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THE APPLICATION OF MONTE CARLO SIMULATION BASED ON NORMAL INVERSE GAUSSIAN DISTRIBUTION IN OPTION PRICING

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Abstract

Options is an important financial derivative products, therefore it is important to reasonable pricing. According to the financial asset returns typically exhibit the feature of aiguilles large remaining part and hypothesis, it obeys normal inverse Gaussian distribution, using Monte Carlo simulation based on normal inverse Gaussian distribution on its pricing and improving it by antithesis variable. Finally, the conclusion is that the improved method is an effective method for option pricing through the empirical test.

1. Introduction

In numerous financial derivatives, the option is an important foundation of derivative products, so how to price the option accurately is an important issue for many scholars. For European option, Black and Scholes gave the famous B-S formula

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$$c(S_t, t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where $d_1 = \dfrac{\ln(S_t / K) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$, $d_2 = d_1 - \sigma\sqrt{T - t}$.

This kind of pricing formula with analytical form can evaluate the option directly. However, there is no such an analytical solution for many other options in general, such as American options, Asian options, etc. In this case, people tend to use numerical methods to simulate the price of the option. The numerical method not only can solve the pricing problem of European option, but also can solve some complex option pricing problems, such as compound options, select options. At present, there are some mature numerical simulation methods, for instance, the binary tree method, the finite method, and Monte Carlo simulation method.

Monte Carlo simulation method assume that the price of the underlying asset follows geometric Brownian motion, then according to the underlying asset price is lognormal distribution assumption, simulate the underlying asset option price movements in the holding period and obtain distribution of underlying asset price at maturity date. The method assumes that the underlying asset price is lognormal distribution, i.e., yields follows a normal distribution. However, in reality, the real rate of return showed spike thick tail characteristics, so normal distribution cannot be well reflect the characteristics. Ma and Wang [4] researched and explored the problem of pricing warrants by using Monte Carlo simulation methods and its improved technique, and set up Monte Carlo models pricing warrants. Ouyang and Xu [5] compared the first to default swaps' prices under Gaussian model and normal Gaussian model with numerical examples. Luo and Jia [6] showed that how to construct and compute equivalent martingale measures on stocks driven by the NIG-Levy process and considered numerical pricing analysis of Asian options by uses low discrepancy sequences. Cao et al. [7] fits the returns of Shanghai Composite using generalized hyperbolic distribution and tests its goodness of fit.

In this paper, we assume that the real rate of return of financial assets obey the normal inverse Gaussian distribution, give the Monte Carlo simulation method based on normal inverse Gaussian distribution, and then improve it by antithesis variable. Finally, the conclusion is that the improved method is an effective method for option pricing through the empirical test.

2. Theoretical Basis

2.1. Monte Carlo simulation method

Monte Carlo simulation is a kind of calculation method based on the theory and methods of probability and statistics. It associate problem with a probability model, and stochastic simulation on the computer, thus draws the approximate solution of the problem. Therefore, the Monte Carlo simulation method is also known as random simulation method or statistical test method.

The basic idea of Monte Carlo simulation method is based on the price of the underlying asset S_t is a lognormal distribution, and meets the following dynamic equation under the risk-neutral conditions:

$$dS_t = rS_t dt + \sigma S_t dZ_t,$$

where r is the risk-free rate, σ is the volatility, and dZ_t is a Wiener process in a risk-neutral word. Then the price of the option is

$$C = e^{-rT} E_q[(S(T) - K)^+],$$

where K is the strike price, T is the expiration time, and E denotes the expected value in the risk-neutral probability q. Due to an unbiased estimate of the expected effective is the average of sample, therefore, the value of option based on the Monte Carlo simulation can be expressed as

$$\widetilde{C} = \frac{1}{n}\sum_{i=1}^{n} C_i = \frac{1}{n} e^{-rT} \sum_{i=1}^{n} E_q[(S_i(T) - K)^+].$$

Here, C_i and $S_i(T)$ is option price and underlying asset price of the i-th simulation, respectively, n is the number of simulations. This is a standard Monte Carlo simulation method.

