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TOTAL NUMBER OF JUMP-POINTS OF ORDERED PURE *P*-EXTENSIONS

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Abstract

Fraenkel and Tanny in [9] introduced Wythoff-like games, denoted by $\mathrm{Wyt}(f)$. We define a class of new games based on a given $\mathrm{Wyt}(f)$: Let Γ_K be the new game obtained from $\mathrm{Wyt}(f)$ by adjoining to it the first K P-positions as additional moves. For an integer $m \geq 1$, if the set of all P-positions of Γ_m does not equal to the set of all P-positions of Γ_{m-1}, we call m a jump-point of $\mathrm{Wyt}(f)$.

The main purpose of this paper is to analyze the structure and total number of jump-points of $\mathrm{Wyt}(f)$ for $f(x) = sx$ with integer coefficient $s \geq 2$. It turns out that 1 is the only jump-point if $s = 2$, and 1 and 2 are the only two jump-points if $s \geq 3$.

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1. Introduction

By game we mean a combinatorial game; we restrict our attention to classical impartial games. There are two conventions: in *normal play convention*, the player first unable to move is the loser (his opponent the winner); in *misère play convention*, the player first unable to move is the winner (his opponent the loser). The positions from which the previous player can win regardless of the opponent's moves are called *P-positions*, and those from which the next player can win regardless of the opponent's moves are called *N-positions*. See [1, 2].

Wythoff's game is played with two piles of tokens. Each player can either remove any positive number of tokens from a single pile (*Nim-rule*) or remove the same positive number of tokens from both piles (*Wythoff-rule*). All *P*-positions of Wythoff's game under normal play convention were given in [12]. All *P*-positions of Wythoff's game under misère play convention were determined in [7].

Given a game Γ, let $\mathscr{M}(\Gamma)$ be the set of all moves of Γ and $\mathscr{P}(\Gamma)$ be the set of all *P*-positions of Γ. If $\mathscr{M}(\Gamma) \subset \mathscr{M}(\Gamma_1)$, we call Γ_1 an *extension* of Γ. In many papers devoted to variations of Wythoff's game, new rules are adjoined to the original ones. In particular, the following extensions can be found in the literature.

1.1. Wythoff-like games

Fraenkel in [5] introduced the so-called (s, t)-Wythoff's game, where the Wythoff-rule is changed into the *More General Wythoff-rule*: Take tokens from both piles, $k > 0$ from one pile and $\ell > 0$ from the other, such that $0 < k \leq \ell < sk + t$. Under normal or misère play convention, all *P*-positions of (s, t)-Wythoff's game were given in [5, 10] for all integers s, $t \geq 1$.

Fraenkel and Tanny in [9] defined an even more general version of the game by replacing $sk + t$, $s, t > 0$, by an arbitrary function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ and requiring that $0 < k \leq \ell < f(k)$, the *Wyt*(f)*-rule*. Hence, for any function f, one has a *Wythoff-like game*, denoted by $\mathrm{Wyt}(f)$.

1.2. Pure *P*-extension

It was pointed out in Section 6 of [4] that interesting games can be obtained by adjoining to a given game an appropriate subset of its P-positions as moves. Such an extension is called to be *pure P-extension*, i.e., all additional moves are P-positions. For example, Wythoff's game is 2-pile Nim with all P-positions adjoined. The idea was also exploited in [8] to examine games which bridge Nim and Wythoff's game. The papers [6] and [11] are devoted to the new games obtained by adjoining to a given game Γ its P-positions as additional moves, where Γ are a-Wythoff's game and (s, t)-Wythoff's game, respectively.

The following questions 1 and 2 are nature:

Question 1. Are there a pair of games Γ and Γ_e with $\mathscr{M}(\Gamma) \subset \mathscr{M}(\Gamma_e)$ such that $\mathscr{P}(\Gamma_e) \neq \mathscr{P}(\Gamma)$, i.e., an extension of Γ do not preserve the same set of P-positions of Γ?

Question 2. Are there a pair of games Γ and Γ_e with $\mathscr{M}(\Gamma) \subset \mathscr{M}(\Gamma_e)$ such that $\mathscr{P}(\Gamma_e) = \mathscr{P}(\Gamma)$, i.e., an extension of Γ gives the same set of P-positions of Γ?

The answer to question 1 is ordinary. Normally, a slight change on the rule-set of Γ would lead to the change of the whole set of P-positions of Γ. As an example, let Γ be Wythoff's game. The partial result of [3] showed us that $\mathscr{P}(\Gamma_r) \neq \mathscr{P}(\Gamma)$ for any restriction Γ_r of Γ, i.e., no strict subset of rules of Wythoff's game gives the same set of P-positions. Note that Γ is an extension Γ_r.

