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THE BALANCED CREDIBILITY ESTIMATORS WITH MULTITUDE CONTRACTS OBTAINED UNDER LINEX LOSS FUNCTION

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Abstract

Considering the target premium, we propose LINEX loss function to solve the problem of high premium by using a balanced loss function in most of classical credibility models. The inhomogeneous and homogeneous credibility estimators with multitude contracts are derived under LINEX loss function. Finally, the simulations have been introduced to show the consistency of the credibility estimators.

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1. Introduction

In insurance practice, credibility theory is a set of quantitative methods, which allows an insurer to adjust premium based on the policyholder's experience and the experience of the entire group of policyholders. It has been widely used in commercial property of liability insurance and group health or life insurance. The well-known credibility formulas obtained are written as a weighted sum of the average experience of the policyholder and the average of the entire collection of policyholders. These formulas are easy to understand and simple to apply due to their linear properties. The modern credibility theory is believed to be attributed to the remarkable contribution by Bühlmann [1], which is the first one that based the credibility theory on modern Bayes statistics. For the recent detailed introduction, see Bühlmann and Gisler [2], which describes modern credibility theory comprehensively.

In classical decision theory, the loss function usually focus on precision of estimation. However, goodness of fit is also a very important criterion. Thus, there is a need to provide of estimation formally. Zellner [11] introduced a general class of balanced loss functions of the form

$$L_1(x, \theta) = w(\delta_0(x) - \theta)^2 + (1 - w)(x - \theta)^2, \tag{1.1}$$

where $0 \leq w \leq 1$, and $\delta_0(x)$ is a pre-determined target estimator of θ. Huang and Wu [5] studied the Bühlmann and Bühlmann-Straub models under balanced loss function, established the credibility premiums with common effects. Furthermore, using balanced loss function, a generalization of the credibility expression in Bühlmann [1] under the distribution free approach is also obtained.

However, in some estimation problems, use of symmetric loss functions may be in-appropriate, see, for example, Varian [9], Berger [3], Fergudon [4], Promislow [6], Promislow and Young [7]. Credibility estimator using symmetric loss functions usually lead to very high maluses. To overcome this problem, using asymmetric loss functions in building credibility estimator is considered. That is, for a policyholder, "paying too much" is more serious than "not paying enough". So in asymmetric loss functions overcharges we should be penalized more than undercharges to satisfy the policyholder. Varian [9] introduced LINEX (linear exponential) loss function, which is a useful asymmetric loss function.

$$L_2(x,\ \theta) = e^{a(x-\theta)} - a(x - \theta) - 1, \quad a > 0. \tag{1.2}$$

The LINEX loss function rises approximately exponentially on one side of zero and approximately linearly on the other side of zero, the property of a LINEX function is usefull in building a credibility estimator. Inspired by these papers, we aim at extending credibility estimators with multitude contracts under LINEX loss function.

The rest of the paper is arranged as follows. In Section 2, model assumptions are introduced and some preliminaries are discussed. Section 3 derives the credibility estimator under LINEX loss function. Finally, the simulations have been done to investigate the consistency of credibility estimator under LINEX loss function.

2. Model Assumptions and Preliminaries

Consider a portfolio of K insured individuals. In this portfolio, each individual I is associated with a claim experience X_{ij} over n time periods $j = 1, 2, \cdots, n$. Write $X_i = (X_{i1}, \cdots, X_{in})'$, $i = 1, 2, \cdots, K$. Our interest is to predict the future claim $X_{i,n+1}$ for each individual, taking into account all observed claim experiences $X_1, X_2, \cdots, X_K$. It is well known from statistical theory that the best LINEX premium based on all the observed claims $X_1, X_2, \cdots, X_K$, denoted by $H(X_{i,n+1}) = \frac{1}{a}\ln$ $\left(\frac{1}{E(e^{-aX_{i,n+1}}|X_1, X_2, \cdots, X_K)}\right)$, is the solution of the minimization problem

$$\min_g E[e^{a(g(X_1, X_2, \cdots, X_K) - X_{i,n+1})} - a(g(X_1, X_2, \cdots, X_K) - X_{i,n+1}) - 1]. \tag{2.1}$$

In the classical credibility theory, we assumed the risk quality of an individual i can be characterized by a risk parameter Θ_i, which is an unobservable random variable. Given Θ_i, the claims $X_{i1}, \cdots, X_{in}$, $X_{i,n+1}$ are independent and identically distributed. Formally, the assumptions of the model are stated as following:

Assumption 2.1. For fixed contract i, given Θ_i, X_{ij} are conditionally independent, with $E(X_{ij}|\Theta_i) = \mu(\Theta_i)$ and $\mathrm{Var}(X_{ij}|\Theta_i) = \dfrac{\sigma^2(\Theta_i)}{m_{ij}}$, where m_{ij} are known weights. We will use the following notations regarding the weights: $M_i^{-1} = \mathrm{diag}(m_{i1}, \cdots, m_{in})$, $m_i = M_i^{-1}\ \mathbf{1}_n$, $\overline{m}_i = \mathbf{1}'_n m_i / n$.

