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ON THE DYNAMICS OF THE RECURSIVE SEQUENCE $x_{n+1} = \frac{x_{n-1}}{p + qx_n^2 + x_{n-1}}$

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Abstract

In this paper, we study the global asymptotic behaviour and the periodic character of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{p + qx_n^2 + x_{n-1}}, \quad n = 0, 1, \dots,$$

where the initial conditions x_{-1}, x_0 are arbitrary non-negative real numbers, and the parameters p, q are positive real numbers.

1. Introduction

In this paper, we investigate the global asymptotic behaviour and the periodic character of solutions of the rational difference equation

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$$x_{n+1} = \frac{x_{n-1}}{p + qx_n^2 + x_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the initial conditions x_{-1}, x_0 are arbitrary non-negative real numbers, and the parameters p, q are positive real numbers.

Recently, there has been a great interest in studying the behaviour of nonlinear difference equations. For example, Camouzis et al. [1] investigated the behaviour of solutions of the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-1}^2},$$

where $\beta \in (0, \infty)$ and initial values $x_{-1}, x_0 \in (0, \infty)$. Kulenovic et al. [2] studied the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{p + qx_n + x_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where $p, q \in (0, \infty)$ and initial values $x_{-1}, x_0 \in (0, \infty)$.

Amleh et al. [3] investigated the global asymptotic behaviour of solutions of some special types of the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

with non-negative parameters and with arbitrary non-negative initial conditions such that the denominator is always positive. For other related works, see [4-8].

The study of these equations is quite challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore that results about such equations offer prototypes for the development of the basic theory of the global behaviour of nonlinear difference equations.

Let us introduce some basic definitions and some theorems that we need in the sequel, let I be an interval of real numbers and let

$$f : I \times I \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $\{x_{-1}, x_0\} \subset I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots, \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 1. A point $\bar{x} \in I$ is called an equilibrium point of Equation (1.2), if

$$\bar{x} = f(\bar{x}, \bar{x}).$$

That is, $x_n = \bar{x}$, for $n \geq 0$, is a solution of Equation (1.2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2. (Stability)

(i) The equilibrium point $\bar{x}$ of Equation (1.2) is locally stable, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-1}, x_0 \in I$ with

$$|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon, \quad \text{for all} \quad n \geq 1.$$

(ii) The equilibrium point $\bar{x}$ of Equation (1.2) is locally asymptotically stable, if $\bar{x}$ is locally stable and there exists $\gamma > 0$, such that for all $x_{-1}, x_0 \in I$ with

$$|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iii) The equilibrium point $\bar{x}$ of Equation (1.2) is called a global attractor, if for every $x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) The equilibrium point $\bar{x}$ of Equation (1.2) is called globally asymptotically stable, if it is locally stable and a global attractor.

(v) The equilibrium point $\bar{x}$ of Equation (1.2) is unstable, if it is not locally stable.

The linearized equation of Equation (1.2) about $\bar{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where

$$c_1 = \frac{\partial f}{\partial x_n}(\bar{x}, \bar{x}), \quad c_2 = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \bar{x}),$$

and the characteristic equation of Equation (1.3) is

$$\lambda^2 - c_1\lambda - c_2 = 0. \tag{1.4}$$

Theorem A ([2]).

(a) *If both roots of the quadratic equation* (1.4) *lie in the open unit disk* $|\lambda| < 1$, *then the equilibrium point* $\bar{x}$ *of Equation* (1.2) *is locally asymptotically stable*.

(b) *If at least one of the roots of Equation* (1.4) *has absolute value greater than one*, *then the equilibrium point* $\bar{x}$ *of Equation* (1.2) *is unstable*.

(c) *A necessary and sufficient condition for both roots of Equation* (1.4) *to lie in the open unit disk* $|\lambda| < 1$ *is*

$$|c_1| < 1 - c_2 < 2.$$

In this case, *the locally asymptotically stable equilibrium* $\bar{x}$ *is also called a sink*.