2.2. The normal inverse Gaussian distribution

Compared with other numerical methods, the Monte Carlo method is easy to implement and improve, and the application scope is wide, and it is not restricted by the dimension of the problem. But in the real financial data, the distribution of asset returns usually present fat tailed and asymmetric, i.e., peak thick tail, so only using the normal distribution to fit the real financial data distribution has great limitations. Therefore, many scholars began to seek a more reasonable distribution hypothesis. Mandelbort proposed stable distribution instead of the financial data of normal distribution assumption, but the tail of the stable distribution is usually thicker than the actual distribution. Therefore, there are a lot of scholars began to use generalized hyperbolic distribution. Barndorff-Nielsen proposed the generalized hyperbolic distribution in 1977, Eberlein and Keller applied it to the financial sector firstly. Due to the tail of the generalized hyperbolic distribution is thicker than the tail of the stable distribution, so it get rapid development in the financial sector. A probability density function of the generalized hyperbolic is given by:

$$GH(x, \lambda, \alpha, \beta, \mu, \delta) = g(\lambda, \alpha, \beta, \mu, \delta)(\delta^2 + (x-\mu)^2)^{\lambda/2-0.25} \times K_{\lambda-0.5}(\alpha\sqrt{\delta^2 - (x-\mu)^2})e^{\beta(x-\mu)},$$

where $K_\nu(\bullet)$ is the modified Bessel function of the second kind, and

$$g(\lambda, \alpha, \beta, \mu, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{(\lambda-0.5)}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})},$$

when $\lambda > 0$, there have $\delta \geq 0, |\beta| < \alpha$; when $\lambda = 0$, there have $\delta > 0$, $|\beta| < \alpha$; when $\lambda < 0$, there have $\delta > 0, |\beta| \leq \alpha$. Especially when $\lambda = -0.5$,

$\delta > 0$, $|\beta| \leq \alpha$, $K_{-0.5}(z) = K_{0.5}(z)$ reduces to $K_{0.5}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$. The probability density function is given by

$$f_{NIG}(x;\ \alpha,\ \beta,\ \mu,\ \delta) = \frac{\alpha\delta}{\pi}\exp\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\}\frac{K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\sqrt{\delta^2 + (x-\mu)^2}},$$

we call it normal inverse Gauss distribution, denoted as $NIG(\alpha, \beta, \mu, \delta)$.

Property 1 ([1])**.** Properties of the normal inverse Gauss distribution:

$$X \sim NIG(\alpha,\ \beta,\ \mu,\ \delta) \Rightarrow cX \sim NIG(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu,\ c\delta),$$

$$X \sim NIG(\alpha,\ \beta,\ \mu_1,\ \delta_1),\ Y \sim NIG(\alpha,\ \beta,\ \mu_2,\ \delta_2) \Rightarrow X + Y$$
$$\sim NIG(\alpha,\ \beta,\ \mu_1 + \mu_2,\ \delta_1 + \delta_2),$$

$$X \sim NIG(\alpha,\ \beta,\ \mu,\ \delta) \Rightarrow E(X) = \mu + \frac{\delta\beta}{\gamma},\ \mathrm{Var}(X) = \frac{\delta\alpha^2}{\gamma^3},$$

$$\text{skewness} = \gamma_1 = \frac{3\beta}{\alpha(\gamma\delta)^{1/2}},$$

$$\text{kurtosis} = \gamma_2 = \frac{3(1 + 4\beta^2/\alpha^2)}{\gamma\delta},\ \text{where } \gamma = \sqrt{\alpha^2 - \beta^2}. \tag{1}$$

If the known samples obey the normal inverse Gaussian distribution, moments (1) have a rather simple form

$$\hat{\gamma} = \frac{3}{s\sqrt{3\bar{\gamma}_2 - 5\bar{\gamma}_1^2}},\quad \hat{\beta} = \frac{\bar{\gamma}_1 s\hat{\gamma}^2}{3},\quad \hat{\delta} = \frac{s^2\hat{\gamma}^3}{\hat{\beta}^2 + \hat{\gamma}^2},\quad \hat{\mu} = \bar{x} - \hat{\beta}\frac{\hat{\delta}}{\hat{\gamma}}, \tag{2}$$

where $\bar{x}$, s^2 are as usual the sample mean and variance, respectively. While $\bar{\gamma}_1 = \frac{\mu_3}{\mu_2^{3/2}}$ and $\bar{\gamma}_2 = \frac{\mu_4}{\mu_2^2} - 3$, with $\mu_k = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^k$, i.e., the sample skewness and kurtosis, respectively. These values will be seen to be quite helpful as initial values.