Generally, under *normal play convention*, adjoining to $\text{Wyt}(f)$ a P-position do not preserve the same set of P-positions of $\text{Wyt}(f)$: (0, 0) is a P-position. Let (A_k, B_k) with $k \geq 1$ be a P-position of $\text{Wyt}(f)$ and Γ_e be the game obtained from $\text{Wyt}(f)$ by adjoining to it the P-position (A_k, B_k) as an additional move. Then $(A_k, B_k) \rightarrow (0, 0) \in \mathscr{P}(\Gamma_e)$, i.e., (A_k, B_k) is an N-position of Γ_e. Hence $\mathscr{P}(\Gamma_e)$ is not the same set of all P-positions of $\text{Wyt}(f)$.

The answer to question 2 is also ordinary. Let Γ be a-Wythoff's game, i.e., $f(x) = x + a$ in $\text{Wyt}(f)$. The authors in [6] considered three new games under normal play convention: Let Γ_1 (resp., Γ_2, Γ_3) be the game obtained from Γ by adjoining to it the P-position $(A_1, B_1) = (1, a+1)$ (resp., two P-positions (A_1, B_1) and (A_{a+3}, B_{a+3})), all P-positions $\bigcup_{n=1}^{\infty}\{(A_n, B_n)\}$ as additional moves. It turns out that $\mathscr{P}(\Gamma_3) = \mathscr{P}(\Gamma_1)$ if $a = 2$, and $\mathscr{P}(\Gamma_3) = \mathscr{P}(\Gamma_2)$ if $a > 2$. These two facts give a positive answer to question 2. The fact $\mathscr{P}(\Gamma_1) \neq \mathscr{P}(\Gamma)$ answers affirmatively question 1.

1.3. Our games and results

We consider the following *ordered pure P-extension*: Given a function f and assume that the set of all P-positions of $Wyt(f)$ under normal play convention is $\mathscr{P}(f) = \bigcup_{n=0}^{\infty}\{(A_n, B_n)\}$ with $(A_0, B_0) = (0, 0)$. Let $K \geq 1$ be an integer. By Γ_K, we denote the new game obtained from $\text{Wyt}(f)$ by adjoining to it the first K P-positions $\cup_{1\leq i\leq K}(A_i, B_i)$ as additional moves. In particular, let $\Gamma_0 = Wyt(f)$.

For an integer $m \geq 1$, if $\mathscr{P}(\Gamma_m) \neq \mathscr{P}(\Gamma_{m-1})$, we call m a *jump-point* of $\text{Wyt}(f)$. Otherwise, we call m a *stable-point* of $Wyt(f)$. Obviously, $\mathscr{P}(\Gamma_1) \neq \mathscr{P}(\Gamma_0)$, i.e., 1 is a jump-point of $\text{Wyt}(f)$.

The main purpose of this paper is to analyze the structure and total number of jump-point of $\text{Wyt}(f)$. It is not easy to give a complete answer as the function f has many different forms. The authors in [6] considered the case $f(x) = x + t$ with integer $t \geq 2$. Their results showed us that 1 and $t + 3$ are the only two jump-points if $t > 2$, and 1 is the only jump-point if $t = 2$. The paper [11] was concerned with the question 2 for $f(x) = sx + t$. It turned out that there are many stable-points, but the total number of jump-points can not be determined.

Let us briefly present the content of this paper. The present paper is devoted to ordered pure P-extensions Γ_K for $f(x) = sx$ with integer coefficient $s \geq 2$, under normal play convention. Theorem 3 gives the set $\mathscr{P}(\Gamma_1)$ of all P-positions of Γ_1 for all integers $s \geq 2$. Theorem 4 shows us that $\mathscr{P}(\Gamma_K) = \mathscr{P}(\Gamma_1)$ for all integers $K \geq 1$ if $s = 2$, i.e., 1 is the only jump-point. Theorem 6 determines the set $\mathscr{P}(\Gamma_2)$ of all P-positions of Γ_2 for all integers $s \geq 3$, and shows that $\mathscr{P}(\Gamma_K) = \mathscr{P}(\Gamma_2)$ for all integers $K \geq 3$ and $s \geq 3$. This fact, together with Remark 1, implies that 1 and 2 are the only two jump-points for all integers $s \geq 3$.

2. All Jump-Points of Γ_K with $f(x) = sx$

Definition 1. Given a function f and assume that the set of all P-positions of $\mathrm{Wyt}(f)$ is $\bigcup_{n=0}^{\infty}\{(A_n, B_n)\}$. Let $K \geq 1$ be an integer. By Γ_K, we denote the new game obtained from $\mathrm{Wyt}(f)$ by adjoining to $\mathrm{Wyt}(f)$ the P-positions $\cup_{1 \leq i \leq K}(A_i, B_i)$ as additional moves. More clearly, three rules of moves of Γ_K are allowed:

Type I. A player may remove any positive number of tokens from a single pile, possibly the entire pile.