(d) *A necessary and sufficient condition for both roots of Equation* (1.4) *to have absolute value greater than one is*

$$|c_2| > 1 \quad \textit{and} \quad |c_1| < |1 - c_2|.$$

In this case, $\bar{x}$ *is a repeller point.*

(e) *A necessary and sufficient condition for one root of Equation* (1.4) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$c_1^2 + 4c_2 > 0 \quad \textit{and} \quad |c_1| > |1 - c_2|.$$

In this case, the unstable equilibrium point $\bar{x}$ *is called a saddle point.*

(f) *A necessary and sufficient condition for a root of Equation* (1.4) *to have absolute value equal to one is*

$$|c_1| = |1 - c_2|,$$

or

$$c_2 = -1 \quad \textit{and} \quad |c_1| \le 2.$$

In this case, the equilibrium point $\bar{x}$ *is called a non-hyperbolic point.*

Theorem B ([2])**.** *Let* $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b],$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ *is non-increasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, *and* $f(x, y)$ *is non-decreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$.

(b) *The difference equation* (1.2) *has no solutions of prime period two in* $[a, b]$.

Then Equation (1.2) *has a unique equilibrium point* $\bar{x} \in [a, b]$ *and every solution of Equation* (1.2) *converges to* $\bar{x}$.

2. Main Results

In this section, we study the asymptotic stability and the periodic character of the non-negative equilibrium points of Equation (1.1). We can see that the non-negative equilibrium points of Equation (1.1) are the non-negative solutions of equation

$$\bar{x} = \frac{\bar{x}}{p + q\bar{x}^2 + \bar{x}}. \tag{1.5}$$

So $\bar{x}_1 = 0$ is always an equilibrium point of Equation (1.1). When $p < 1$, Equation (1.1) also possesses the unique positive equilibrium

$$\bar{x}_2 = \frac{-1 + \sqrt{1 + 4q(1-p)}}{2q}.$$

Let $f : [0, \infty)^2 \to [0, \infty)$ be a function defined by

$$f(x, y) = \frac{y}{p + qx^2 + y}. \tag{1.6}$$

Therefore, it follows that

$$f_x(x, y) = \frac{-2qxy}{(p + qx^2 + y)^2}, \tag{1.7}$$

$$f_y(x, y) = \frac{p + qx^2}{(p + qx^2 + y)^2}. \tag{1.8}$$

We see that

$$c_1 = f_x(\bar{x}, \bar{x}) = \frac{-2q\bar{x}^2}{(p + q\bar{x}^2 + \bar{x})^2}, \tag{1.9}$$

$$c_2 = f_y(\bar{x}, \bar{x}) = \frac{p + q\bar{x}^2}{(p + q\bar{x}^2 + \bar{x})^2}. \tag{1.10}$$

The linearized equation of Equation (1.1) about $\bar{x}$ is

$$y_{n+1} + \frac{2q\bar{x}^2}{(p + q\bar{x}^2 + \bar{x})^2} y_n - \frac{p + q\bar{x}^2}{(p + q\bar{x}^2 + \bar{x})^2} y_{n-1} = 0, \quad n = 0, 1, 2, \ldots,$$

whose characteristic equation

$$\lambda^2 + \frac{2q\bar{x}^2}{(p + q\bar{x}^2 + \bar{x})^2} \lambda - \frac{p + q\bar{x}^2}{(p + q\bar{x}^2 + \bar{x})^2} = 0, \quad n = 0, 1, 2, \ldots.$$

2.1. Stability of the zero equilibrium point

In this subsection, we investigate the stability of the zero equilibrium point of Equation (1.1).

Theorem 1. *For Equation* (1.1), *we have the following results*:

(i) *Assume that* $p > 1$, *then the zero equilibrium point of equation* (1.1) *is locally asymptotically stable.*

(ii) *Assume that* $p < 1$, *then the zero equilibrium point of Equation* (1.1) *is unstable.*

Proof. The linearized equation associated with Equation (1.1) about $\bar{x}_1 = 0$ has the form

$$y_{n+1} - \frac{1}{p} y_{n-1} = 0, \quad n = 0, 1, 2, \ldots,$$

so, the characteristic equation of Equation (1.1) about $\bar{x}_1 = 0$, is

$$\lambda^2 - \frac{1}{p} = 0,$$

then the proof of (i), (ii) follows immediately from Theorem A.

In the following theorem, we prove the global asymptotic stability of the zero equilibrium point $\bar{x}_1 = 0$, when $p > 1$.

