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ZERO-DIVISOR GRAPHS OF FINITE DIRECT PRODUCTS OF FINITE NON-COMMUTATIVE RINGS AND SEMIGROUPS

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Abstract

We determine the number of edges of the zero-divisor graph of the direct product of finitely many finite non-commutative rings or semigoups.

1. Introduction

This paper is a follow-up to Birch et al. [8]. Throughout this paper, R will denote a ring. In this paper, all rings are finite with identity and are not necessarily commutative.

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Let R be a ring with identity $1 \neq 0.$ Let $Z^*(R)$ denote the set of non-zero zero-divisors of R. The *zero-divisor graph* of R, denoted $\Gamma(R)$, is a directed graph, whose vertices are labelled by the elements of $Z^*(R)$. There is an edge in $\Gamma(R)$ from r to s if and only if $rs = 0.$ In this case, we say that r is *adjacent to s* and s is *adjacent from r*. Using the notation of graph theory, we say that the set of vertices of $\Gamma(R)$ is $V(\Gamma(R)) = Z^*(R)$ and the set of edges of $\Gamma(R)$ is $E(\Gamma(R)) = \{(r, s) \mid r, s \in Z^*(R) \text{ and } rs = 0\}$. In contrast to the standard definition of zero-divisor graph, if $r^2 = 0$, then we allow an edge from r to itself in the zero-divisor graph. Such an edge is called a *loop*. For a reference on graph theory, see [10].

Zero-divisor graphs were first defined for commutative rings by Beck [7] who studied graph colouring. Papers by Anderson and Naseer [3] and Anderson and Livingston [6] followed. In the last several years, there has been a large number of papers on this topic; see the survey papers by Anderson et al. [5] and Coykendall et al. [11]. Each survey contains an extensive bibliography. The latter survey also has material on the zero-divisor graphs of semigroups and posets as well as generalizations of zero-divisor graphs.

As examples of more recent research, Anderson and Badawi [4] studied the zero-divisor graph of rings, which are generalizations of valuation rings. De-Meyer et al. [12] studied zero-divisor graphs of semigroups.

Redmond [17, 18] introduced the concept of the zero-divisor graph for a non-commutative ring. Bozic and Petrovic [9] studied the zero-divisor graph of a ring of matrices over a commutative ring. Akbari and Mohammadian [1] studied the problem of determining when the zero-divisor graphs of rings are isomorphic, given that the zero-divisor graphs of their matrix rings are isomorphic, as did Redmond [19]. Li [16] and Li and Tucci [15] studied the zero-divisor graphs of upper triangular matrix rings. Dolzan and Oblak [13] studied zero-divisor graphs of semirings as well as those of rings.

In this paper, we determine a formula for the number of edges of the zero-divisor graph of a direct product of non-commutative rings or semigroups $R_1 \times \cdots \times R_t$, given the zero-divisor graphs of each R_i. This problem was solved for finite commutative rings without nonzero nilpotent elements by Lagrange [14]. In Birch et al. [8], the problem was solved for finite commutative rings in general. The techniques and results in Redmond [19] are similar to those in this paper.

Although the results in this paper and in [8] are stated for rings, they hold true for semigroups as well. It is well-known that in a finite ring every element is a unit or a zero-divisor. This is not true for semigroups; however, the only fact we need is that every element in a ring or a semigroup is either a zero-divisor or not.

For any set X, let $|X|$ denote the cardinality of X. Let U denote the set of units of R. Then $|R| - 1 = |U| + |Z^*|$. We will use this fact without explicit mention when needed.

2. The Zero-Divisor Graph of a Direct Product of Rings

In this section, we determine a formula for the number of edges in the zero-divisor graph of a direct product $R_1 \times \cdots \times R_t$ of non-commutative rings, given complete information about each $\Gamma(R_i)$ and each R_i. We develop a recursive formula for an arbitrary direct product and then we derive a non-recursive version of this formula. Finally, we give a MatLab implementation of this latter formula.

Let $R = R_1 \times R_2$. Let E be the set of edges of $\Gamma(R)$. For $i = 1, 2,$ let Z_i^* be the set of nonzero zero-divisors of R_i, let E_i denote the set of edges in $\Gamma(R_i)$, and let U_i be the set of non-zero-divisors of R_i.

In order to count the number of edges in $\Gamma(R_1 \times \cdots \times R_t)$, we first count the number of edges in $\Gamma(R_1 \times R_2)$, and then we extend this result to $\Gamma(R_1 \times \cdots \times R_t)$ by induction. Since any ring consists of non-zero-divisors and zero-divisors, the set of nonzero elements of $(R_1 \times R_2)$ is $\bigcup(A_1, A_2)$, where $A_i = Z_i^*$ or $A_i = U_i$ or $A_i = \{0\}$ for $i = 1, 2$ and either $A_1 \neq \{0\}$ or $A_2 \neq \{0\}$. To count the number of edges in $\Gamma(R_1 \times R_2)$, we construct the graph in Figure 1.