Assumption 2.2. The risk parameter $\Theta_1, \Theta_2, \cdots, \Theta_K$ are independent and identically distributed as the same structure distribution function $\pi(\theta)$.

Assumption 2.3. The random vectors (X_i, Θ_i) are independent for $i = 1, 2, \cdots, K$.

Under the assumptions above, we can define the risk premiums of $X_{i,n+1}$ under LINEX loss function for $i = 1, 2, \cdots, K$.

Definition 1.1. The risk premiums are given by

$$H(X, \Theta_i) = \frac{1}{a} \ln\left(\frac{1}{\mu(\Theta_i)}\right), \quad \alpha > 0, \tag{2.2}$$

where $\mu(\Theta_i) = E(e^{-aXi,n+1}|\Theta_i)$.

From the credibility estimators of $\mu(\Theta_i)$, the estimators of $H(X, \Theta_i)$ can be easily derived by $\widehat{H(X, \Theta_i})= \dfrac{1}{a} \ln\left(\dfrac{1}{\widehat{\mu(\Theta_i)}}\right)$. So we firstly consider the inhomogeneous estimators of $\mu(\Theta_i)$ by means of credibility idea, i.e., to solve the following optimal problem:

$$\min_{c_0, c_{ij} \in R} E[w(e^{-a\delta_{0i}(X)} - c_0 - \sum_{i=1}^{K}\sum_{j=1}^{n} c_{ij} e^{-aX_{ij}})^2 + (1-w)(\mu(\Theta_i) - c_0 - \sum_{i=1}^{K}\sum_{j=1}^{n} c_{ij} e^{-aX_{ij}})^2], \tag{2.3}$$

where $e^{-a\delta_{0i}(X)}$ is a priori chosen target predictor of $\mu(\Theta_i)$. For statistic $e^{-a\delta_{0i}(X)}$, introduce the following more notations:

$$E[e^{-a\delta_{0i}(X)}] = \mu, \ \mathrm{Cov}[e^{-a\delta_{0i}(X)}, e^{-aX_{ij}}] = d_{ij}, \ d_i' = (d_{i1}, d_{i2}, \cdots, d_{iK}),$$

$$\lambda_i = \frac{n\overline{m_j}}{\sigma^2 + n\overline{m_j}\tau^2}, \ \lambda = \sum_{i=1}^{K}\lambda_i, \ \overline{X}_i^e = \frac{1}{n}\sum_{j=1}^{n} e^{-aX_{ij}}, \ \overline{\overline{X}}_{d_i}^e = \frac{1}{\sum_{t=1}^{K} d_{it}\lambda_t}\sum_{t=1}^{K} d_{it}\lambda_t \overline{X}_t^e.$$

To simplify, we set $\delta_{0i}(Y^*) = e^{-a\delta_{0i}(X)}$, $Y_{ij} = e^{-aX_{ij}}$, $i = 1, 2, \cdots, K$, $j = 1, 2, \cdots, n$ and $Y = (Y_1', Y_2', \cdots, Y_K')'$, here $Y_i = (Y_{i1}, Y_{i2}, \cdots, Y_{in})'$. We can get the following lemma:

Lemma 2.1. *Under the Assumptions* 2.1-2.3, *we have*

(1) *The means of* Y_i *and* Y *are given by*

$$E(Y_i) = \mu \mathbf{1}_n, \quad i = 1, 2, \cdots, K, \quad E(Y) = \mu \mathbf{1}_{nK}, \tag{2.4}$$

where $\mathbf{1}_n$ *is an n-vector with* 1 *in all of the n entries.*

(2) *The covariance between* $\mu(\Theta_i)$ *and* Y *is given by*

$$\sum\nolimits_{\mu(\Theta_i)Y} \mathrm{Cov}(\mu(\Theta_i), Y) = \tau^2 e_i' \otimes \mathbf{1}_n', \tag{2.5}$$

where e_i *is a vector with* 1 *in the i-th entry and* 0 *in the other entries. Here*, "$\otimes$" *indicates the Kronecker product of matrices.*