Theorem 2. *Assume that* $p > 1$, *then the zero equilibrium point of Equation* (1.1) *is globally asymptotically stable.*

Proof. We know by Theorem 1 that $\bar{x}_1 = 0$ is locally asymptotically stable equilibrium point of Equation (1.1), and so it suffices to show that $\bar{x}_1 = 0$ is a global attractor of Equation (1.1) as follows

$$0 \leq x_{n+1} = \frac{x_{n-1}}{p + qx_n^2 + x_{n-1}} \leq \frac{1}{p} x_{n-1}.$$

Since $p > 1$, then

$$\lim_{n \to \infty} x_n = 0.$$

This completes the proof. In the following theorem, we prove the local stability of the zero equilibrium point $\bar{x}_1 = 0$ when $p = 1$.

Theorem 3. *Assume that* $p = 1$, *then the zero equilibrium point of Equation* (1.1) *is locally stable.*

Proof. Let $\epsilon > 0$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Equation (1.1) such that

$$|x_{-1}| + |x_0| < \epsilon.$$

It suffices to show that

$$|x_1| < \epsilon.$$

Now,

$$0 < x_1 = \frac{x_{-1}}{1 + qx_0^2 + x_{-1}} \leq x_{-1} < \epsilon,$$

and so the proof is complete.

The next theorem shows that the zero equilibrium point of Equation (1.1) is globally asymptotically stable when $p = 1$.

Theorem 4. *Assume* $p = 1$, *then the zero equilibrium point of Equation* (1.1) *is globally asymptotically stable.*

Proof. We know by Theorem 3 that $\bar{x}_1 = 0$ is locally stable, and so it suffices to show that $\bar{x}_1 = 0$ is a global attractor of Equation (1.1)

$$0 \leq x_{n+1} = \frac{x_{n-1}}{1 + qx_n^2 + x_{n-1}} \leq x_{n-1},$$

so, the even terms of this solution decrease to a limit $(\text{say } L_1 \geq 0)$, and the odd terms decrease to a limit $(\text{say } L_2 \geq 0)$. Thus,

$$L_1 = \frac{L_1}{1 + qL_2^2 + L_1}, \quad L_2 = \frac{L_2}{1 + qL_1^2 + L_2},$$

which implies that

$$L_1 = L_2 = 0,$$

so

$$\lim_{n \to \infty} x_n = 0.$$

This completes the proof.

2.2. Stability of the positive equilibrium point

In this subsection, we investigate the stability of the positive equilibrium point of Equation (1.1). In the following theorem, we determine more precisely necessary conditions (on parameters) for $\bar{x}_2$ to be locally asymptotically stable and for $\bar{x}_2$ to be unstable.

Theorem 5. *Assume that* $p < 1$, *then we have the following results*:

(1) *If* $q < \frac{3}{4(1-p)}$, *then the positive equilibrium point* $\bar{x}_2$ *of Equation* (1.1) *is locally asymptotically stable.*

(2) *If* $q > \frac{3}{4(1-p)}$, *then the positive equilibrium point* $\bar{x}_2$ *of Equation* (1.1) *is unstable* (*a saddle point*).

Proof. By using the identity

$$q\bar{x}_2^{\,2} + \bar{x}_2 + p = 1,$$

we see that

$$c_1 = \frac{-2q\bar{x}_2^{\,2}}{(p + q\bar{x}_2^{\,2} + \bar{x}_2)^2} = -2q\bar{x}_2^{\,2},$$

$$c_2 = \frac{p + q\bar{x}_2^{\,2}}{(p + q\bar{x}_2^{\,2} + \bar{x}_2)^2} = p + q\bar{x}_2^{\,2}.$$

So,

$$|c_1| + c_2 - 1 = 3q\bar{x}_2^{\,2} + p - 1 = 3(1 - p - \bar{x}_2) + p - 1 = 2(1 - p) - 3\bar{x}_2$$

$$= 2(1 - p) - 3\left(\frac{-1 + \sqrt{1 + 4q(1 - p)}}{2q}\right)$$

$$= \frac{1}{2q}\left(4q(1 - p) + 3 - 3\sqrt{1 + 4q(1 - p)}\right) < 0 \Leftrightarrow 4q(1 - p)$$

$$+ 3 < 3\sqrt{1 + 4q(1 - p)},$$

which is valid iff

$$q < \frac{3}{4(1 - p)}.$$

Also,

$$1 - c_2 = 1 - (p + q\bar{x}_2^{\,2}) < 2.$$

So, $\bar{x}_2$ is locally asymptotically stable when $q < \dfrac{3}{4(1 - p)}$.

