Journal of Algebra, Number Theory: Advances and Applications
Volume 14, Number 1, 2015, Pages 11-26
Available at http://scientificadvances.co.in
DOI: http://dx.doi.org/10.18642/jantaa_7100121522

POSITIVE LINEAR FUNCTIONALS WITH CONVEX AND PREINVEX FUNCTIONS

ZLATKO PAVIĆ

Mechanical Engineering Faculty in Slavonski Brod
University of Osijek
Trg Ivane Brlić Mažuranić 2
35000 Slavonski Brod
Croatia
e-mail: Zlatko.Pavic@sfsbhr

Abstract

The paper considers the functional form of Jensen's inequality and its applications. The paper offers the extension of this functional form and its usage in creating the most important inequalities for convex functions. In particular, the Jensen and Hermite-Hadamard inequality are applied to preinvex functions.

1. Introduction

The concept of convexity plays a significant role in many fields of pure and applied mathematics. Its generalization, the concept of invexity is applied to problems of variational inequalities, equilibrium, nonlinear programming, and optimization.

The aim of this paper is to establish the basic functional inequality which can be widely applied to convex and preinvex functions. With this intention, we will discuss the Jensen (see [4]) and Hermite-Hadamard (see [2] and [1]) inequality.

2010 Mathematics Subject Classification: 26D15, 46E40.
Keywords and phrases: positive linear functional, convex function, preinvex function.
Received July 13, 2015

The basic structure that we use in this research is the real vector space marked with $\mathbb{X}$. As the domain of observed convex and preinvex functions, we will use bounded closed intervals in $\mathbb{R}$ and line segments in $\mathbb{R}^k$.

1.1. Convex set and convex function.

We recall the basic notions of the concept of convexity.

Definition 1. A set $C \subseteq \mathbb{X}$ is said to be *convex* if the inclusion

$$(1-t)x + ty = x + t(y-x) \in C, \tag{1}$$

holds for all points $x, y \in C$ and coefficients $t \in [0, 1]$.

Definition 2. Let $C \subseteq \mathbb{X}$ be a convex set. A function $f : C \to \mathbb{R}$ is said to be *convex* if the inequality

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y), \tag{2}$$

holds for all points $x, y \in C$ and coefficients $t \in [0, 1]$.

The expression $(1-t)x + ty$ in formula (1) is called the binomial convex combination of points x and y with coefficients $1-t$ and t. The convex hull of a set $S \subseteq \mathbb{X}$ is the smallest convex set in $\mathbb{X}$ containing S, and it consists of all binomial convex combinations of points of S. The convex hull of S is denoted with $\operatorname{conv} S$.

1.2. Invex set and preinvex function.

A notion of preinvex function was introduced in [14] and [13], and came from the notion of invex function. Some prominent properties of preinvex functions can be found in [15]. We briefly present the concept of preinvexity, referring to a preinvex function on the invex set.

Definition 3. A set $K \subseteq \mathbb{X}$ is said to be *invex* respecting a vector function $v : K \times K \to \mathbb{X}$ if the inclusion

$$(1-t)x + t(x + v(y, x)) = x + tv(y, x) \in K, \tag{3}$$

holds for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

The invex set K contains the line segment between points x and $x + v(y, x)$ for every pair of points x and y of K, because

$$x + tv(y, x) = (1 - t)x + t(x + v(y, x)). \tag{4}$$

Any subset $K \subseteq \mathbb{R}^k$ is invex respecting the vector function v identically equal to null vector.

Every convex set K is invex respecting the mapping $v(y, x) = y - x$. The following example demonstrates that the reverse statement is not true.

