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IDENTITIIES AND PARSEVAL TYPE RELATIONS FOR THE $\mathcal{L}_4$-TRANSFORM

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Abstract

In the present paper, the Laplace transform, its generalizations the $\mathcal{L}_2$ and $\mathcal{L}_4$ transforms, the Widder transform, and the $\mathcal{P}_4$ transform are examined. The main results of the paper, Parseval-Goldstein type theorems and corollaries, are proven in Section 2. Some illustrative examples for these relations are given in Section 2 and Section 3. The theorems and the lemmas that are proven in this article are new useful relations for evaluating indefinite integrals of special functions. In Section 3, the author derives some infinite integrals which include the error function, the complementary error function, elementary functions, and some special functions.

1. Introduction

The Laplace transform is defined as

$$\mathcal{L}\{f(x);\ y\} = \int_0^\infty e^{-xy} f(x) dx. \tag{1}$$

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Sadek and Yürekli [10] introduced the $\mathcal{L}_2$-transform as a generalization of the Laplace transform as follows:

$$\mathcal{L}_2\{f(x);\ y\} = \int_0^\infty x e^{-x^2y^2} f(x)dx. \tag{2}$$

Dernek et al. [2] presented the $\mathcal{L}_4$-transform

$$\mathcal{L}_4\{f(x);\ y\} = \int_0^\infty x^3 e^{-x^4y^4} f(x)dx. \tag{3}$$

The Laplace transform, the $\mathcal{L}_2$-transform, and the $\mathcal{L}_4$-transform are related by the formula

$$\mathcal{L}_4\{f(x);\ y\} = \frac{1}{2}\mathcal{L}_2\left\{f\left(x^{1/2}\right);\ y^2\right\} = \frac{1}{4}\mathcal{L}\left\{f\left(x^{1/4}\right);\ y^4\right\}. \tag{4}$$

The Widder transform, the $\mathcal{P}_4$-transform [2], the $\mathcal{K}$-transform of order ν are defined, respectively, by

$$\mathcal{P}\{f(x);\ y\} = \int_0^\infty \frac{xf(x)}{x^2 + y^2}dx, \tag{5}$$

$$\mathcal{P}_4\{f(x);\ y\} = \int_0^\infty \frac{x^3f(x)}{x^4 + y^4}dx, \tag{6}$$

$$\mathcal{K}_\nu\{f(x);\ y\} = \int_0^\infty (xy)^{1/2} K_\nu(xy)f(x)dx, \tag{7}$$

where K_ν denotes the modified Bessel function of second kind of order ν.

The error function and the complementary error function are defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du, \tag{8}$$

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du, \tag{9}$$

respectively. It is well-known that $erf(x) + erfc(x) = 1$. In Dernek et al. [2], using the following identities involving the $\mathcal{L}_4$-transform and the Widder transform:

$$\int_0^\infty x^3 \mathcal{L}_4\{h(y);\, x\} \mathcal{L}_4\{g(u);\, x\} dx = \int_0^\infty y^3 f(y) \mathcal{L}_4\{\mathcal{L}_4\{g(u);\, x\};\, y\} dy, \quad (10)$$

$$\mathcal{L}_4^2\{f(x);\, y\} = \frac{1}{4} \mathcal{P}_4\{f(x);\, y\}, \quad (11)$$

several transform pairs, including some other integral transforms were evaluated.

The objective of this paper is firstly to show that the Parseval-Goldstein-type relations yield some simple techniques for evaluating infinite integrals involving special functions. The second aim is to establish new identities involving the $\mathcal{L}_2$-transform, the $\mathcal{L}_4$-transform, the error function, the complementary error function, the $\mathcal{P}_4$-transform, and the $\mathcal{K}$-transform. The identities generalize some of the earlier formulas in ([1], [2], [4], [11]). Using these new identities, we can extend the table of integral transforms.

2. The Main Theorems

Lemma 2.1. *The Parseval-Goldstein-type identity*

$$\int_0^\infty x^3 \mathcal{L}_4\left\{f(y);\, (x^4 + z^4)^{1/4}\right\} \mathcal{L}_4\{g(u);\, x\} dx = \frac{1}{4} \mathcal{L}_4\{f(y);\, \mathcal{P}_4\{g(u);\, y\};\, z\} \quad (12)$$

holds true, *provided that the integrals involved converge absolutely*.