(3) *The covariance of* Y *is given by*

$$\sum_{\mathrm{YY}} = \mathrm{Cov}(Y, Y) = I_K \otimes \mathrm{diag}(\sigma_0^2 M_i + \tau^2 m_i m_i'). \tag{2.6}$$

(4) *The inverse of the variance matrix of* Y *is given by*

$$\sum_{YY}^{-1} = \frac{1}{\sigma^2} I_K \otimes (M_i^{-1} - \frac{\tau^2 m_i m_i'}{\sigma^2 + n\overline{m_i}\tau^2}). \tag{2.7}$$

3. The Credibility Estimator Under LINEX Loss Function

In this section, we proceed to drive the credibility estimator of $H(X, \Theta_i)$ under the LINEX loss function. We state the following theorem:

Theorem 3.1. *Under Assumptions* 2.1-2.3, *the inhomogeneous credibility estimators* $H(X, \Theta_i)$ *under LINEX loss function are given by*

$$\widehat{H(X, \Theta_i)} = \frac{1}{\alpha} \ln\left(\frac{1}{Z_{i1}\overline{X}_i^e + Z_{i2}\overline{\overline{X}}_{d_i}^e + (1 - Z_{i1} - Z_{i2})\mu}\right), \tag{3.2}$$

where $Z_{i1} = \dfrac{n\overline{m_i}\tau^2}{\sigma^2 + n\overline{m_i}\tau^2}$, $Z_{i2} = \sum_{i=1}^{K} d_{it}\lambda_t$.

Proof. For each fixed i, introduce $Z_i = I\delta_{0i}(Y^*) + (1 - I)\mu(\Theta_i)$, where I is an auxiliary random variable statistically independent of all the other random variables in this system and distributed as $P(I - 1) = 1 - P(I = 0) = w$, the weight in (2.3). Thus, we can rewrite the optimization problem as

$$\min_{c_0, c_s} E[Z_i - c_0 - \sum_{i=s}^{K} c_s' Y_s)^2], \tag{3.3}$$

where $c_0 \in R$, $c_s \in R^{n_s}$. So, the inhomogeneous credibility estimator of $\mu(\Theta_i)$ is exactly the orthogonal projection of Z_i on linear space: $L(Y, 1)$ $= \{c_0 + \sum_{s=1}^{K} c_s' Y_s,\ c_0 \in R,\ c_s \in R^{n_s}\}$, i.e., $\widehat{\mu(\Theta_i)} = \mathrm{proj}(Z_i \mid L(Y, 1))$ (see, for example, Wen et al. [10]). From the relationship between orthogonal projection and credibility estimator, we have $\widehat{\mu(\Theta_i)} = E(Z_i) + \sum_{Z_i Y} \sum_{YY}^{-1}(Y - E(Y))$. From the definition of Z_i, the mean $E(Z_i)$ can be computed by $E(Z_i) = \mu$. Moreover, in view of the fact $E(Y|I) = E(Y)$ is a constant, yielding the equality $\mathrm{Cov}(E(Z_i|I), E(Y|I)) = 0$. The covariance matrix $\sum_{Z_i Y}$ can be computed by

$$\sum_{Z_i Y} = \mathrm{Cov}(Z_i, Y) = w d_i' \otimes \mathbf{1}_n' + (1 - w)\sum_{\mu(\Theta_i)Y}. \tag{3.4}$$

By Lemma 2.1, we need to get the following terms:

$$w d_i' \otimes \mathbf{1}_n' \sum_{YY}^{-1}(Y - E(Y)) = w\sum_{t=1}^{K} d_{it} \frac{n\overline{m_t}}{\sigma^2 + n\overline{m_t}\tau^2}(\overline{\overline{X}}{}^e_{d_i} - \mu), \tag{3.5}$$

and

$$(1 - w)\sum_{\mu(\Theta_i)Y}\sum_{YY}^{-1}(Y - E(Y)) = \frac{n(1 - w)\overline{m_i}\tau^2}{\sigma^2 + n\overline{m_i}\tau^2}(\overline{X}{}^e_i - \mu). \tag{3.6}$$

Then

$$\widehat{\mu(\Theta_i)} = w\sum_{t=1}^{K} d_{it} \frac{n\overline{m_t}}{\sigma^2 + n\overline{m_t}\tau^2}\overline{\overline{X}}{}^e_{d_i} + \frac{n(1 - w)\overline{m_i}\tau^2}{\sigma^2 + n\overline{m_i}\tau^2}\overline{X}{}^e_i$$

$$+ (1 - w\sum_{i=1}^{K} d_{it} \frac{n\overline{m_t}}{\sigma^2 + n\overline{m_t}\tau^2} - \frac{n(1 - w)\overline{m_i}\tau^2}{\sigma^2 + n\overline{m_i}\tau^2})\mu.$$