Type II. A player may take tokens from both piles, $k > 0$ from one pile and $\ell > 0$ from the other. This move is restricted by the condition

$$0 < k \leq \ell < f(k). \tag{1}$$

Type III. A player takes tokens from both piles, A_i tokens from one pile and B_i tokens from the other, where $i \in \{1, 2, \ldots, K\}$.

When analyzing the P-positions we frequently use the mex function: Let S be a finite set of nonnegative integers. Then mex(S) is defined to be the least nonnegative integer not in S. In particular, $\mathrm{mex}(\emptyset) = 0$.

In [9], the authors gave the following results: Let $f(x) = sx$ with $s \geq 2$. Under normal play convention, we have

$$\mathscr{P}(\Gamma_0) = \bigcup_{n=0}^{\infty} \{(A_n, B_n), (B_n, A_n)\}, \tag{2}$$

where for $n \geq 0$,

$$\begin{cases} A_n = \text{mex}\{A_i, B_i \mid 0 \leq i < n\}, \\ B_n = sA_n. \end{cases} \tag{3}$$

For the case $s = 1$, i.e., $f(x) = x$, the moves of Type II cannot be done and the original game reduces to *Nim on Two Piles*. This special case is omitted by this paper. We define two sequences C_n and D_n which will be used to present the formula of $\mathscr{P}(\Gamma_1)$.

Lemma 2. *Given any $f(x) = sx$ with $s \geq 2$, we define two sequences C_n and D_n :*

$$\begin{cases} C_0 = D_0 = 0 \textit{ and for integer } n \geq 1, \\ C_n = mex\{C_i, D_i \mid 0 \leq i < n\}, \\ D_n = (s-1)C_n + 2n. \end{cases} \tag{4}$$

Then

(1) *Both C_n and D_n are strictly increasing sequences.*

(2) *$D_n \geq C_n + 2n > G_n$ for any integer $n \geq 1$.*

(3) *$C_n - C_{n-1} \in \{1, 2\}$.*

(4) *$D_n - D_{n-1} \in \{s+1, 2s\}$. Moreover, $D_n - D_{n-1} = s+1$ if and only if $C_n - C_{n-1} = 1$; $D_n - D_{n-1} = 2s$ if and only if $C_n - C_{n-1} = 2$.*

(5) *For integers $n \geq i \geq 0$, we have $n - i \leq C_n - C_i \leq 2(n-i)$.*

(6) *For integers $n > i \geq 0$, we have $D_n - D_i \geq s(C_n - C_i)$.*

(7) *For integers $n > i \geq 0$, we have*

$$D_n - C_i \geq D_n - D_i > C_n - C_i \geq C_n - D_i. \tag{5}$$

Proof. (1) By the definition of mex, C_n is strictly increasing sequence. Also for $n \geq 1$,

$$D_n - D_{n-1} = (s-1)(C_n - C_{n-1}) + 2 > 0.$$

(2) The condition $s \geq 2$ implies that $D_n = (s-1)C_n + 2n \geq C_n + 2n > C_n$ for $n \geq 1$.

(3) By (1), C_n is strictly increasing sequence, i.e., $C_n - C_{n-1} > 0$. Suppose $C_n - C_{n-1} > 2$. Then

$$C_{n-1} < C_{n-1} + 1 < C_{n-1} + 2 < C_n = \text{mex}\{C_k, D_k \mid 0 \leq k < n\},$$

i.e., $C_{n-1} + 1, C_{n-1} + 2 \in \{C_k, D_k \mid 0 \leq k < n\}$. Then we have $C_{n-1} + 1$, $C_{n-1} + 2 \in \bigcup_{k=0}^{n-1} D_k$, i.e., there exist two integers $i, t \in \{0, 1, \ldots, n-1\}$ with $i < t$ such that $D_i = C_{n-1} + 1$ and $D_t = C_{n-1} + 2$. Thus,

$$1 = D_t - D_i = (s-1)(C_t - C_i) + 2(t-i) > (s-1)(C_t - C_i) \geq s - 1 \geq 1,$$

a contradiction. Hence $C_n - C_{n-1} \in \{1, 2\}$.

(4) If $C_n - C_{n-1} = 1$, then $D_n - D_{n-1} = (s-1)(C_n - C_{n-1}) + 2 = s + 1$. If $C_n - C_{n-1} = 2$, then $D_n - D_{n-1} = (s-1)(C_n - C_{n-1}) + 2 = 2s$.

(5) By (3), we have $n - i \leq C_n - C_i = \sum_{j=i+1}^{n} (C_j - C_{j-1}) \leq 2(n-i)$.

(6) For integers $n > i \geq 0$, by (5), we have

$$\begin{aligned} D_n - D_i &= (s-1)C_n + 2n - (s-1)C_i - 2i \\ &= (s-1)(C_n - C_i) + 2(n-i) \\ &\geq (s-1)(C_n - C_i) + C_n - C_i \\ &= s(C_s - C_i). \end{aligned}$$

(7) By (6), we have $D_n - D_i \geq s(C_n - C_i) > C_n - C_i$. Then by (2), $D_n - C_i \geq D_n - D_i > C_n - C_i \geq C_n - D_i$ for integers $n > i \geq 0$.