It is clear that $c_1^2 + 4c_2 > 0$ and

$$|c_1| > |1 - c_2| \Leftrightarrow \left|-2q\bar{x}_2^{\,2}\right| > \left|1 - p - q\bar{x}_2^{\,2}\right| = |1 - 1 + \bar{x}_2|$$

$$\Leftrightarrow 2q\bar{x}_2^{\,2} > \bar{x}_2 \Leftrightarrow 2q\left(\frac{-1 + \sqrt{1 + 4q(p - 1)}}{2q}\right) > 1$$

$$\Leftrightarrow -1 + \sqrt{1 + 4q(p-1)} > 1$$

$$\Leftrightarrow \sqrt{1 + 4q(p-1)} > 2 \Leftrightarrow 1 + 4q(1-p) > 4 \Leftrightarrow q > \frac{3}{4(1-p)}.$$

Thus $\bar{x}_2$ is unstable (saddle point) when $q > \frac{3}{4(1-p)}$.

2.3. Existence of prime period two solutions

This subsection is devoted to discuss the condition under which there exist prime period two solutions.

Theorem 6. *For Equation* (1.1), *we have the following results*:

(i) *Equation* (1.1) *possesses the prime-period two solutions*

$$\ldots, 0, 1-p, 0, 1-p, \ldots,$$

when

$$p < 1.$$

(ii) *Equation* (1.1) *possesses the prime-period two solutions*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots,$$

where the values of Φ *and* Ψ *are the two positive and distinct roots of the quadratic equation*

$$t^2 - \frac{1}{q} t + \frac{(p-1)q + 1}{q^2} = 0,$$

when

$$p < 1 \quad \textit{and} \quad \frac{3}{4(1-p)} < q \le \frac{1}{1-p}.$$

Proof. Let

$$\ldots \Phi, \Psi, \Phi, \Psi, \ldots,$$

be a prime period two solution of Equation (1.1). Then

$$\Phi = \frac{\Phi}{p + q\Psi^2 + \Phi}, \quad \text{and} \quad \Psi = \frac{\Psi}{p + q\Phi^2 + \Psi}.$$

If $\Phi = 0$, then $\Psi = 1 - p > 0$, which implies that $p < 1$. If $\Phi \neq 0$ and $\Psi \neq 0$, then

$$p + q\Psi^2 + \Phi = 1, \tag{1.11}$$

and

$$p + q\Phi^2 + \Psi = 1. \tag{1.12}$$

Subtracting (1.11) from (1.12) gives

$$q(\Phi^2 - \Psi^2) = (\Phi - \Psi).$$

Since $\Phi \neq \Psi$, it follows that

$$\Phi + \Psi = \frac{1}{q}. \tag{1.13}$$

Again adding (1.11) and (1.12) yields

$$q(\Phi^2 + \Psi^2) + (\Phi + \Psi) = 2(1 - p). \tag{1.14}$$

It follows by (1.13) and (1.14) and the relation

$$\Phi^2 + \Psi^2 = (\Phi + \Psi)^2 - 2\Phi\Psi \quad \text{for all} \quad \Phi, \Psi \in \mathbb{R},$$

that

$$\Phi\Psi = \frac{(p-1)q + 1}{q^2}. \tag{1.15}$$

So, Φ and Ψ are the roots of the quadratic equation

$$t^2 - \frac{1}{q}t + \frac{(p-1)q + 1}{q^2} = 0.$$

Since Φ and Ψ must be non-negative and distinct real numbers, so

$$0 < 1 - 4[1 + q(p-1)] \leq 1,$$

which implies that

$$p < 1 \quad \text{and} \quad \frac{3}{4(1-p)} < q \leq \frac{1}{1-p}.$$

This completes the proof.

Theorem 7. *The prime-period two solutions*

$$\ldots, 0, 1-p, 0, 1-p, \ldots,$$

occurs iff

$$(x_{-1}, x_0) = (0, 1-p) \quad \text{or} \quad (x_{-1}, x_0) = (1-p, 0).$$

Proof. Consider the case $(x_{-1}, x_0) = (0, 1-p)$ (The proof when $(x_{-1}, x_0) = (1-p, 0)$ is similar and will be omitted.), then $x_1 = 0$ and $x_2 = 1-p$, and the proof follows by induction. Now suppose that there exists $N \geq 1$ such that

$$x_N = 0, \quad x_{N+1} = 1-p.$$

Then from Equation (1.1), we see that

$$1 - p = \frac{x_{N-1}}{p + x_{N-1}} \Rightarrow x_{N-1} = 1-p,$$

and

$$0 = \frac{x_{N-2}}{p + q(1-p)^2 + x_{N-2}} \Rightarrow x_{N-2} = 0,$$

and the proof follows by induction.

