{\mathbf{B}_2, \mathbf{B}_1}(g)$ be the matrices of f and g, respectively, with respect to the bases $\mathbf{B}_2$ and $\mathbf{B}_1$, and let A (respectively, B) be the matrix representing the action of X on M_1 with respect to the base $\mathbf{B}_1$ (respectively, M_2 with respect to the base $\mathbf{B}_2$).

Corollary 2.4. *The following conditions are equivalent*:

(1) $\overline{(i_1, M_1 \times_f M_2, p_2)} = \overline{(i_1, M_1 \times_g M_2, p_2)}$.

(2) *There exists* $H \in M_{n_1, n_2}(R)$ *such that*, $AH - HB = C_f - C_g$.

Proof. Just take $H = Mat_{\mathbf{B}_2, \mathbf{B}_1}(h)$. □

In the rest of this section, we give some interesting results on the $R[X]$-modules $M_1 \times_f M_2$, where we use the following notations:

$$\mathrm{Ker}(f) = \{x \in M_1 \mid f(x) = 0\} \text{ and } \mathrm{Im}(f) = \{f(x) \mid x \in M_1\}.$$

Proposition 2.5. *The following assertions are true*:

(1) *Each direct summand of* M_2 *included in* $\mathrm{Ker}(f)$ *is a direct summand of* $M_1 \times_f M_2$.

(2) *If* N_1 *is a direct summand of* M_1 *and* $\mathrm{Im}(f) \subseteq N_1$, *then* $N_1 \times_f M_2$ *is a direct summand of* $M_1 \times_f M_2$.

Proof. (1) If $M_2 = M_2' \oplus M_2''$, where M_2' and M_2'' are two $R[X]$-submodules of M_2 such that $M_2' \subset \mathrm{Ker}(f)$, then $M_1 \times_f M_2 = M_1 \times_f M_2'' \oplus \{0\} \times_f M_2'$.

(2) If N_1 is an $R[X]$-submodule of M_1 such that $\mathrm{Im}(f) \subseteq N_1$, and there exists an $R[X]$-submodule of M_1 such that $M_1 = N_1 \oplus N_1'$, then $M_1 \times_f M_2 = N_1' \times_f \{0\} \oplus N_1 \times_f M_2$. □

Corollary 2.6. (1) *If* f *is zero*, *then* $M_1 \times_f M_2$ *is a decomposable* $R[X]$-*module.*

(2) *If* $\mathrm{Im}(f)$ *is a direct summand of* M_1, *then* $\mathrm{Im}(f) \times_f M_2$ *is a direct summand of* $M_1 \times_f M_2$.

Proof. Obvious. □

3. Isomorphism Classes

Let R be a commutative ring. Let $M_{n,m}(R)$, $M_n(R)$, and $Gl_n(R)$ denote, respectively, the set of the $n \times m$ matrices, the set of the $n \times n$ matrices, and the set of the $n \times n$ invertible matrices, with entries in R, where m and n are two nonzero natural numbers.

Let M_1 and M_2 be two $R[X]$-modules that are R-free of respective finite rank n and m. We assume that $M_1 = R^n$ and $M_2 = R^m$ (as R-modules). Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the respective canonical bases of M_1 and M_2. Let $f, g \in Hom_R(M_2, M_1)$, let $C_f = Mat_{\mathbf{B}_2, \mathbf{B}_1}(f)$ and

$C_g = Mat_{\mathbf{B}_2, \mathbf{B}_1}(g)$, and let A (respectively, B) be the matrix representing the action of X on M_1 with respect to the base $\mathbf{B}_1$ (respectively, M_2 with respect to the base $\mathbf{B}_2$).

Remark 3.1. (1) Let $\widetilde{\mathbf{B}}$ be the canonical base of $M_1 \times_f M_2$ seen as R-module. Then, the matrix representing the action of X on $M_1 \times_f M_2$ with respect to the base $\widetilde{\mathbf{B}}$ is $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$.

(2) The modules $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are isomorphic if and only if $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$ are similar.

The study of isomorphism classes of modules $M_1 \times_f M_2$ led us to introduce an equivalence relation on the matrices that is stronger than that of the similarity, and that is the object of the following definition.