The proof is completed. □

Theorem 3. *Given* $f(x) = sx$ *with* $s \geq 2$ *by* $\mathscr{P}_K(s)$ *we denote the set of all P-positions of* Γ_K. *Then for* $K = 1$,

$$\mathscr{P}_1(s) = \bigcup_{n=0}^{\infty} \{(C_n, D_n), (D_n, C_n)\}, \tag{6}$$

where C_n *and* D_n *are determined by Equation* (4).

Proof. We use the notation $(x_1, y_1) \to (x_2, y_2)$ if there is a legal move from (x_1, y_1) to (x_2, y_2).

Proof of Fact I. Every move from any position $u \in \mathscr{P}_1(s)$ results in a position not in $\mathscr{P}_1(s)$ by any legal move of Γ_1. Let $u = (C_n, D_n)$ be a position in $\mathscr{P}_1(s)$.

(1) Suppose that $u = (C_n, D_n) \to (C_i, D_i)$ be a move of Type I.

If $C_n = C_i$ and $D_n > D_i$, then $C_n = C_i$ implies $n = i$ since C_n is strictly increasing sequence. Thus $D_n = D_i$, which contradicts $D_n > D_i$.

If $D_n = D_i$ and $C_n > C_i$, then $D_n = D_i$ implies $n = i$ since D_n is strictly increasing sequence. Thus $C_n = C_i$, which contradicts $C_n > C_i$.

(2) Suppose that $u = (C_n, D_n) \to (C_i, D_i)$ with $0 \leq i < n$ be a move of Type II. By Lemma 2(7), we have $D_n - C_i \geq D_n - D_i > C_n - C_i \geq C_n - D_i$ for integers $n > i \geq 0$. It suffices to consider the following two possibilities:

(2.1) $(C_n, D_n) \to (C_i, D_i)$ with $k = C_n - C_i > 0$ and $\ell = D_n - D_i > 0$.

Now $\ell > k > 0$. By Lemma 2(6), we have $\ell = D_n - D_i \geq s(C_n - Ci) = sk = f(k)$, which contradicts $0 < k \leq \ell < f(k)$.

(2.2) $(C_n, D_n) \to (D_i, C_i)$ with $k_1 = C_n - D_i$ and $\ell_1 = D_n - C_i$.

By Lemma 2(6) and (7), we have $\ell_1 > k_1 > 0$ and

$$\ell_1 = D_n - C_i \geq D_n - D_i \geq s(C_n - C_i) \geq s(C_n - D_i) = sk_1 = f(k_1),$$

which contradicts $0 < k_1 \leq \ell_1 < f(k_1)$.

(3) Suppose that $u = (C_n, D_n) \to (C_i, D_i)$ with $0 \leq i < n$ be a move of Type III. By Equation (5) and $1 = A_1 < B_1 = s$, we only need to consider the following two cases:

(3.1) $(C_n, D_n) \to (C_i, D_i)$ with $C_n - C_i = A_1$ and $D_n - D_i = B_1$.

It follows from Lemma 2(4) that

$$\begin{aligned} B_1 = D_n - D_i &= (D_n - D_{n-1}) + (D_{n-1} - D_{n-2}) + \cdots + (D_{i+1} - Di) \\ &\geq (n - i)(s + 1) \\ &> s = B_1, \end{aligned}$$

a contradiction.

(3.2) $(C_n, D_n) \to (C_i, D_i)$ with $C_n - D_i = A_1$ and $D_n - C_i = B_1$.

It follows from Lemma 2(2) and (4) that $B_1 = D_n - C_i \geq D_n - D_i \geq (n - i)(s + 1) > s = B_1$. This is a contradiction.

Proof of Fact II. For any position $u = (x, y) \notin \mathscr{P}_1(s)$, we can move from u to $v \in \mathscr{P}_1(s)$ by a legal move of Γ_1.

If $y > x = 0$, then we move $(x, y) = (0, y) \to (0, 0) \in \mathscr{P}_1(s)$ by a legal move of Type I. Without loss of generality, we assume $y \geq x \geq 1$. We have either $x = C_n$ or $x = D_n$ for some integer $n \geq 1$.

Case (i) $x = D_n$. Now $y \geq x = D_n > C_n$, we can move $(x, y) = (D_n, y) \to (D_n, C_n) \in \mathscr{P}_1(s)$ by taking $y - C_n > 0$ tokens from the heap of size y.

Case (ii) $x = C_n$. Now $y \neq D_n$. We consider four following possibilities: (1) $y > D_n$; (2) $y = D_n - 1$; (3) $sC_n \leq y < D_n - 1$; (4) $C_n = x \leq y < sC_n$.