2.3. Monte Carlo simulation based on normal inverse Gaussian distribution for option pricing

When the financial asset returns rate show a peak thick tail, and assuming that the asset returns follow a normal inverse Gaussian distribution, as indicated by the reference [2], under the risk neutral probability measure, the price of the underlying asset $\overline{S}(T)$ in the moments T meet

$$\overline{S}(T) = S_0 \exp(rT + L_T - \overline{\omega}T),$$

where L_T is a normal inverse Gaussian process, and $\overline{\omega} = \mu + \delta\gamma - \delta\sqrt{\alpha^2 - (1+\beta)^2}$; $\gamma = \sqrt{\alpha^2 - \beta^2}$; $\delta_T = \delta T$; $\mu_T = \mu T$. Hence, the European options price is given by

$$\overline{C} = e^{-rT} E_q[(\overline{S}(T) - K)^+].$$

Option price based on normal inverse Gaussian distribution using Monte Carlo simulation is given by

$$\widetilde{C} = \frac{1}{n}\sum_{i=1}^{n} \overline{C}_i = \frac{1}{n} e^{-rT} \sum_{i=1}^{n} E_q[(\overline{S}_i(T) - K)^+],$$

where $\overline{S}_i(T)$ is the price of the i-th simulation, and n is the number of simulations.

3. Antithetic Variable Technique

Monte Carlo simulation accuracy is closely related to the number of simulation, the higher the number of simulation times, the more accurate, but the number of times increasing will increase the amount of calculation. Antithetic variable is a kind of simple and effective variance reduction technique, and it is applied widely. Now, we apply it into the Monte Carlo simulation based on the normal inverse Gaussian distribution to improve the simulation efficiency.

The method of antithetic variates attempts to reduce variance by introducing negative dependence between pairs of replications. In the antithetic variable technique, a simulation trial involves calculating two values of the derivative. The first value is calculated in the usual way; the second value is calculated by changing the sign of all the random samples. The final result is the average of two simulation results. The method can take various forms; the most broadly applicable is based on the observation that if U is uniformly distributed over $[0, 1]$, then $1 - U$ is too. Hence, if we generate a path using as inputs $U_1, \cdots, U_n$, we can generate a second path using $1 - U_1, \cdots, 1 - U_n$ without changing the law of simulated process. The variables U_i and $1 - U_i$ form an antithetic pair in the sense that a large value of one is accompanied by a small value of the other. This suggests that an unusually large or small output computed from the first path may be balanced by the value computed from the antithetic path, resulting in a reduction in variance.

Now, let $L_{T_i}(i = 1, 2, \cdots, n)$ denote the random numbers in normal inverse Gaussian distribution random variables, therefore, the price of underlying asset is

$$\overline{S}_{i1}(T) = S_0 \exp(rT + L_{T_i} - \overline{\omega} T),$$

then we can get the simulation value

$$\hat{C}_{i1} = e^{-rT} E_q[(\overline{S}_{i1}(T) - K)^+](i = 1, 2, \cdots, n).$$

Hence, the Monte Carlo simulation is

$$\hat{C}_1 = \frac{1}{n} \sum_{i=1}^{n} \hat{C}_{i1}.$$

According to properties of the normal inverse Gauss distribution, we can construct antithetic random number $\widetilde{L}_{T_i} = -L_{T_i}(i = 1, 2, \cdots, n)$ based on $L_{T_i}(i = 1, 2, \cdots, n)$. Then the price of the underlying assets generated by the dual random number is

$$\overline{S}_{i2}(T) = S_0 \exp(rT + \widetilde{L}_{T_i} - \overline{\omega}T) = S_0 \exp(rT - L_{T_i} - \overline{\omega}T).$$

The corresponding option price is given by

$$\widetilde{C}_{i2} = e^{-rT} E_q[(\overline{S}_{i2}(T) - K)^+](i = 1, 2, \cdots, n),$$

and the corresponding Monte Carlo simulation is

$$\widetilde{C}_2 = \frac{1}{n}\sum_{i=1}^{n} \widetilde{C}_{i2}.$$

The antithetic variates estimator is simply the average of $\hat{C}_1$ and $\widetilde{C}_2$

$$C = \frac{1}{2}(\hat{C}_1 + \widetilde{C}_2) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\hat{C}_{i1} + \widetilde{C}_{i2}}{2}\right).$$

4. Data Analysis and Empirical

4.1. Normality test of the sequence of returns

We select the day closing price of baosteel (600019.ss) from the date December 12, 2000 to February 21, 2006, as the sample, a total of 1356 data, and data from yahoo finance database. Let X_t denotes the day's closing price, $Y_t = \ln(X_t) - \ln(X_{t-1})$ denotes the daily logarithmic return rate, a total of 1355 data. Using Matlab to test normality of the sequence of yield, we obtain Figure 1 and Table 1:

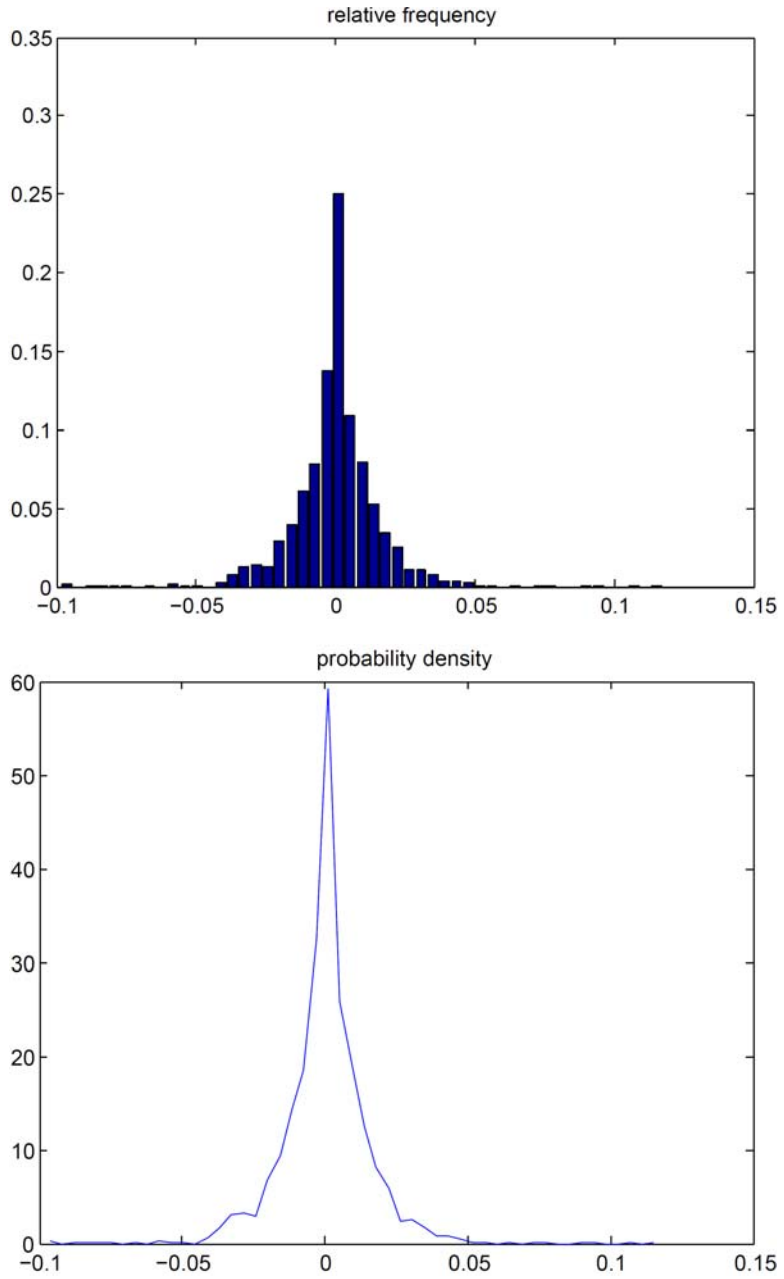

Figure 1. Histogram of baosteel's daily returns.

Table 1. Statistical characteristics of the sequence of yield of the baosteel

Mean	2.5685e-004	Skewness	0.1516
Variance	2.6191e-004	Kurtosis	11.9879
Standard deviation	0.0162	Jacques-Bera test value	4566.0269

As can be seen from Figure 1, the distribution of baosteel's daily returns show slightly skewed to the right, high kurtosis; Table 1 obtain normality test results. From Table 1, skewness is $S = \frac{E[(x-\mu_x)^3]}{\sigma^3} = 0.1516$, where μ_x, σ are the mean and standard deviation of the sample of x, respectively, and skewness is a positive. This means that the sequence distribution has a long right tail. Kurtosis is $K = \frac{E[(x-\mu_x)^4]}{\sigma^4} = 11.9879$, its value is significantly higher than the kurtosis of the normal distribution of 3, this means that the tail of daily returns sequence distribution is thicker than the tail of normal distribution, and its distribution density curve where the mean distance far is located above the normal distribution curve, this means the probability of outliers greater than normal distribution, indicating that the return series have high peak and fat tail characteristic. Through the Jarque-Bera normality tests, we have $JB = \frac{N}{6}[S^2 + \frac{1}{4}(K-3)^2] = 4566.0269$, it shows that baosteel's yield sequence is significantly different from normal distribution.