Definition 3.2. Let $A, A' \in M_n(R)$, $B, B' \in M_m(R)$, and $C, C' \in M_{n,m}(R)$, where n and m are two nonzero natural numbers. The matrices $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A' & C' \\ 0 & B' \end{pmatrix}$ are called strongly similar if there exists $(U, V) \in Gl_n(R) \times Gl_m(R)$ and $T \in M_{n,m}(R)$ such that

$$\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A' & C' \\ 0 & B' \end{pmatrix}\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}.$$

For two matrices M and M', we put $M \,\tilde{s}\, M'$ (respectively, $M \,\tilde{\tilde{s}}\, M'$) to mean that M and M' are similar (respectively, strongly similar).

Remark 3.3. For $U, V \in M_n(R) \times M_m(R)$ and $T \in M_{n,m}(R)$, it is easy to see that $\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}$ is invertible if and only if U and V are invertible.

Definition 3.4. Let $f, g \in Hom_R(M_2, M_1)$. $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are called globally isomorphic, if there exists an isomorphism of $R[X]$-modules $\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2$, such that $\varphi(M_1 \times \{0\}) \subset M_1 \times \{0\}$.

We have the following interesting result:

Proposition 3.5. *The modules $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic if and only if $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$ are strongly similar.*

Proof. We denote by $\widetilde{X}_f$ (respectively, $\widetilde{X}_g$) the endomorphism which represents the action of X on $M_1 \times_f M_2$ (respectively, $M_1 \times_g M_2$). The matrix of $\widetilde{X}_f$ (respectively, $\widetilde{X}_g$) in the canonical base of $M_1 \times_f M_2$ (respectively, $M_1 \times_g M_2$) is $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$ (respectively, $\begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$).

Assume that $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic by an isomorphism φ. In the canonical bases of $M_1 \times_f M_2$ and $M_1 \times_g M_2$, φ admits a matrix of the form $\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}$, where $(U, V) \in Gl_n(R) \times Gl_m(R)$ and $T \in M_{n,m}(R)$. As for all $m \in M_1 \times_f M_2$, $\varphi(X.m) = X.\varphi(m)$, then $\varphi o \widetilde{X}_f = \widetilde{X}_g o \varphi$. So, $\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}$.

Therefore, $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$ are strongly similar.

Reciprocally, assume that $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$ are strongly similar. So, there exist $U, V \in Gl_n(R) \times Gl_m(R)$ and $T \in M_{n,m}(R)$, such that $\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}$. Let $\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2$ be the morphism of R-modules, which is represented by the matrix $\begin{pmatrix} U & T \\ 0 & V \end{pmatrix}$. Then, we have $\varphi o \widetilde{X}_f = \widetilde{X}_g o \varphi$. So, for all $m \in M_1 \times_f M_2$, $\varphi(X.m) = X.\varphi(m)$. As φ is obviously bijective and $\varphi(M_1 \times \{0\}) \subset M_1 \times \{0\}$, then $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic. □

It is clear that if $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic, then they are isomorphic. In the following, we show the equivalence in special cases.

Proposition 3.6. *The following conditions are equivalent*:

(1) *The modules* $M_1 \times_f M_2$ *and* $M_1 \oplus M_2$ *are globally isomorphic.*

(2) *The modules* $M_1 \times_f M_2$ *and* $M_1 \oplus M_2$ *are isomorphic.*

(3) *There exists* $T \in M_{n,m}(R)$ *such that* $AT - TB = C_f$.

(4) $\overline{(i_1, M_1 \times_f M_2, p_2)} = \overline{(i_1, M_1 \oplus M_2, p_2)}$.

Proof. (3) $\Leftrightarrow$ (4): Comes from Corollary 2.4. Then, it suffices to show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2): Evident.

(2) $\Rightarrow$ (3): Let $\varphi : M_1 \times_f M_2 \to M_1 \oplus M_2$ be an isomorphism of $R[X]$-modules. Then, we have $\varphi o \widetilde{X} = \widetilde{X} o \varphi$. This implies that $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are similar. By [5] or [6], the equation $AX - XB = C_f$ has a solution in $M_{n,m}(R)$.