(1) $y > D_n$. We move $(x, y) = (C_n, y) \to (C_n, D_n) \in \mathscr{P}_1(s)$, by a legal move of Type I.

(2) $y = D_n - 1$. Now we move

$$(x, y) = (C_n, D_n - 1) \to (C_{n-1}, D_{n-1}) \in \mathscr{P}_1(s),$$

with $k = C_n - C_{n-1}$ and $\ell = D_n - 1 - D_{n-1}$. This is a legal move. Indeed,

$$\ell = D_n - 1 - D_{n-1} = (s-1)(C_n - C_{n-1}) + 1 = (s-1)k + 1.$$

By Lemma 2(3), $k = C_n - C_{n-1} \in \{1, 2\}$. If $k = 1 = A_1$, then $\ell = s = B_1$, giving a legal move of Type III. If $k = 2$, then $k = 2 < 2s - 1 = \ell < 2s = f(k)$, giving a legal move of Type II.

(3) $sC_n \leq y < D_n - 1$. Now we move

$$(x, y) = (C_n, y) \to (C_i, D_i) \in \mathscr{P}_1(s),$$

where $k = C_n - C_i$ and $\ell = y - D_i$ and $i = y - (s-1)C_n - n + 1$. This is a legal move of Type II: Note that $i \geq C_n - n + 1 \geq 1$.

(a) $k > 0$. Indeed, $y < D_n - 1$ implies that $i < D_n - 1 - (s-1)C_n - n + 1 = n$. Thus $k = C_n - C_i > 0$.

(b) $\ell \geq k$. By the definition of i, we have $y = (s-1)C_n + n + i - 1$. Thus,

$$\ell = y - D_i = (s-1)(C_n - C_i) + n - i - 1 \geq C_n - C_i = k.$$

(c) $\ell < sk$. It follows from Lemma 2(5) that

$$\begin{aligned}\ell = y - D_i &= (s-1)(C_n - C_i) + n - i - 1 \\ &\leq (s-1)(C_n - C_i) + (C_n - C_i) - 1 \\ &< sk.\end{aligned}$$

(4) $C_n = x \leq y < sC_n$. We move $(x, y) = (C_n, y) \to (0, 0) \in \mathscr{P}_1(s)$, by a legal move of Type II: $0 < k = C_n \leq \ell = y < sC_n = f(k)$.

The proof is complete. □

Theorem 4. *Given* $f(x) = sx$ *with* $s = 2$, *by* $\mathscr{P}_K(s)$ *we denote the set of all P-positions of* Γ_K. *Then for all integers* $K \geq 2$,

$$\mathscr{P}_K(s) = \mathscr{P}_1(s),$$

where $\mathscr{P}_1$ *is determined by Equations* (6) *and* (4).

Proof. Recall that Γ_K is obtained by adjoining to Γ_1 the moves $\bigcup_{i=2}^{K}(A_i, B_i)$. We show that the addition of these moves leaves the P-positions of Γ_1 invariant. The proof is based on Theorem 3.

Proof of Fact I. Let (C_n, D_n) be a position in $\mathscr{P}_1(s)$, the proof of Fact I in Theorem 3 implies that $u = (C_n, D_n)$ lands in a position not in $\mathscr{P}_1(s)$ by any legal move of Types I or II, or the move $(A_1, B_1) = (1, 2)$ of Type III.

It suffices to show that the move $(C_n, D_n) \to (C_i, D_i)$, for every $n > i \geq 0$, can not be (A_k, B_k) for any integer $k \in \{2, 3, \dots, K\}$:

Case 1. Suppose that there exists an integer $k \geq 2$ such that $A_k = C_n - C_i$ and $B_k = D_n - D_i$. Let $C = \bigcup_{j=1}^{\infty}\{C_j\}$, $D = \bigcup_{j=1}^{\infty}\{D_j\}$. By $\sharp(U)$, we denote the number of elements in a given set U. We define

$$\begin{cases} S = \{x \mid x \text{ is an integer and } C_i < x \leq C_n\}, \\ n_C = \sharp(S \cap C), \\ n_D = \sharp(S \cap D). \end{cases}$$

Then $\sharp(S) = n_C + n_D$, $\sharp(S) = C_n - C_i = A_k$ and $n_C = n - i$. Thus

$$n - i = A_k - n_D. \tag{7}$$

Let $r := n_D$ be the number of elements in D between C_i and C_n, i.e., $A_k = n - i + r$.