Example 1.1. The set $K = (-\infty, -a] \cup [a, +\infty) \subset \mathbb{R}$, where $a \geq 0$, is invex respecting the mapping $v(y, x) = x$ because it contains the combinations

$$x + tv(y, x) = (1 + t)x, \tag{5}$$

for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

Definition 4. Let $K \subseteq \mathbb{X}$ be an invex set respecting a vector function $v : K \times K \to \mathbb{X}$. A function $f : K \to \mathbb{R}$ is said to be *preinvex* respecting v if the inequality

$$f(x + tv(y, x)) \leq (1 - t)f(x) + tf(y), \tag{6}$$

holds for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

Every convex function f on the convex set K is preinvex respecting the mapping $v(y, x) = y - x$. As the following example (see [14]) shows, the converse is not true.

Example 1.2. The function $f(x) = -|x|$ observed on the set $K = \mathbb{R}$ is preinvex respecting the mapping

$$v(y, x) = \begin{cases} y - x, & xy \geq 0, \\ x - y, & xy < 0. \end{cases} \tag{7}$$

In the case $xy \geq 0$, we obtain formula (6) with the sign of equality. In the case $xy < 0$, we obtain formula (6).

2. Main Results

We aspire to determine the appropriate and applicable inequality with positive linear functionals. In that pursuit, we rely on the Jessen functional form of Jensen's inequality.

Let X be a nonempty set and $\mathbb{X}$ be a subspace of the linear space of all real functions on the domain X. We assume that the space $\mathbb{X}$ contains the unit function u defined by $u(x) = 1$ for every $x \in X$. Such space contains every real constant c because $c = cu$. Consequently, the composite function $f(g) \in \mathbb{X}$ for every affine function $f(x) = c_1 x + c_2$ and every function $g \in \mathbb{X}$.

The space of all linear functionals on the space $\mathbb{X}$ will be denoted with $\mathbb{L}(\mathbb{X})$. The functional L is positive (nonnegative) if $L(g) \geq 0$ for every nonnegative function $g \in \mathbb{X}$. The functional L is unital (normalized) if $L(u) = 1$. Such functional has the property $L(c) = c$ because $L(c) = L(cu) = cL(u) = c$.

We start with unital functionals.

Lemma 2.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an affine function*, $g \in \mathbb{X}$ *be a function*, *and* $L \in \mathbb{L}(\mathbb{X})$ *be a unital functional*.

Then

$$f(L(g)) = L(f(g)). \tag{8}$$

Proof. Using the affine equation $f(x) = c_1 x + c_2$, and the unital property of L, we obtain

$$f(L(g)) = c_1 L(g) + c_2 = L(c_1 g + c_2 u) = L(f(g)) \tag{9}$$

proving the equality in (8). □

Now we use a function $g \in \mathbb{X}$, whose image is contained in the bounded closed interval of real numbers. In this case and throughout the paper, $[a, b] \subset \mathbb{R}$ will be the closed interval with endpoints $a < b$.

Lemma 2.2. *Let* $g \in \mathbb{X}$ *be a function such that its image is in* $[a, b]$ *and* $L \in \mathbb{L}(\mathbb{X})$ *be a unital positive functional.*

Then

$$L(g) \in [a, b]. \tag{10}$$

Proof. Applying the positive and unital functional L to the function image assumption $a \leq g(x) \leq b$, we get $a \leq L(g) \leq b$ proving the inclusion in (10). □