(3) $\Rightarrow$ (1): By Corollary 2.4 and Remark 2.3,

$$\varphi : M_1 \times_f M_2 \to M_1 \oplus M_2$$

$$(m_1, m_2) \mapsto (m_1 + Tm_2, m_2)$$

is an isomorphism of $R[X]$-modules leaving M_1 globally invariant. □

Remark 3.7. If we assume that the conditions of Proposition 3.6 are satisfied, then

$$\varphi : M_1 \times_f M_2 \to M_1 \oplus M_2$$

$$(m_1, m_2) \mapsto (m_1 + Tm_2, m_2)$$

is an isomorphism of $R[X]$-modules leaving M_1 globally invariant. Its reciprocal isomorphism is defined by

$$\varphi^{-1} : M_1 \oplus M_2 \to M_1 \times_f M_2$$

$$(m_1, m_2) \mapsto (m_1 - Tm_2, m_2).$$

Proposition 3.8. *If one of the following conditions is true*:

(i) *The equation* $BX - XA = 0$ *admits* 0 *as a unique solution.*

(ii) *There existe* $\alpha \in R$ *such that* $A = \alpha I_n$ *and* C_g *is invertible.*

(iii) *There existe* $\beta \in R$ *such that* $B = \beta I_m$ *and* C_f *is invertible*,

then the following assertions are equivalent:

(1) $M_1 \times_f M_2$ *and* $M_1 \times_g M_2$ *are isomorphic.*

(2) $M_1 \times_f M_2$ *and* $M_1 \times_g M_2$ *are globally isomorphic.*

Proof. (1) $\Rightarrow$ (2): Assume that $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are isomorphic by an isomorphism φ. Then, by (2) of Remark 3.1, there exist $(U, V) \in M_n(R) \times M_n(R)$, $T \in M_{n,m}(R)$, and $R \in M_{m,n}(R)$ such that $\begin{pmatrix} U & R \\ T & V \end{pmatrix}$ is invertible, and

$$\begin{pmatrix} U & R \\ T & V \end{pmatrix}\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}\begin{pmatrix} U & R \\ T & V \end{pmatrix}.$$

But,

$$\begin{pmatrix} U & R \\ T & V \end{pmatrix}\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}\begin{pmatrix} U & R \\ T & V \end{pmatrix} \Leftrightarrow \begin{cases} TA = BT; \\ UA = AU + C_gT; \\ TC_f + VB = BV; \\ UC_f + RB = AR + C_gV. \end{cases}$$

If one of the conditions (i), (ii) or (iii) is true, then $T = 0$. Therefore, $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} \underset{s}{\approx} \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$. At last, by Proposition 3.5, $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic.

(2) $\Rightarrow$ (1): Obvious. □

Remark 3.9. (1) If $m = n$ and there exists $\alpha \in R$ such that $A = \alpha I_n$ and $B - \alpha I_n$ is invertible or there exists $\beta \in R$ such that $B = \beta I_n$ and $A - \beta I_n$ is invertible, then the condition (i) in Proposition 3.8 is satisfied.

(2) In a field the equation $BX - XA = 0$ admits 0 as a unique solution means that A and B have no eigenvalue in common.

Lemma 3.10. *Let* $(m_1, m_2) \in M_1 \times_f M_2$ *and* n *be a nonzero natural number. Then,*

$$X^n.(m_1, m_2) = (X^n.m_1 + \sum_{k=0}^{n-1} X^{n-1-k} f(X^k.m_2), X^n.m_2).$$

Proof. We use an induction on n. Let $(m_1, m_2) \in M_1 \times_f M_2$. For $n = 1$, we have

$$X.(m_1, m_2) = (X.m_1 + f(m_2), X.m_2)$$

$$= (X^1.m_1 + \sum_{k=0}^{1-1} X^{1-1-k} f(X^k.m_2), X^1.m_2).$$

Assume that

$$X^n.(m_1, m_2) = (X^n.m_1 + \sum_{k=0}^{n-1} X^{n-1-k} f(X^k.m_2), X^n.m_2).$$

Then,

$$X^{n+1}.(m_1, m_2) = X(X^n.m_1 + \sum_{k=0}^{n-1} X^{n-1-k} f(X^k.m_2), X^n.m_2)$$

$$= (X^{n+1}.m_1 + \sum_{k=0}^{n-1} X^{n-k} f(X^k.m_2) + f(X^n.m_2), X^{n+1}.m_2)$$

$$= (X^{n+1}.m_1 + \sum_{k=0}^{n} X^{n-k} f(X^k.m_2), X^{n+1}.m_2).$$

□

Proposition 3.11. *Let* $\varphi : M_1 \times_f M_2 \to M_1 \times_g M_2$ *be a homomorphism of* $R[X]$*-modules. If* g *is an injective homomorphism of* $R[X]$*-modules, and there exists a nonzero natural number* l *such that* l *is invertible in* R *and both* M_1 *and* M_2 *are annihilated by* $X^l - 1$*, then* $\varphi(M_1 \times \{0\}) \subset M_1 \times \{0\}$.