Proof. In formula (10), that was previously obtained in [2], we set

$$h(y) = e^{-y^4 z^4} f(y). \quad (13)$$

Using the definition of $\mathcal{L}_4$-transform (3), we deduce the following property:

$$\mathcal{L}_4\left\{e^{-y^4 z^4} f(y);\ x\right\} = \mathcal{L}_4\left\{f(y);\ (x^4 + z^4)^{1/4}\right\}. \tag{14}$$

Substituting the relations (13) and (14) into (10) and using the relation (11), we obtain the assertion (12).

Remark 2.2. If we set $f(y) = 1$ into (12), we have

$$\mathcal{P}_4\{\mathcal{L}_4\{g(u);\ x\};\ z\} = \mathcal{L}_4\{\mathcal{P}_4\{g(u);\ y\};\ z\}, \tag{15}$$

provided that the integrals involved converge absolutely. The formula (15) was previously obtained in [2].

Lemma 2.3. *We have*

$$\mathcal{K}_\nu\left\{y^{\nu+1/2} e^{-y^4 z^4} f(y);\ a\right\} = a^{1/2}\mathcal{L}_4\{y^{\nu-2} K_\nu(ay) f(y);\ z\}. \tag{16}$$

Proof. The relation (16) is obtained from the definitions of the $\mathcal{L}_4$-transform and the $\mathcal{K}$-transform.

Theorem 2.4. *The following identities hold true*:

$$\mathcal{L}_4\{f(y) y^{2\nu} K_\nu(a^2 y^2);\ z\} = \frac{a^{2\nu}}{2^{\nu-1}} \int_0^\infty \frac{x^{-4\nu-1}}{e^{a^4/4x^4}} \mathcal{L}_4\left\{f(y);\ (x^4 + z^4)^{1/4}\right\} dx, \tag{17}$$

and

$$\mathcal{K}_\nu\left\{\frac{y^{\nu+1/2}}{e^{z^4 y^2}} f\left(y^{1/2}\right);\ a^2\right\} = \frac{a^{2\nu+1}}{2^{-\nu-2}} \mathcal{L}_4\left\{t^{4(\nu-1)} \mathcal{L}_4\left\{f(y);\ \frac{(1+4t^4 z^4)^{1/4}}{\sqrt{2}t}\right\};\ a\right\}, \tag{18}$$

where $\mathcal{R}(\nu) > -1$ and the integrals involved converge absolutely.

Proof. We set

$$g(u) = u^{2\nu} J_\nu(a^2 u^2), \tag{19}$$

in the relation (12) of Lemma 2.1, using the relationship (11) and the formula ([5], p. 146, Entry (29)), we get

$$\mathcal{P}_4\{u^{2\nu} J_\nu(a^2 u^2);\ y\} = \frac{1}{2}\, y^{2\nu} K_\nu(a^2 y^2). \tag{20}$$

Using the relation (4) and the formula ([5], p. 185, (30)), we have

$$\mathcal{L}_4\{u^{2\nu} J_\nu(a^2 u^2);\ x\} = \frac{1}{4}\,\mathcal{L}\left\{u^{\nu/2} J_\nu\left(a^2 u^{1/2}\right);\ x^4\right\} = \frac{a^{2\nu}}{2^{\nu+2}}\,\frac{x^{-4\nu-4}}{e^{a^4/4x^4}}. \tag{21}$$

Substituting (19), (20), and (21) into (12), we obtain the assertion (17). Identity (18) follows upon changing the variable of the integration on the right-hand side of (17) from x to t, where $x = 1/\sqrt{2}t$ and using the relation (16) of Lemma 2.3.

Remark 2.5. Setting $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$ into (17), we obtain

$$\mathcal{L}_4\left\{f(y) e^{-a^2 y^2};\ z\right\} = \frac{2a^2}{\sqrt{\pi}} \int_0^\infty x^{-3} e^{-a^4/4x^4}\, \mathcal{L}_4\left\{f(y);\ (x^4 + z^4)^{1/4}\right\} dx, \tag{22}$$

and

$$\mathcal{L}_4\left\{y^{-2} f(y) e^{-a^2 y^2};\ z\right\} = \frac{4}{\sqrt{\pi}} \int_0^\infty x e^{-a^4/4x^4}\, \mathcal{L}_4\left\{f(y);\ (x^4 + z^4)^{1/4}\right\} dx, \tag{23}$$

where each of the integrals converge absolutely.