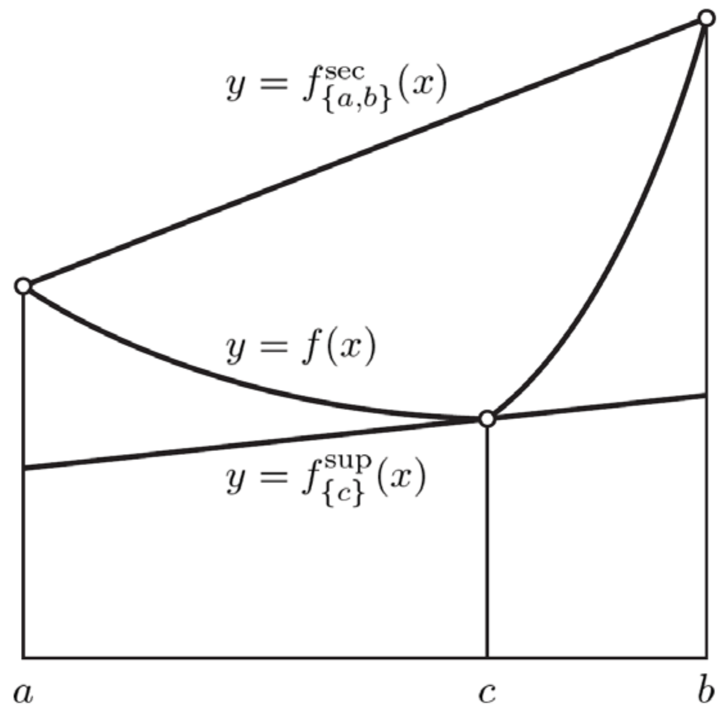

Figure 1. Support and secant line of a convex function.

Let $c \in (a, b)$ be a point. A convex function $f : [a, b] \to \mathbb{R}$ satisfies the support-secant inequality

$$f^{\text{sup}}_{\{c\}}(x) \leq f(x) \leq f^{\text{sec}}_{\{a, b\}}(x), \tag{11}$$

for every $x \in [a, b]$. The support lines pass through the graph point $C(c, f(c))$. Each support line is specified by the slope coefficient $\kappa \in [f'(c-), f'(c+)]$ and the corresponding equation:

$$f_{\{c\}}^{\sup}(x) = \kappa(x - c) + f(c). \tag{12}$$

The secant line passes through the graph points $A(a, f(a))$ and $B(b, f(b))$, and its equation is

$$f_{\{a,b\}}^{\sec}(x) = \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b). \tag{13}$$

A visual perception of the support-secant inequality can be seen in Figure 1.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function, and $g \in \mathbb{X}$ be a function such that its image is in $[a, b]$ and that the composite function $f(g)$ is in $\mathbb{X}$. Let $L \in \mathbb{L}(\mathbb{X})$ be a unital positive functional.*

Then

$$f(L(g)) \le L(f(g)) \le f_{\{a,b\}}^{\sec}(L(g)). \tag{14}$$

Proof. Let $c = L(g)$. Then $c \in [a, b]$ by Lemma 2.2. We sketch the proof in two steps depending on the location of c.

If $c \in (a, b)$, we use any support line of f at c, and the secant line of f. Since $g(x) \in [a, b]$ for every $x \in X$, we have that

$$f_{\{c\}}^{\sup}(g(x)) \le f(g(x)) \le f_{\{a,b\}}^{\sec}(g(x)) \tag{15}$$

by formula (11). Acting with the functional L to the above double inequality, and applying formula (8) to affine functions $f_{\{c\}}^{\sup}$ and $f_{\{a,b\}}^{\sec}$, we obtain the double inequality in formula (14). We also use the equality

$$f_{\{c\}}^{\sup}(L(g)) = f(L(g)). \tag{16}$$

If $c \in \{a, b\}$, we rely on the continuity of f using the support line at a point that is close enough to c. □

In 1931, Jessen (see [5] and [6]) stated the left-hand side of the inequality in formula (2.3) for a convex function f on the interval $I \subseteq \mathbb{R}$. In 1988, Raşa (see [12]) pointed out that I must be closed, otherwise it could happen that $L(g) \notin I$, and that f must be continuous, otherwise it could happen that the left-hand side of the inequality in formula (14) does not apply. Some generalizations of the functional form of Jensen's inequality can be found in [11].

3. Application to the Jensen and Hermite-Hadamard Inequality

To take advantage of Theorem 2.3, we will use $\mathbb{X}$ as the linear space of applicable real functions on the domain $X = [a, b]$.