In particular, if furthermore φ *is an isomorphism, then* $M_1 \times_f M_2$ *and* $M_1 \times_g M_2$ *are globally isomorphic by the isomorphism* φ.

Proof. Let $(m_1, 0) \in M_1 \times \{0\}$. We put $\varphi(m_1, 0) = (m_1', m_2')$. Then, we have

$$\begin{aligned}\varphi(m_1, 0) &= \varphi(X^l.m_1, 0) \\ &= \left(X^l.m_1' + \sum_{k=0}^{l-1} X^{l-1-k} g(X^k.m_2'),\ X^l.m_2'\right) \text{ (By Lemma 3.10)} \\ &= (m_1', m_2').\end{aligned}$$

So,

$$\sum_{k=0}^{l-1} X^{l-1-k} g(X^k.m_2') = 0.$$

$$\Rightarrow \sum_{k=0}^{l-1} g(X^{l-1}.m_2') = 0. \text{ (Since } g \text{ is a homomorphism of } R[X]\text{-modules)}$$

$$\Rightarrow lg(X^{l-1}.m_2') = 0.$$

$$\Rightarrow g(X^{l-1}.m_2') = 0. \text{ (Since } l \text{ is invertible in } R)$$

$$\Rightarrow X^{l-1}.m_2' = 0. \text{ (Since } g \text{ is injective)}$$

$$\Rightarrow X.X^{l-1}.m_2' = X^l.m_2' = m_2' = 0.$$

So, $\varphi(m_1, 0) = (m_1', 0)$. Therefore, $\varphi(M_1 \times \{0\}) \subset M_1 \times \{0\}$. □

Corollary 3.12. *Let* $A \in M_n(R)$, $B \in M_m(R)$ *and* $C, C' \in M_{n,m}(R)$, *where* n *and* m *are two nonzero natural numbers. Let* $(U, V) \in M_n(R) \times M_m(R)$, $R \in M_{n,m}(R)$ *and* $T \in M_{m,n}(R)$ *such that* $\begin{pmatrix} U & R \\ T & V \end{pmatrix}$

$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C' \\ 0 & B \end{pmatrix} \begin{pmatrix} U & R \\ T & V \end{pmatrix}$. *If* C' *is invertible,* $AC = C'B$ *and there exists a nonzero natural number* l *such that* l *is invertible in* R, $A^l = I_n$ *and* $B^l = I_m$, *then* $T = 0$.

In particular, if $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tilde{s} \begin{pmatrix} A & C' \\ 0 & B \end{pmatrix}$, *then* $\begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} \tilde{\tilde{s}} \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix}$.

Proof. Just take $M_1 = R^n$, $M_2 = R^m$, $\mathbf{B}_1$ and $\mathbf{B}_2$ the respective canonical bases of M_1 and M_2, $C = C_f = Mat_{\mathbf{B}_2, \mathbf{B}_1}(f)$, $C' = C_g = Mat_{\mathbf{B}_2, \mathbf{B}_1}(g)$ and A (respectively, B) the matrix representing the action of X on M_1 with respect to the base $\mathbf{B}_1$ (respectively, M_2 with respect to the base $\mathbf{B}_2$). □

Lemma 3.13. *If* $(U, V) \in M_n(R) \times M_m(R), R \in M_{n,m}(R), T \in M_{m,n}(R)$ *and* $\begin{pmatrix} U & R \\ T & V \end{pmatrix} \begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix} \begin{pmatrix} U & R \\ T & V \end{pmatrix}$, *then* $\begin{pmatrix} A & UC_f \\ 0 & B \end{pmatrix} \tilde{\tilde{s}} \begin{pmatrix} A & C_g V \\ 0 & B \end{pmatrix}$.