If we constrain the estimator of $\mu(\Theta_i)$ to be a homogeneous linear class of Y, we can derive the homogeneous credibility estimator. Hence we should solve the following problem:

$$\min_{c_0, c_s} E[Z_i - \sum_{i=s}^{K} c_s' Y_s)^2], \text{ with } E(Z_i) = E(\sum_{i=s}^{K} c_s' Y_s). \tag{3.7}$$

Then we obtain the following theorem:

Theorem 3.2. *The homogeneous credibility estimators of* $H(X, \Theta_i)$ *are*

$$\widehat{H(X, \Theta_i)}^* = \frac{1}{a}\ln\left(\frac{1}{Z_{i1}\overline{\overline{X}}_{d_i}^e + Z_{i2}\overline{X}_i^e + (1 - Z_{i1} - Z_{i2})\overline{\overline{X}}^e}\right),$$

where Z_{i1}, Z_{i2} *are the same as in Theorem* 3.1 *and* $\overline{\overline{X}}^e = \frac{1}{K}\sum_{i=1}^{K}\overline{X}_i^e$.

Proof. Write $Le(Y) = \sum_{i=s}^{K} c_s' Y_s$, with $E(Z_i) = E(\sum_{i=s}^{K} c_s' Y_s)$. Then the homogeneous credibility estimator of $\mu(\Theta_i)$ is exactly the orthogonal projection in the linear space $Le(Y)$, i.e., $\widehat{\mu(\Theta_i)}^* = \text{proj}(Z_i \mid L_e(Y))$ (see, Wen et al. [10]). Since $Le(Y) \in L(Y, 1)$, from the iteratively of projection operator (see Bühlmann and Gisler [2]), one can obtain

$$\widehat{\mu(\Theta_i)}^* = \text{proj}(\text{proj}(Z_i \mid L(Y, 1) \mid Le(Y)) = Z_{i1}\overline{X}_i^e + Z_{i2}\overline{\overline{X}}_{d_i}^e + (1 - Z_{i1} - Z_{i2})\,\text{proj}(\mu \mid Le(Y)).$$

Recall the formula

$$\text{proj}(\mu | Le(Y) = \frac{\mu \in (Y')\sum_{YY}^{-1} Y}{E(Y')\sum_{YY}^{-1} E(Y)}. \tag{3.8}$$

The proof can be referred to (see, Wen et al. [10]).

Inserting (2.4) and (2.7) into (3.8), we get

$$\text{proj}(\mu | Le(Y) = \frac{\mu^2 \mathbf{1}_{nK}' \frac{1}{\sigma^2} I_K \otimes (M_i^{-1} - \frac{\tau^2 m_i m_i'}{\sigma^2 + n\overline{m}_i\tau^2})Y}{\mu^2 \mathbf{1}_{nK}' \frac{1}{\sigma^2} I_K \otimes (M_i^{-1} - \frac{\tau^2 m_i m_i'}{\sigma^2 + n\overline{m}_i\tau^2})\mathbf{1}_{nK}} = \overline{\overline{X}}^e.$$

Remark. For fixed i, if $n \to \infty$, the $\overline{X}_i^e \to \mu(\Theta_i)$, a.s. from the central limit theorem.

4. Numerical Example

Here, we give an example to show the credibility esyimator under LINEX loss function and check the consistency of credibility estimator $\widehat{H(X, \Theta_i)}$ given as in Theorem 3.1.

We assume that the claim of the i-th contract in j-th year X_{ij} is distributed as (Θ_i, σ_0^2), and the risk parameter Θ_i is exponential variable with density function $\pi(\theta) = \frac{1}{\mu} e^{\frac{1}{\mu}\theta}$. In order to compare the estimators in Theorem 3.1 in (3.2), the following simulations are needed. First, we take $K = 10$, $n = 100$, $a = 0.05$, $\sigma_0^2 = 0.81$ and $m_{ij} = 1$ for $i = 1, 2, \cdots, 10$, $j = 1, 2, \cdots, 100$. In this simulation, we assume that $\delta_{0i}(X) = -\frac{1}{a}\ln(\overline{X}_i^e)$, then $E(e^{-a\delta_{0i}(X)}) = \mu$. We can get

$$d_{ij} = \mathrm{Cov}(e^{-a\delta_{0i}(X)}, e^{-aX_{jt}}) = \frac{1}{n}\sum_{l=1}^{n} \mathrm{Cov}(e^{-a\delta_{il}}, e^{-aX_{jt}}) = \begin{cases} \frac{\sigma^2}{n} + \tau^2 & i = j \\ 0 & i \neq j \end{cases},$$

and

$$d_i = \frac{\sigma^2}{n} + \tau^2, \quad \overline{\overline{X}}_{d_i}^e = \overline{X}_i^e.$$