Each point $x \in [a, b]$ can be presented by the unique binomial convex combination

$$x = \alpha a + \beta b, \tag{17}$$

where

$$\alpha = \frac{b - x}{b - a}, \quad \beta = \frac{x - a}{b - a}. \tag{18}$$

As the first consequence of Theorem 2.3, we affirm the following symmetric form of the extended Jensen's inequality.

Corollary 3.1. *Let* $g : [a, b] \to [a, b]$ *be a function,* $\sum_{i=1}^{n} \lambda_i x_i$ *be a convex combination of points* $x_i \in [a, b]$, *and* $\alpha a + \beta b$ *be the convex combination that is equal to* $\sum_{i=1}^{n} \lambda_i g(x_i)$.

Then every convex function $f : [a, b] \to \mathbb{R}$ *satisfies the double inequality*

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} \lambda_i f(g(x_i)) \leq \alpha f(a) + \beta f(b). \tag{19}$$

Proof. Let $\mathbb{X}$ be the space of all real functions on the domain $X = [a, b]$. Then the summarizing linear functional

$$L(h) = \sum_{i=1}^{n} \lambda_i h(x_i), \tag{20}$$

where $h \in \mathbb{X}$, is positive and unital. Using the given functions g and f, we obtain

$$L(g) = \sum_{i=1}^{n} \lambda_i g(x_i) = \alpha a + \beta b, \tag{21}$$

$$L(f(g)) = \sum_{i=1}^{n} \lambda_i f(g(x_i)), \tag{22}$$

and

$$f_{\{a,b\}}^{\mathrm{sec}}(L(g)) = \frac{b - L(g)}{b - a} f(a) + \frac{L(g) - a}{b - a} f(b) = \alpha f(a) + \beta f(b). \tag{23}$$

Assuming that f is continuous, and arranging the above items into the functional inequality in formula (14), we achieve the discrete inequality in formula (19). The same is true for the convex function f, which is not continuous at endpoints because the one-sided limits satisfy $f(a+) \leq f(a)$ and $f(b-) \leq f(b)$. $\square$

Taking the identity function $g(x) = x$, it follows that $\alpha a + \beta b = \sum_{i=1}^{n} \lambda_i x_i$, and the left-hand side of the inequality in formula (19) represents the classical form of the Jensen inequality.

As the second consequence of Theorem 2.3, we state the following generalization of the Hermite-Hadamard inequality.

Corollary 3.2. *Let* $g : [a, b] \to [a, b]$ *be an integrable function and* $\alpha a + \beta b$ *be the convex combination that is equal to* $(b-a)^{-1}\int_a^b g(x)dx.$

Then every convex function $f : [a, b] \to \mathbb{R}$ *satisfies the double inequality*

$$f(\alpha a + \beta b) \leq \frac{1}{b-a}\int_a^b f(g(x))dx \leq \alpha f(a) + \beta f(b). \tag{24}$$

Proof. Let $\mathbb{X}$ be the space of all integrable functions over the domain $X = [a, b]$. The composition $f(g)$ is integrable over $[a, b]$ because it is bounded, and continuous almost everywhere in $[a, b]$. The integrating linear functional

$$L(h) = \frac{1}{b-a}\int_a^b h(x)dx, \tag{25}$$

where $h \in \mathbb{X}$, is positive and unital. Applying the functional L to the given functions g and f, we get

$$L(g) = \frac{1}{b-a}\int_a^b g(x)dx = \alpha a + \beta b, \tag{26}$$

$$L(f(g)) = \frac{1}{b-a}\int_a^b f(g(x))dx, \tag{27}$$

and

$$f_{\{a,b\}}^{\sec}(L(g)) = \alpha f(a) + \beta f(b), \tag{28}$$

as in formula (23).

The functional inequality in formula (14) with the above findings stands as the integral inequality in formula (24) for any convex function f. $\square$

Taking $g(x) = x$, it follows that $\alpha = \beta = 1/2$, and the inequality in formula (24) represents the classical form of the Hermite-Hadamard inequality.