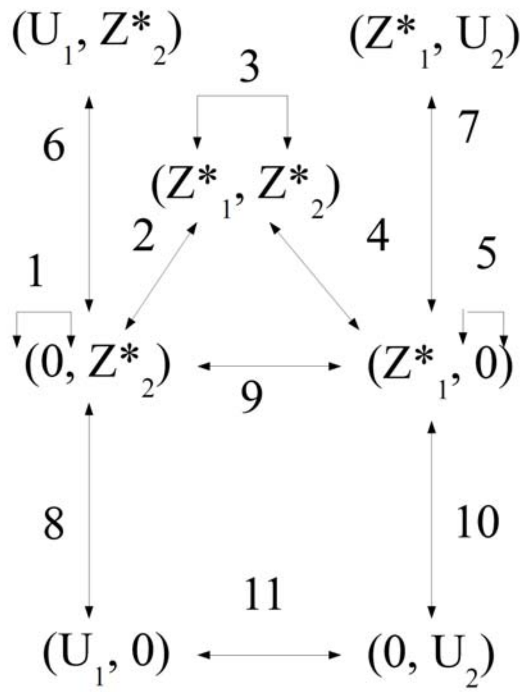

Figure 1. Sets of zero-divisors in $R_1 \times R_2$.

The numbers on the edges are labels. The vertices of this graph are the sets $(A_1, A_2) \subseteq R_1 \times R_2$. We draw an edge from (A_1, A_2) to $(A_1', A_2') \subseteq R_1 \times R_2$ precisely when there are elements $(0, 0) \neq (a_1, a_2) \in (A_1, A_2)$ and $(0, 0) \neq (a_1', a_2') \in (A_1', A_2')$ such that $(a_1, a_2)(a_1', a_2') = (0, 0)$; that is, each edge in Figure 1 from (A_1, A_2) to (A_1', A_2') represents the set of all edges in $\Gamma(R_1 \times R_2)$ from elements of (A_1, A_2) to (A_1', A_2').

If R_1 is a domain, then $Z_1^* = \emptyset$, and hence the vertices (Z_1^*, Z_2^*), $(Z_1^*, 0)$, (Z_1^*, U_2) do not appear in the graph. Likewise, if R_2 is a domain, then $Z_2^* = \emptyset$, and the vertices (Z_1^*, Z_2^*), (U_1, Z_2^*), $(0, Z_2^*)$ do not appear in the graph.

Lemma 2.1. *For each edge labelled by* n, $1 \leq n \leq 11$, *in Figure* 1, *let* $Card(n)$ *denote the number of edges in* $\Gamma(R_1 \times R_2)$ *represented by this edge. Then the values of* $Card(n)$ *are given as follows*:

n	1	2	3	4	5	6	7
$Card(n)$	$\lvert E_2 \rvert$	$2\lvert Z_1^* \rvert \lvert E_2 \rvert$	$\lvert E_1 \rvert \lvert E_2 \rvert$	$2\lvert Z_2^* \rvert \lvert E_1 \rvert$	$\lvert E_1 \rvert$	$2\lvert U_1 \rvert \lvert E_2 \rvert$	$2\lvert U_2 \rvert \lvert E_1 \rvert$

n	8	9	10	11
$Card(n)$	$2\lvert U_1 \rvert \lvert Z_2^* \rvert$	$2\lvert Z_1^* \rvert \lvert Z_2^* \rvert$	$2\lvert Z_1^* \rvert \lvert U_2 \rvert$	$2\lvert U_1 \rvert \lvert U_2 \rvert$

Proof. Clearly, the values for $Card(1)$ and $Card(5)$ are correct.

For $Card(2)$, let $z_1 \in Z_1^*$ and let $z_2, z_2' \in Z_2^*$, where $z_2 z_2' = 0$. Then, there are distinct edges $(z_1, z_2) \to (0, z_2')$ and $(0, z_2) \to (z_1, z_2')$. Therefore, $Card(2) = 2|Z_1^*||E_2|$. A similar argument holds for $Card(4)$.

For $Card(3)$, let $z_i, z_i' \in Z_i^*$ for $i = 1, 2$, where $z_i z_i' = 0$. These elements give rise to the edge $(z_1, z_2) \to (z_1', z_2')$. The number of such edges is $|E_1||E_2|$.

For $Card(6)$, let $u_1 \in U_1, z_2, z_2' \in Z_2^*$, where $z_2 z_2' = 0$. These elements give rise to the edges $(u_1, z_2) \to (0, z_2')$ and $(0, z_2) \to (u_1, z_2')$. The number of such edges is $2|U_1||E_2|$. A similar argument holds for $Card(7)$.