By the definitions of A_n and B_n, $A_0 = B_0 = 0$, $A_1 = 1$, $B_1 = s = 2$. Thus $A_k \geq A_2 = \text{mex}\{0, 1, 2\} = 3$ for any integer $k \geq 2$. If $n - i = 1$, $3 \leq A_k = C_n - C_i = C_{i+1} - C_i \leq 2$, a contradiction. If $n - i > 1$, there are $D_{j+1}, D_{j+2}, \dots, D_{j+r}$ between C_i and C_n such that $D_{j+1} \geq C_i + 1$ and $D_{j+r} \leq C_n - 1$. Then,

$$D_{j+r} - D_{j+1} \leq C_n - 1 - (C_i + 1) = A_k - 2 = n - i + r - 2.$$

On the other hand, by Lemma 2(4), we have $D_{j+r} - D_{j+1} \geq (r-1)(s+1) = 3(r-1)$. Hence $3(r-1) \leq n - i + r - 2$, i.e., $r < n - i$.

By Lemma 2(4), we have

$$\begin{aligned} D_n - D_i &= (D_n - D_{n-1}) + (D_{n-1} - D_{n-2}) + \dots + (D_{i+1} - D_i) \\ &= (n - i - r)(s + 1) + 2rs \\ &= (n - i + r)s + n - i - r \\ &= sA_k + n - i - r \\ &> sA_k = B_k, \end{aligned}$$

which contradicts $B_k = D_n - D_i$.

Case 2. Suppose that $A_k = C_n - D_i$ and $B_k = D_n - C_i$ for any $n > i \geq 1$. By Lemma 2(6) and (7), we have $D_n - C_i > D_n - D_i \geq s(C_n - C_i) \geq s(C_n - C_i) = sA_k = B_k$, which contradicts $B_k = D_n - C_i$.

Proof of Fact II. Let u be a position which is not in $\mathscr{P}_1(s)$. The proof of Fact II in Theorem 3 implies that we can move from u to $v \in \mathscr{P}_1(s)$ by a legal move of Γ_1. We note that the set of legal moves of Γ_1 is a subset of legal moves of Γ_K. Hence, we can move from u to $v \in \mathscr{P}_1(s)$ by a legal move of Γ_K.

The proof is completed. □

Remark 1. If $f(x) = sx$ with $s \geq 3$, it follows from Equation (3) that $A_0 = B_0 = 0$, $A_1 = 1$, $B_1 = s \geq 3$. Thus $A_k \geq A_2 = \text{mex}\{0, 1, B_1\} = 2$ for $k \geq 2$. In the proof of Case 1 in Theorem 4, if $n - i = 1$, then $2 = A_2 = n - i + r$ implies $r = 1$, i.e., there exists an integer j such that $C_i < D_j < C_{i+1}$. Thus $C_{i+1} - C_i = 2 = A_2$ and

$$D_{i+1} - D_i = (s-1)(C_{i+1} - C_i) + 2 = 2s = f(A_2) = B_2.$$

Hence adjoining (A_2, B_2) to Γ_1 will change the set $\mathscr{P}_1(s)$. □

Theorem 6 presents the explicit formula of $\mathscr{P}_2(s)$ for all integers $s \geq 3$ by two new sequences G_n and H_n defined in Lemma 5. Moreover, Theorem 6 shows that $\mathscr{P}_K(s) = \mathscr{P}_2(s)$ for all integers $K \geq 3$ and $s \geq 3$.

Lemma 5. *Given $f(x) = sx$ with $s \geq 3$, we define two sequences G_n and H_n :*

$$\begin{cases} G_0 = H_0 = 0 \textit{ and for integer } n \geq 1, \\ G_n = mex\{G_i, H_i \mid 0 \leq i < n\}, \\ H_n = sG_n + n. \end{cases} \tag{8}$$

Then

(1) *Both G_n and H_n are strictly increasing sequences.*

(2) *$H_n > G_n$ for integer $n \geq 1$.*

(3) *$G_n - G_{n-1} \in \{1, 2\}$.*

(4) *$H_n - H_{n-1} \in \{s+1, 2s+1\}$. Moreover, $H_n - H_{n-1} = s+1$ if and only if $G_n - G_{n-1} = 1$; $H_n - H_{n-1} = 2s+1$ if and only if $G_n - G_{n-1} = 2$.*

(5) *For integers $n > i \geq 0$, we have*

$$H_n - G_i \geq H_n - H_i > G_n - G_i \geq G_n - H_i. \tag{9}$$

Proof. (1) By the definition of mex, G_n is strictly increasing sequence. Also for $n \geq 1$, $H_n - H_{n-1} = s(G_n - G_{n-1}) + 1 > 0$.

(2) The condition $s \geq 3$ gives $H_n = sG_n + n > G_n$ for $n \geq 1$.