Extensions and generalizations of the above famous inequalities can be found in [9] and [10]. An interesting historical story about the Hermite-Hadamard inequality can be read in [8].

4. Application to Preinvex Functions

We want to apply the Jensen and Hermite-Hadamard inequality to preinvex functions on the invex set $K \subseteq \mathbb{R}^k$. For this purpose, we will formulate the extended Jensen's inequality (Lemma 4.1) and the Hermite-Hadamard inequality (Lemma 4.2) for convex functions on the line segment in space $\mathbb{R}^k$.

Let $a \neq b$ be a pair of points in $\mathbb{R}^k$. The line segment between points a and b will be written as the convex hull

$$\operatorname{conv}\{a, b\} = \{\alpha a + \beta b : \alpha, \beta \in [0, 1], \alpha + \beta = 1\}. \tag{29}$$

Each point $x \in \operatorname{conv}\{a, b\}$ can be presented by the unique binomial convex combination

$$x = \alpha a + \beta b, \tag{30}$$

where (using the norm $\|\ \|$)

$$\alpha = \frac{\|b - x\|}{\|b - a\|}, \quad \beta = \frac{\|x - a\|}{\|b - a\|}. \tag{31}$$

Lemma 4.1. *Let* $\sum_{i=1}^{n} \lambda_i x_i$ *be a convex combination of points* $x_i \in \operatorname{conv}\{a, b\}$, *and* $\alpha a + \beta b$ *be the convex combination that is equal to* $\sum_{i=1}^{n} \lambda_i x_i$.

Then every convex function $f : \operatorname{conv}\{a, b\} \to \mathbb{R}$ *satisfies the double inequality*

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{32}$$

Lemma 4.2. *Every convex function* $f : \operatorname{conv}\{a, b\} \to \mathbb{R}$ *satisfies the double inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\|b-a\|} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{33}$$

Using the segment equation $x = a + t(b-a)$ through the real parameter $t \in [0, 1]$, the middle term of formula (33) can be expressed by

$$\int_0^1 f(a + t(b-a))dt. \tag{34}$$

A little more about invex sets which will now be considered in space $\mathbb{R}^k$. If a set $K \subseteq \mathbb{R}^k$ is invex respecting v, and if $a, b \in K$, then the generated line segment $\operatorname{conv}\{a, a + v(b, a)\}$ is not necessarily invex respecting v. The requirement that the generated segments of the invex set be invex provides the condition introduced in [7]. It is known as condition C, and its consequence is Lemma 4.3 which among other things allows the application of the important inequalities to preinvex functions.

Definition 5. Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a vector function $v : K \times K \to \mathbb{R}^k$. It is said that the function v satisfies condition C if the equalities

$$v(x, x + tv(y, x)) = -tv(y, x), \tag{35}$$

$$v(y, x + tv(y, x)) = (1-t)v(y, x), \tag{36}$$

hold for all points $x, y \in K$ and coefficients $t \in [0, 1]$.

A consequence of condition C is the equality

$$v(x + t_2 v(y, x), x + t_1 v(y, x)) = (t_2 - t_1) v(y, x), \tag{37}$$

which holds for all points $x, y \in K$ and coefficients $t_1, t_2 \in [0, 1]$.

Assuming the presence of condition C, the following lemma shows where the preinvexity coincides with convexity.

Lemma 4.3. *Let $K \subseteq \mathbb{R}^k$ be an invex set respecting a vector function v that satisfies condition C and $f : K \to \mathbb{R}$ be a preinvex function respecting v.*

Then the function f is convex on the generated segment $\operatorname{conv}\{a, a + v(b, a)\}$ *for every pair of points $a, b \in K$.*