Theorem 8. *Assume* $p < 1$, $q > \frac{1}{1-p}$ *and*

$$px_0 > (1-p)\left(p + \frac{qx_{-1}^2}{(p + qx_0^2 + x_{-1})^2}\right). \tag{1.16}$$

Then every solution of Equation (1.1) *converges to a period-two solution with a basin*

$$S = (0, 1-p) \times (1-p, \infty).$$

Proof. Let $\{x_n\}$ be a solution of Equation (1.1) with initial conditions $(x_{-1}, x_0) \in S$. Then from Equation (1.1), we see that

$$x_1 = \frac{x_{-1}}{p + qx_0^2 + x_{-1}} < \frac{x_{-1}}{p + \frac{1}{1-p}(1-p)^2 + x_{-1}} = \frac{x_{-1}}{1 + x_{-1}} < x_{-1} < 1 - p,$$

$$x_2 = \frac{x_0}{p + qx_1^2 + x_0} < \frac{x_0}{p + qx_1^2 + 1 - p} = \frac{x_0}{qx_1^2 + 1} < x_0.$$

Also from Equation (1.16), we have

$$x_2 = \frac{x_0}{p + qx_1^2 + x_0} = \frac{x_0}{p + q\left(\frac{x_{-1}}{p + qx_0^2 + x_{-1}}\right)^2 + x_0} > \frac{x_0}{p + \frac{px_0}{1-p} - p + x_0} = 1 - p.$$

Thus by induction, we see that

$$(x_{2n-1}, x_{2n}) \in S, \quad n = 0, 1, 2, \dots,$$

and

$$x_{2n+2} < x_{2n}, \; x_{2n+1} < x_{2n-1}, \quad n = 0, 1, 2, \dots.$$

So the sequence $\{x_{2n}\}$ decreases to a limit (say $L_1 \geq 1 - p$), and the sequence $\{x_{2n+1}\}$ decreases to a limit (say $L_2 \leq 1 - p$), thus,

$$L_1 = \frac{L_1}{p + qL_2^2 + L_1}, \quad L_2 = \frac{L_2}{p + qL_1^2 + L_2},$$

which implies that

$$L_1 = 1 - p, \quad L_2 = 0.$$

This completes the proof.

Theorem 9. *Let* $p < 1$, *and* $q < \frac{3}{4(1-p)}$, *then the positive equilibrium point* $\bar{x}_2$ *of Equation* (1.1) *is globally asymptotically stable with basin* $(0, \infty)^2$.

Proof. We know by Theorem 5 that $\bar{x}_2$ is locally asymptotically stable, and so it suffices to show that $\bar{x}_2$ is a global attractor of Equation (1.1). From Equations (1.7) and (1.8), we have $f(x, y)$ defined by Equation (1.6) is decreasing in $x \in (0, \infty)$ for each $y \in (0, \infty)$ and increasing in $y \in (0, \infty)$ for each $x \in (0, \infty)$. Recall by Theorems 6 and 7 that there exist no solutions of Equation (1.1) with prime period two when $q < \frac{3}{4(1-p)}$ and $(x_{-1}, x_0) \in (0, \infty)^2$. Also

$$0 < f(x, y) = \frac{y}{p + qx^2 + y} < 1, \quad \text{for all} \quad x, y \in (0, \infty).$$

So by Theorem B,

$$\lim_{n \to \infty} x_n = \bar{x}.$$

Conjecture. Assume $p < 1$, and $q > \frac{1}{1-p}$. Show that the solution of Equation (1.1) converges to a period two solution $0, 1-p$ when

$$(x_{-1}, x_0) = (1-p, \infty) \times (0, 1-p) \cup (0, 1-p)^2 \cup (1-p, \infty)^2.$$

Open problem. Investigate the global behaviour of Equation (1.1) when $p < 1$ under the condition $q = \frac{3}{4(1-p)}$.

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