(3) By (1), G_n is strictly increasing sequence, i.e., $G_n - G_{n-1} > 0$. If $G_n - G_{n-1} > 2$, then

$$G_{n-1} < G_{n-1} + 1 < G_{n-1} + 2 < G_n = \text{mex}\{G_k, H_k \mid 0 \leq k < n\},$$

i.e., $G_{n-1} + 1, G_{n-1} + 2 \in \{G_k, H_k | 0 \leq k < n\}$. Then $G_{n-1} + 1, G_{n-1} + 2 \in \bigcup_{k=0}^{n-1} H_k$, i.e., there exist two integers $i, t \in \{0, 1, \ldots, n-1\}$ with $t > i$ such that $H_i = G_{n-1} + 1$ and $H_t = G_{n-1} + 2$. Now, $1 = H_t - H_i = s(G_t - G_i) + (t - i) \geq s + 1 > 1$, giving a contradiction. Hence, $G_n - G_{n-1} \in \{1, 2\}$.

(4) If $G_n - G_{n-1} = 1$, then $H_n - H_{n-1} = s(G_n - G_{n-1}) + 1 = s + 1$. If $G_n - G_{n-1} = 2$, then $H_n - H_{n-1} = s(G_n - G_{n-1}) + 1 = 2s + 1$.

(5) The condition $s \geq 3$ gives $H_n - H_i = s(G_n - G_i) + (n - i) > G_n - G_i$. By (2), we have $H_n - G_i \geq H_n - H_i > G_n - G_i \geq G_n - H_i$ for integers $n > i \geq 0$.

The proof is completed. □

Theorem 6. *Given $f(x) = sx$ with $s \geq 3$, by $\mathscr{P}_K(s)$ we denote the set of all P-positions of Γ_K. Then*

(A) *For $K = 2$ and $s \geq 3$,*

$$\mathscr{P}_2(s) = \bigcup_{n=0}^{\infty} \{(G_n, H_n), (H_n, G_n)\}, \tag{10}$$

where G_n and H_n are determined by Equation (8).

(B) *For* $K \geq 3$ *and* $s \geq 3$,

$$\mathscr{P}_K(s) = \mathscr{P}_2(s), \tag{11}$$

where $\mathscr{P}_2(s)$ *is determined by Equation* (10).

Proof of (A).

(A.1) Every move from any position $u \in \mathscr{P}_2(s)$ results in a position not in $\mathscr{P}_2(s)$ by any legal move of T_2. Let $u = (G_n, H_n)$ be a position in $\mathscr{P}_2(s)$.

(1) Similar to the proof of Fact I of Theorem 3, Lemma 5(1) means that $u = (G_n, H_n) \rightarrow (G_i, H_i)$ can not be a legal move of Type I.

(2) Suppose that $u = (G_n, H_n) \rightarrow (G_i, H_i)$ with $0 \leq i < n$ be a move of Type II. By Lemma 5(5), we have $H_n - G_i \geq H_n - H_i > G_n - G_i \geq G_n - H_i$ for integers $n > i \geq 0$. So, we need only consider the following two possibilities:

(2.1) $(G_n, H_n) \rightarrow (G_i, H_i)$ with $k = G_n - G_i > 0$ and $\ell = H_n - H_i > 0$. Then

$$\ell = H_n - H_i = s(G_n - G_i) + (n - i) > s(G_n - G_i) = sk = f(k) > k, \tag{12}$$

which contradicts $0 < k \leq l < f(k)$.

(2.2) $(G_n, H_n) \rightarrow (H_i, G_i)$ with $k_2 = G_n - H_i$ and $\ell_2 = H_n - G_i$. It follows from Lemma 5(2) and Equation (12) that

$$\ell_2 = H_n - G_i \geq H_n - H_i > s(G_n - G_i) \geq s(G_n - H_i) = sk_2 = f(k_2) > k_2,$$

which contradicts $0 < k_2 \leq \ell_2 < f(k_2)$.

(3) Suppose that $u = (G_n, H_n) \rightarrow (G_i, H_i)$ with $0 \leq i < n$ be a move of Type III: (A_1, B_1) or (A_2, B_2).

By Equation (9) and $A_1 < B_1$ and $A_2 < B_2$, we need only consider two cases:

(3.1) $A_j = G_n - G_i$ and $B_j = H_n - H_i$ for some $j \in \{1, 2\}$. Then $B_j = H_n - H_i = s(G_n - G_i) + (n - i) > s(G_n - G_i) = sA_j = B_j$, giving a contradiction.

(3.2) $A_j = G_n - H_i$ and $B_j = H_n - G_i$ for some $j \in \{1, 2\}$. Then $B_j = H_n - G_i \geq H_n - H_i > s(G_n - G_i) \geq s(G_n - H_i) = sA_j = B_j$. This is a contradiction.

(A.2) For any position $u = (x, y) \notin \mathscr{P}_2(s)$, we can move from u to $v \in \mathscr{P}_2(s)$ by a legal move of Γ_2.

Without loss of generality, we assume $0 \leq x \leq y$. If $y > x = 0$, then we move $(x, y) = (0, y) \to (0, 0) \in \mathscr{P}_2(s)$ by a legal move of Type I. For $y \geq x \geq 1$, we have either $x = G_n$ or $x = H_n$ for some integer $n \geq 1$.