Proof. Let $a, b \in K$ be a pair of set points, $x, y \in \operatorname{conv}\{a, a + v(b, a)\}$ be a pair of segment points, and $t \in [0, 1]$ be a coefficient. We will verify the equality of combinations $(1 - t)x + ty$ and $x + tv(y, x)$. Using the representations

$$x = a + t_1 v(b, a), \quad y = a + t_2 v(b, a)$$

via formula (37), we get

$$\begin{aligned}(1 - t)x + ty &= (1 - t)(a + t_1 v(b, a)) + t(a + t_2 v(b, a)) \\ &= a + t_1 v(b, a) + t(t_2 - t_1) v(b, a) \\ &= a + t_1 v(b, a) + tv(a + t_2 v(b, a), a + t_1 v(b, a)) \\ &= x + tv(y, x). \end{aligned} \tag{38}$$

Taking into account the above equality, and applying the preinvexity of f to the invex combination $x + tv(y, x)$, we obtain the inequality

$$f((1 - t)x + ty) = f(x + tv(y, x)) \le (1 - t)f(x) + tf(y), \tag{39}$$

which proves the convexity of f on the segment $\operatorname{conv}\{a, a + v(b, a)\}$. $\square$

Formula (38) specifies the vector function v, it follows that $v(y, x) = y - x$ for all points x and y of the generated segment $\operatorname{conv}\{a, a + v(b, a)\}$.

The type of convexity given in Lemma 4.3 enables us to apply the convex function inequalities to preinvex functions. First and foremost, it refers to fundamental inequalities for convex functions on the line segment, which are prepared in Lemmas 4.1 and 4.2.

Extended version of the Jensen inequality for preinvex functions is the first that follows.

Corollary 4.4. *Let* $K \subseteq \mathbb{R}^k$ *be an invex set respecting a vector function* v *that satisfies condition* C *and* $f : K \to \mathbb{R}$ *be a preinvex function respecting* v. *Let* $\lambda_1, \ldots, \lambda_n \in [0, 1]$ *be coefficients such that* $\sum_{i=1}^n \lambda_i = 1$, *let* $t_1, \ldots, t_n \in [0, 1]$ *be coefficients, and let* $t = \sum_{i=1}^n \lambda_i t_i$.

Then the double inequality

$$f(a + tv(b, a)) \leq \sum_{i=1}^{n} \lambda_i f(a + t_i v(b, a)) \leq (1 - t)f(a) + tf(a + v(b, a)), \quad (40)$$

holds for every pair of points $a, b \in K$.

Proof. Let a and b be a pair of points of K and $\operatorname{conv}\{a, a + v(b, a)\}$ be the line segment with endpoints a and $a + v(b, a)$. The points $a + t_i v(b, a)$ belong to $\operatorname{conv}\{a, a + v(b, a)\}$, as well as their convex combination

$$\sum_{i=1}^{n} \lambda_i (a + t_i v(b, a)) = \sum_{i=1}^{n} \lambda_i a + \sum_{i=1}^{n} \lambda_i t_i v(b, a) = a + tv(b, a) \quad (41)$$

$$= (1 - t)a + t(a + v(b, a)).$$

The function f is convex on the generated segment $\operatorname{conv}\{a, a + v(b, a)\}$ by Lemma 4.3. Therefore, respecting the above equalities, we can employ formula (32) by using a as a, $a + v(b, a)$ as b, $a + t_i v(b, a)$ as x_i, and t as β, we obtain formula (40). □

Since

$$(1-t)f(a)+tf(a+v(b,\ a)) \leq (1-t)f(a)+tf(b), \tag{42}$$

the inequality in formula (40) can be extended to the right side. If $v(b,\ a)=0,$ the inequality in formula (40) is reduced to $f(a) \leq f(a) \leq f(a).$

The left-hand side of the inequality in formula (40) representing the Jensen inequality for preinvex functions can be written in the form

$$f\left(\sum_{i=1}^{n}\lambda_i(a+t_iv(b,\ a))\right) \leq \sum_{i=1}^{n}\lambda_i f(a+t_iv(b,\ a)). \tag{43}$$

It remains to specify the Hermite-Hadamard inequality for preinvex functions.