For $Card(8)$, let $u_1 \in U_1$ and $z_2 \in Z_2^*$. These elements give rise to the edges $(u_1, 0) \to (0, z_2)$ and $(0, z_2) \to (u_1, 0)$. The number of such edges is $2|U_1||Z_2^*|$. A similar argument holds for $Card(9)$, $Card(10)$, and $Card(11)$. □

Proposition 2.2. *The number of edges in* $\Gamma(R_1 \times R_2)$ *is*

$$|E| = (2|R_1| + |E_1| - 1)|E_2| + (2|R_2| - 1)|E_1| + 2(|R_1| - 1)(|R_2| - 1). \tag{1}$$

Proof. From Lemma 2.1, add $Card(1) + Card(2) + Card(3) + Card(6)$ to obtain

$$|E_2| + 2|Z_1^*||E_2| + |E_1||E_2| + 2|U_1||E_2| = |E_2| + 2(|R_1| - 1)|E_2| + |E_1||E_2| = (2|R_1| + |E_1| - 1)|E_2|.$$

In a similar manner, from Lemma 2.1, add $Card(4) + Card(5) + Card(7)$ to obtain $(2|R_2| - 1)|E_1|$. Finally, add the remaining terms in Lemma 2.1 to obtain $2(|R_1| - 1)(|R_2| - 1)$. □

We now describe the general case. Let $R_{1,\ldots,t} = R_1 \times \cdots \times R_t$ for some t and let $R = R_1 \times \cdots \times R_t \times R_{t+1}$. Let $E_{1,\ldots,t}$ denote the edges of $\Gamma(R_{1,\ldots,t})$ and let E denote the edges of $\Gamma(R_{1,\ldots,t+1})$. Suppose that we know $|E_{1,\ldots,t}|$, $|E_{t+1}|$, and $|R_i|$ for each $1 \leq i \leq t+1$.

Proposition 2.3. *The number of edges in* $\Gamma(R)$ *is*

$$|E| = (2|R_{1,\ldots,t}| + |E_{1,\ldots,t}| - 1)|E_{t+1}| + (2|R_{t+1}| - 1)|E_{1,\ldots,t}| + 2(|R_{1,\ldots,t}| - 1)(|R_{t+1}| - 1). \tag{2}$$

Proof. In Equation (3) replace E_1 and R_1, by $E_{1,\ldots,t}$ and $R_{1,\ldots,t}$, respectively, and replace E_2 and R_2, by E_{t+1} and R_{t+1}, respectively. □

We now present a non-recursive version of Proposition 2.3.

Theorem 2.4. *Let* $c_i = 2|R_{i+1}| + |E_{i+1}| - 1,$ *and let* $b_i = 2|R_{1,\dots,t}|(|R_{t+1}| + |E_{t+1}| - 1) - 2|R_{t+1}| - |E_{t+1}| + 2.$ *Then Equation* (2) *becomes*

$$|E_{1,\dots,t+1}| = \left(\prod_{i=1}^{t} c_i\right) E_1 + b_t + \sum_{i=1}^{t-1}\left(b_i \prod_{j=i+1}^{t} c_j\right). \tag{3}$$

Proof. Write Equation (2) as $a_{i+1} = c_i a_i + b_i,$ where $a_{i+1} = |E_{1,\dots,t+1}|,$ $a_i = |E_{1,\dots,i}|,$ and b_i, c_i are given above. The solution to this equation is given in Allen [2, p.6, 1.8]. □

Following is a MATLAB implementation of formula 3.

```
Function ZD = calc_ZD(E, R)

% The inputs E and R are vectors containing the number of edges in the
% zero-divisor graphs of each R(i) and the cardinality of each R(i),
% respectively. The output ZD is the number of edges in the zero-divisor
% graph of the direct product of the R(i)'s.

for i = 1 : length (E) - 1
    c(i) = 2 * R(i + 1) + E(i + 1) - 1;
end
for i = 1 : length (E) - 1
b(i) = 2 * R_prod (R, i) * (R(i + 1) + E(i + 1) - 1) - 2 * R(i + 1) - E(i + 1) + 2;
end
    ZD = c_prod(c, 1) * E(1) + b(length(b));
for i = 1 : length(b) - 1
    ZD = ZD + b(i) * c_prod(c, i + 1);
end
```

```
%-----------------------------------------------------------------------------------------------
    function c_prod = c_prod(c, i)
    % This function calculates the products containing the (c, i)'s
    % defined in Theorem 2.4.
c_prod = c(i);
for j = i : length(c) - 1
    c_prod = c_prod * c(j + 1);
end
%-----------------------------------------------------------------------------------------------
    function R_prod = R_prod(R, i)
    % This function calculates the products of the cardinalities
    % of the R(i)'s as required in Equation (3).
    R_prod = R(1);
for j = 1 : i - 1
    R_prod = R_prod * R(j + 1);
end.
```

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