4.2. Normal inverse Gaussian distribution fitting

Using normal distribution and normal inverse Gaussian distribution fit the daily returns, the results are as follows:

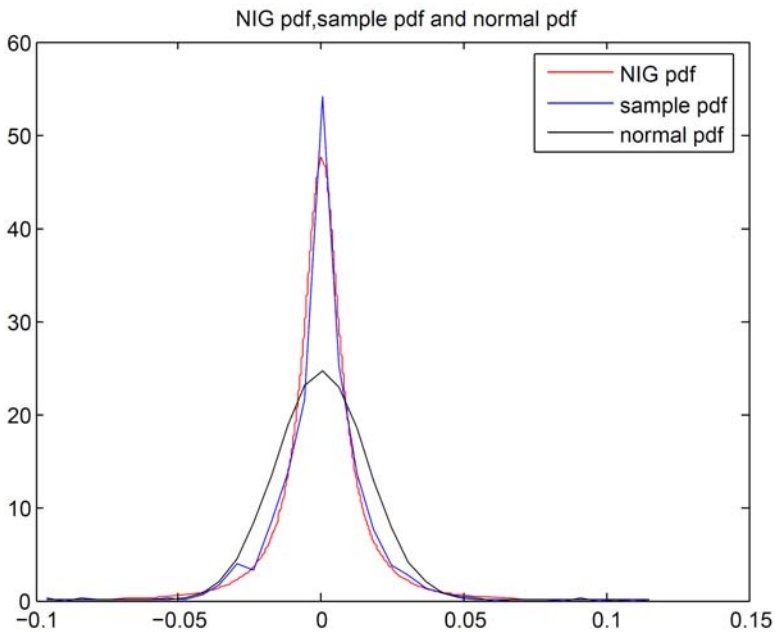

Figure 2. Sample probability density function, the normal inverse Gaussian probability density function and probability density function of normal distribution.

It can be seen from Figure 2 that the fitting results of the normal inverse Gaussian distribution is better than the normal distribution. It can be better reflect the real situation of the heavy tail of the sample data.

4.3. The empirical test

Consider a European call which the underlying assets is baosteel's stock. Suppose that the European call option which is issued by an institution for a period of half a year, the stock initial price is 4.5, the strike price of the stock is 4.5, the risk-free rate is 5%, and using daily

returns of December 12, 2000 to February 21, 2006 to estimate the parameters of normal inverse Gaussian distribution, the parameters can be obtained from Equation (2). Using the Monte Carlo method based on the normal inverse Gauss distribution and the antithetic variables of to solve the problem, the results are as follows:

Table 2. Comparison of Monte Carlo method based on the normal inverse Gauss distribution and the antithetic variables

Simulation times	Normal inverse Gaussian distribution of Monte Carlo simulation		Antithetic variables	
	Simulation value	Standard error	Simulation value	Standard error
5000	0.4497	0.0090	0.3747	0.0045
10000	0.4571	0.0063	0.3743	0.0032
15000	0.4572	0.0051	0.3721	0.0026
20000	0.4539	0.0045	0.3760	0.0031
100000	0.4553	0.0020	0.3709	9.8149e – 004
150000	0.4556	0.0016	0.3729	8.0819e – 004
500000	0.4560	8.9476e – 004	0.3724	4.4220e – 004
1000000	0.4530	6.3010e – 004	0.3722	3.1296e – 004

Table 2 shows that the results using normal inverse Gaussian distribution of Monte Carlo simulation method to price the Baosteel warrants. when simulation times up to 100000 times, the antithetic variables technique is better. Option price of the standard error is decreased obviously, and convergence speed is significantly accelerated.

5. Conclusion Analysis

For financial yield assets which typically exhibit fat tail characteristics, we use normal inverse Gaussian distribution to study the yields of warrants. By empirical analysis, we can get the following conclusions:

(1) If the yield of the underlying stock warrants disobey the assumption of traditional normal distribution, but exist the phenomenon of peak and fat tails, we can see that normal inverse Gaussian distribution can be better to fit the sample data, and better reflect the characteristics of fat tail.

(2) In warrants pricing, using the variance reduction techniques of antithetic variable which based on Monte Carlo method of normal inverse Gaussian distribution is better than normal inverse Gaussian distribution simulation directly in simulation efficiency and convergence rate.

Although the article uses the normal inverse Gaussian distribution to fit rate of return of financial assets, but there are still some problems to be improved. In Warrants market, most is a kind of warrant between European and American, and Monte Carlo simulation is used to simulate the general European style warrants. However, with further study of the Monte Carlo method, financial derivatives pricing will also be continuous improvement and development.

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