Remark 2.6. Setting $x = 1/\sqrt{2}t$ on the right-hand side of (22) and (23), we obtain

$$\mathcal{L}_4\left\{e^{-a^2 y^2} f(y);\ z\right\} = \frac{4a^2}{\sqrt{\pi}}\, \mathcal{L}_4\left\{\frac{1}{t^2}\, \mathcal{L}_4\left\{f(y);\ \frac{(1 + 4t^4 z^4)^{1/4}}{\sqrt{2}t}\right\};\ a\right\}, \tag{24}$$

and

$$\mathcal{L}_4\left\{y^{-2}e^{-a^2y^2}f(y);\ z\right\} = \frac{2}{\sqrt{\pi}}\mathcal{L}_4\left\{\frac{1}{t^6}\mathcal{L}_4\left\{f(y);\ \frac{(1+4t^4z^4)^{1/4}}{\sqrt{2}t}\right\};\ a\right\}. \quad (25)$$

Remark 2.7. Substituting $z = 0$ in (24) and (25), then using the definition of the $\mathcal{L}_2$-transform, we find

$$\mathcal{L}_2\{y^2f(y);\ a\} = \frac{4a^2}{\sqrt{\pi}}\mathcal{L}_4\left\{\frac{1}{t^2}\mathcal{L}_4\left\{f(y);\ \frac{1}{\sqrt{2}t}\right\};\ a\right\}, \quad (26)$$

$$\mathcal{L}_2\{f(y);\ a\} = \frac{2}{\sqrt{\pi}}\mathcal{L}_4\left\{\frac{1}{t^6}\mathcal{L}_4\left\{f(y);\ \frac{1}{\sqrt{2}t}\right\};\ a\right\}. \quad (27)$$

Example 2.8. Suppose $-1 < \mathcal{R}(\nu) < 1$, then we have

$$\mathcal{K}_\nu\left\{y^{-1/2}e^{-z^4y^2};\ a^2\right\} = \pi^{1/2}\frac{a2^{\nu-3}}{z^2}\sec\left(\frac{\pi\nu}{2}\right)K_{\nu/2}\left(\frac{a^4}{8z^4}\right). \quad (28)$$

Proof. We set

$$f(y) = y^{-2(\nu+1)}, \quad (29)$$

in the relation (18) of Theorem 2.4. Using the relation (4), we get

$$\mathcal{L}_4\left\{y^{-2(\nu+1)};\ \frac{(1+4t^4z^4)^{\frac{1}{4}}}{\sqrt{2}t}\right\} = 2^{\nu-3}\Gamma\left(\frac{1-\nu}{2}\right)\left(\frac{z}{t}\right)^{2\nu-2}\left(t^4+\frac{1}{4z^4}\right)^{\frac{\nu-1}{2}}, \quad (30)$$

where $\mathcal{R}(\nu) < 1.$ Substituting (29) and (30) into (18), we obtain

$$\mathcal{K}_\nu\left\{e^{-z^4y^2};\ a^2\right\} = \frac{a^{2\nu+1}}{2^{-\nu+2}}\Gamma\left(\frac{1-\nu}{2}\right)z^{2\nu-2}\mathcal{L}_4\left\{t^{2\nu-2}\left(t^4+\frac{1}{4z^4}\right)^{\frac{\nu-1}{2}};\ a\right\}. \quad (31)$$

Using the relation (4) once again and the formula ([5], p. 138, Entry (13)), we evaluate the $\mathcal{L}_4$-transform on the right-hand side of (31):

$$\mathcal{L}_4\left\{t^{2\nu-2}\left(t^4+\frac{1}{4z^4}\right)^{\frac{\nu-1}{2}};\ a\right\} = \frac{1}{\sqrt{\pi}4(2a^2z^2)^\nu}\Gamma\left(\frac{\nu+1}{2}\right)K_{\nu/2}\left(\frac{a^4}{8x^4}\right), \quad (32)$$

where $\mathcal{R}(\nu) > -1$. Now the assertion (28) follows upon substituting (32) into (31) and using the identity

$$\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right) = \pi \sec\left(\frac{\pi\nu}{2}\right). \tag{33}$$

Lemma 2.9. *The following identity holds true*:

$$\mathcal{L}_2\{\mathcal{L}_4\{f(x);\ u\};\ y\} = \frac{\sqrt{\pi}}{4}\int_0^\infty x f(x) e^{y^4/4x^4} erfc\left(\frac{y^2}{2x^2}\right)dx, \tag{34}$$

provided that the integrals involved converge absolutely.