Case (i) $x = H_n$. Now $y \geq x = H_n > G_n$. We move $(x, y) = (H_n, y) \to (H_n, G_n) \in \mathscr{P}_2(s)$, by taking $y - G_n > 0$ tokens from the heap of size y.

Case (ii) $x = G_n$. Now $y \neq H_n$ and we proceed by distinguishing four possibilities: (1) $y > H_n$; (2) $y = H_n - 1$; (3) $sG_n \leq y < H_n - 1$; (4) $G_n = x \leq y < sG_n$.

(1) $y > H_n$. We can move $(x, y) = (G_n, y) \to (G_n, H_n) \in \mathscr{P}_2(s)$ by taking $y - H_n > 0$ tokens from the heap of size y.

(2) $y = H_n - 1$. We move

$$(x, y) = (G_n, H_n - 1) \to (G_{n-1}, H_{n-1}) \in \mathscr{P}_2(s),$$

with $k = G_n - G_{n-1}$ and $\ell = H_n - 1 - H_{n-1}$. This is a legal move of Type III:

Indeed, $\ell = H_n - 1 - H_{n-1} = s(G_n - G_{n-1}) = sk$. By Lemma 5(3), $k = G_n - G_{n-1} \in \{1, 2\}$. If $k = 1$, then $k = A_1$ and $\ell = s = B_1$. If $k = 2$, then $k = A_2$ and $\ell = 2s = B_2$. Both are legal moves of Type III.

(3) $sG_n \le y < H_n - 1$. We move

$$(x, y) = (G_n, y) \to (G_i, H_i) \in \mathscr{P}_2(s),$$

where $k = G_n - G_i$ and $\ell = y - H_i$ and $i = y - sG_n + 1$. We will show that this is a legal move of Type II. Note that $i = y - sG_n + 1 \ge 1 > 0$. It suffices to check the following facts:

(a) $k > 0$. Indeed, $y < H_n - 1$ implies that $i = y - sG_n + 1 < H_n - 1 - sG_n + 1 = n$. Thus $k = G_n - G_i > 0$.

(b) $\ell \ge k$. Now $y = sG_n - 1 + i$. Then $\ell = y - H_i = sG_n - 1 + i - (sG_i + i) = s(G_n - G_i) - 1 \ge 3k - 1 \ge k$.

(c) $\ell < sk$. By (b), we have $\ell = s(G_n - G_i) - 1 < sk$.

(4) $G_n = x \le y < sG_n$. We move $(x, y) = (G_n, y) \to (0, 0) \in \mathscr{P}_2(s)$, by a legal move of Type II: $0 < k = G_n \le \ell = y < sG_n = f(k)$.

Proof of (B). Γ_K is obtained by adjoining to Γ_2 the moves $\bigcup_{i=3}^{K}(A_k, B_k)$. We show that the addition of these moves leaves the P-positions of Γ_2 invariant. The proof is based on Theorem 6(A).

(B.1) Every move from any position $u \in \mathscr{P}_2(s)$ results in a position not in $\mathscr{P}_2(s)$ by any legal move of Γ_K.

Let (G_n, H_n) be a position in $\mathscr{P}_2(s)$, the proof of (A.1) in Theorem 6 implies that $u = (G_n, H_n)$ lands in a position not in $\mathscr{P}_2(s)$ by any legal move of Types I or II, or two moves (A_1, B_1) and (A_2, B_2) of Type III. It suffices to show that the move $(G_n, H_n) \to (G_i, H_i)$, for every $n > i \ge 0$, can not be (A_k, B_k) for any integer $k \in \{3, 4, \ldots, K\}$.

Suppose that there exists an integer $k \in \{3, 4, \ldots, K\}$ such that $A_k = G_n - G_i$ and $B_k = H_n - H_i$. Then $B_k = H_n - H_i = s(G_n - G_i) + (n - i) > s(G_n - G_i) = sA_k = B_k$. This is a contradiction.

Suppose that there exists an integer $k \in \{3, 4, \ldots, K\}$ such that $A_k = G_n - H_i$ and $B_k = H_n - G_i$. By Lemma 5(2), we have $B_k = H_n - G_i \geq H_n - H_i > s(G_n - G_i) \geq s(G_n - H_i) = sA_k = B_k$. This is another contradiction.

(B.2) For any position $u = (x, y) \notin \mathscr{P}_2(s)$, we can move from u to $v \in \mathscr{P}_2(s)$ by a legal move of Γ_K. Let u be a position which is not in $\mathscr{P}_2(s)$. The proof of (A.2) in Theorem 6 implies that we can move from u to $v \in \mathscr{P}_2(s)$ by a legal move of Γ_2. We note that the set of legal moves of Γ_2 is a subset of legal moves of Γ_K. Hence, we can move from u to $v \in \mathscr{P}_2(s)$ by a legal move of Γ_K.

The proof is completed. □

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