Then the credibility estimators of $H(X, \Theta_i)$ are given by

$$\widehat{H(X, \Theta_i)} = -\frac{1}{a}\ln\left[\frac{w\sigma^2 + n\tau^2}{\sigma^2 + n\tau^2}\overline{X}_i^e + \frac{(1-w)\sigma^2}{\sigma^2 + n\tau^2}\mu\right].$$

The corresponding quantities defined in Section 2 can be derive as

$$\mu = \frac{e^{0.5a^2\sigma_0^2}}{1 + a\mu}, \quad \tau^2 = \frac{e^{a^2\sigma_0^2}}{1 + 2a\mu} - \frac{e^{a^2\sigma_0^2}}{(1 + a\mu)^2}, \quad \sigma^2 = \frac{e^{2a^2\sigma_0^2} - e^{a^2\sigma_0^2}}{1 + 2a\mu},$$

and

$$H(X, \Theta_i) = -a\Theta_i + 0.5a^2\sigma^2.$$

Nine different values $\Theta_i = 0.1, 0.2, 0.3, \cdots, 0.7,$ and three weights $w = 0.1, w = 0.5, w = 0.7,$ are considered. For each combination of values of parameters Θ_i and w, we carry out a simulation of 10000 times. We dirive the simulation results are listed in the following tables:

Table 1. The results with $w = 0.1$

Θ_i	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$H(X, \Theta_i)$	0.0925	0.1925	0.2925	0.3925	0.4925	0.5925	0.6925
$H\widehat{(X, \Theta_i)}$	0.0948	0.1949	0.2946	0.3942	0.4939	0.5935	0.6933
std	0.0548	0.0545	0.0548	0.0545	0.0546	0.0546	0.0546

Table 2. The results with $w = 0.5$

Θ_i	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$H(X, \Theta_i)$	0.0925	0.1925	0.2925	0.3925	0.4925	0.5925	0.6925
$H\widehat{(X, \Theta_i)}$	0.0938	0.1937	0.2936	0.3936	0.4932	0.5931	0.6934
std	0.0546	0.0547	0.0546	0.0546	0.0547	0.0546	0.0543

Table 3. The results with $w = 0.7$

Θ_i	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$H(X, \Theta_i)$	0.0925	0.1925	0.2925	0.3925	0.4925	0.5925	0.6925
$H\widehat{(X, \Theta_i)}$	0.0932	0.1931	0.2932	0.3931	0.4929	0.5928	0.6928
std	0.0546	0.0546	0.0545	0.0547	0.0547	0.0545	0.0546

where std indicates the mean square error for the estimator $H\widehat{(X, \Theta_i)}$. We can see from the tables above, that $H\widehat{(X, \Theta_i)}$ is consistent with the premium $H(X, \Theta_i)$.

References

[1] H. Bühlmann, Experience rating and credibility, J. Astin Bulletin 4 (1967), 199-207.

[2] H. Bühlmann and A. Gisler, A Course in Credibility Theory and its Application, Springer, The Netherlands, 2005.

[3] J. O. Berger, Statistical Decision Theory: Foundations, Concepts and Methods, Academic Press, New York, 1980.

[4] T. S. Fergudon, A Decision Theoretic Approach, Academic Press, New York, 1967.

[5] W. Z. Huang and X. Y. Wu, The credibility premiums with common effects obtained under balanced loss functions, Chinese Journal of Applied Probability and Statistics 28(2) (2012), 203-216.

[6] S. D. Promislow, Measurement of equity, Transaction of the Society of Actuaries 39 (1987), 215-256.

[7] S. D. Promislow and V. R. Young, Equity and exact credibility, ASTIN Bulletin 30(1) (2000), 3-13.

[8] R. Rao and H. Toutenburg, Linear Models, Springer, New York, 1995.

[9] H. R. Varian, A Bayesian approach to real estate assessment, Studies in Bayesian econometrics and statistics in honor of Leonard J. Savage, Amsterdam, North-Holland, (1975), 195-208.

[10] L. M. Wen, X. Y. Wu and X. Zhou, The credibility premiums for models with dependence induced by common effects, Insurance: Mathematics and Economics 44(1) (2009), 19-25.

[11] A. Zellner, Bayesian and non-Bayesian estimation using balanced loss function, Statistical Decision Theory and Related Topics V (J. O. Berger and S. S. Gupta Eds.), Springer-Verlag, New York, (1994), 377-390.

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