Proof.

$$\begin{pmatrix} U & R \\ T & V \end{pmatrix} \begin{pmatrix} A & C_f \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C_g \\ 0 & B \end{pmatrix} \begin{pmatrix} U & R \\ T & V \end{pmatrix}$$

$$\Rightarrow UC_f + RB = AR + C_g V.$$

Then, we have

$$\begin{pmatrix} I_n & R \\ 0 & I_m \end{pmatrix}\begin{pmatrix} A & UC_f \\ 0 & B \end{pmatrix}\begin{pmatrix} I_n & -R \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} A & C_gV \\ 0 & B \end{pmatrix}.$$

As $\begin{pmatrix} I_n & R \\ 0 & I_m \end{pmatrix}^{-1} = \begin{pmatrix} I_n & -R \\ 0 & I_m \end{pmatrix}$, then $\begin{pmatrix} A & UC_f \\ 0 & B \end{pmatrix} \underset{s}{\approx} \begin{pmatrix} A & C_gV \\ 0 & B \end{pmatrix}$.

□

Proposition 3.14. *Assume that* $M_1 \times_f M_2$ *and* $M_1 \times_g M_2$ *are isomorphic by an isomorphism* φ. *Let* $f' = p_1 o \varphi o i_1 o f$ *and* $g' = g o p_2 o \varphi o i_2$. *Then,* $M_1 \times_{f'} M_2$ *and* $M_1 \times_{g'} M_2$ *are globally isomorphic.*

Proof. Comes from Lemma 3.13. □

Remark 3.15. In general, $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic does not necessarily imply that $\overline{(i_1, M_1 \times_f M_2, p_2)} = \overline{(i_1, M_1 \times_g M_2, p_2)}$. In particular, $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are isomorphic does not necessarily imply that $\overline{(i_1, M_1 \times_f M_2, p_2)} = \overline{(i_1, M_1 \times_g M_2, p_2)}$. Indeed, let $R = \mathbb{R}$ and let $\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$ (respectively, $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$) be the matrix representing the action of X on $M_1 \times_f M_2$ (respectively, $M_1 \times_g M_2$) with respect to its canonical base (as R-module). We have $\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \underset{s}{\approx} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. So, by Proposition 3.5, $M_1 \times_f M_2$ and $M_1 \times_g M_2$ are globally isomorphic. But, the equation $2 \times X - X \times 2 = 3 - 1$ has not any solution. By Corollary 2.4, $\overline{(i_1, M_1 \times_f M_2, p_2)} \neq \overline{(i_1, M_1 \times_g M_2, p_2)}$.

Lemma 3.16. *If* $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix} \in M_2(R)$ *and* R *is an integral domain, then*

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \widetilde{s} \begin{pmatrix} a & c' \\ 0 & v \end{pmatrix} \Rightarrow \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \widetilde{\widetilde{s}} \begin{pmatrix} a & c' \\ 0 & v \end{pmatrix}.$$

Proof. If $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \widetilde{s} \begin{pmatrix} a & c' \\ 0 & v \end{pmatrix}$, then there exists $\begin{pmatrix} u & r \\ t & v \end{pmatrix} \in Gl_2(R)$ such that

$$\begin{pmatrix} u & r \\ t & v \end{pmatrix}\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix}\begin{pmatrix} u & r \\ t & v \end{pmatrix}.$$

Or

$$\begin{pmatrix} u & r \\ t & v \end{pmatrix}\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix}\begin{pmatrix} u & r \\ t & v \end{pmatrix} \Leftrightarrow \begin{cases} ta = bt; \\ ua = au + c't; \\ tc + vb = bv; \\ uc + rb = ar + c'v. \end{cases}$$

$$\Rightarrow t = 0 \text{ or } c = c' = 0.$$

It is clear that if $t = 0$ or $c = c' = 0$, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \widetilde{\widetilde{s}} \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix}$. □

If R is an integral domain and $n = m = 1$, then we have the following proposition:

Proposition 3.17. *The following conditions are equivalent*:

(1) $M_1 \times_f M_2$ *and* $M_1 \times_g M_2$ *are isomorphic.*

(2) $M_1 \times_f M_2$ *and* $M_1 \times_g M_2$ *are globally isomorphic.*

Proof. Clear by (2) of Remark 3.1, Lemma 3.16 and Proposition 3.5.

□

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