Corollary 4.5. *Let* $K \subseteq \mathbb{R}^k$ *be an invex set respecting a vector function* v *that satisfies condition* C *and* $f : K \to \mathbb{R}$ *be a preinvex function respecting* v.

Then the double inequality

$$f\left(a+\frac{v(b,\ a)}{2}\right) \leq \frac{1}{\|v(b,\ a)\|}\int_a^{a+v(b,\,a)} f(x)dx \leq \frac{f(a)+f(a+v(b,\ a))}{2}, \tag{44}$$

holds for every pair of points $a,\ b \in K$ *such that* $v(b,\ a) \neq 0.$

Proof. Employing formula (33) by using a as a and $a+v(b,\ a)$ as b, we obtain formula (44). □

The middle term of the inequality in formula (44) can be replaced with

$$\int_0^1 f(a+tv(b,\ a))dt. \tag{45}$$

The type of the Hermite-Hadamard inequality involving the Riemann-Liouville integrals and gamma function were considered in [3], wherein some results were achieved for positive preinvex functions on the open invex set $K \subseteq \mathbb{R}.$

5. Conclusion

The functional inequality in formula (14) can be refined in the following way. First we take a point $c \in (a, b)$. Then, we consider functionals L_1 and L_2 of the space $L(\mathbb{X})$ such that $L_1(g) \in [a, c]$ and $L_2(g) \in [c, b]$ for each function g belonging to some class of functions in $\mathbb{X}$. The approach of two functionals enables us to use secant lines $y = f^{\text{sec}}_{\{a, c\}}(x)$ and $y = f^{\text{sec}}_{\{c, b\}}(x)$ together with a certain convex combination $\alpha L_1 + \beta L_2$ of functionals L_1 and L_2. Thus, we achieve the refinement of the inequality in formula (14) and all other inequalities discussed in this paper.

Acknowledgements

The author would like to thank Velimir Pavić who has graphically prepared Figure 1.

References

[1] J. Hadamard, Étude sur les propriétés des fonctions entiéres et en particulier d'une function considerée par Riemann, J. Math. Pures Appl. 58 (1893), 171-215.

[2] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3 (1883), 82.

[3] I. Işcan, Hermite-Hadamard's inequalities for preinvex functions via fractional integrals and related fractional inequalities, AJMA 1 (2013), 33-38.

[4] J. L. W. V. Jensen, Om konvekse Funktioner og Uligheder mellem Middelværdier, Nyt Tidsskr. Math. B 16 (1905), 49-68.

[5] B. Jessen, Bemærkninger om konvekse Funktioner og Uligheder imellem Middelværdier I, Matematisk Tidsskrift B (1931), 17-28.

[6] B. Jessen, Bemærkninger om konvekse Funktioner og Uligheder imellem Middelværdier II, Matematisk Tidsskrift B (1931), 84-95.

[7] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901-908.

[8] C. P. Niculescu and L. E. Persson, Old and new on the Hermite-Hadamard inequality, Real Anal. Exchange 29 (2003), 663-685.

[9] Z. Pavić, Extension of Jensen's inequality to affine combinations, J. Inequal. Appl. 2014 (2014), Article ID 298.

[10] Z. Pavić, Functions like convex functions, J. Funct. Spaces 2015 (2015), Article ID 919470.

[11] Z. Pavić, Generalizations of the functional form of Jensen's inequality, Adv. Inequal. Appl. 2014 (2014), Article ID 33.

[12] I. Raşa, A Note on Jensen's Inequality, Itinerant Seminar on Functional Equations, Approximation and Convexity, Universitatea Babes-Bolyai, Cluj-Napoca, Romania, 1988.

[13] T. Weir and V. Jeyakumar, A class of nonconvex functions and mathematical programming, Bull. Austral. Math. Soc. 38 (1988), 177-189.

[14] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization, J. Math. Anal. Appl. 136 (1988), 29-38.

[15] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.

■