Proof. By the definition (2) and the relation (4), we find

$$\mathcal{L}_2\{\mathcal{L}_4\{f(x);\ u\};\ y\} = \int_0^\infty u e^{-u^2y^2}\left[\int_0^\infty x^3 e^{-u^4y^4} f(x)dx\right]du. \tag{35}$$

Changing the order of integration, which is permisible by absolute converge of the integrals involved, we obtain

$$\mathcal{L}_2\{\mathcal{L}_4\{f(x);\ u\};\ y\} = \int_0^\infty x^3 f(x)\left[\int_0^\infty u e^{-u^4x^4-y^2u^2} du\right]dx$$

$$= \int_0^\infty x^3 f(x) e^{\frac{y^4}{2x^4}} \int_0^\infty u e^{-x^4\left(u^2+\frac{y^2}{2x^4}\right)^2} du dx. \tag{36}$$

In the inner integral on the right-hand side of (36), making the change of variable

$$t = x^2\left(u^2 + \frac{y^2}{2x^4}\right), \tag{37}$$

and using the definition of complementary error function (9), we get the assertion (34).

Example 2.10. We show

$$\int_0^\infty \frac{1}{x}\cos(x^2) e^{\frac{y^2}{2x^2}} erfc\left(\frac{y^2}{2x^2}\right)dx = 2\sqrt{\pi}y^2(2y^4+3)e^{y^4} erfc(y^2) + 4y^4. \tag{38}$$

Proof. If we set

$$f(x) = \frac{\cos(x^2)}{x^2}, \tag{39}$$

in (34) of Lemma 2.9, we get

$$\mathcal{L}_2\left\{\mathcal{L}_4\left\{\frac{\cos(x^2)}{x^2};\ u\right\};\ y\right\} = \frac{\sqrt{\pi}}{4}\int_0^\infty x\,\frac{\cos(x^2)}{x^2}\,e^{y^4/4x^4}\,erfc\left(\frac{y^2}{2x^2}\right)dx. \tag{40}$$

Using the relation (4) and the known formulas ([5], p. 129, Entry (6); p. 146, Entry (21)], we obtain

$$\mathcal{L}_4\left\{\frac{\cos(x^2)}{x^2};\ u\right\} = \frac{\sqrt{\pi}}{4u^2}\,e^{-1/4u^4}, \tag{41}$$

$$\mathcal{L}_2\left\{\frac{1}{u^2}\,e^{-1/4u^4};\ y\right\} = 2\sqrt{\pi}y^2(2y^4 + 3)e^{y^4}erfc(y^2) + 4y^4. \tag{42}$$

Substituting the result (42) into (40), we deduce the assertion (38).

Lemma 2.11. *The following identity holds true*:

$$\mathcal{L}_4\{\mathcal{L}_2\{f(x);\ u\};\ y\} = \frac{1}{4y^4}\int_0^\infty xf(x)dx$$

$$-\frac{\sqrt{\pi}}{8y^6}\int_0^\infty x^{-5}f\left(\frac{1}{x}\right)e^{\frac{1}{4x^4y^4}}erfc\left(\frac{1}{2x^2y^2}\right)dx, \tag{43}$$

provided that each member of the assertion (43) *exits*.

Proof. By the definitions (2) and (3) of the $\mathcal{L}_2$-transform and the $\mathcal{L}_4$-transform, we have

$$\mathcal{L}_4\{\mathcal{L}_2\{f(x);\ u\};\ y\} = \int_0^\infty u^3e^{-y^4u^4}\left[\int_0^\infty xe^{-u^2x^2}f(x)dx\right]du. \tag{44}$$

Changing the order of integration, which is permisible by absolute converge of the integrals involved, it follows from (44),

$$\mathcal{L}_4\{\mathcal{L}_2\{f(x);\ u\};\ y\} = \int_0^\infty xf(x)\left[\int_0^\infty u^3 e^{-y^4u^4 - u^2x^2} du\right]dx. \tag{45}$$

Substituting

$$-y^4u^4 - u^2x^2 = -y^4\left(u^2 + \frac{x}{2y^4}\right)^2 + \frac{x^4}{4y^4}, \tag{46}$$

in the inner integral on the right-hand side of (45), we get

$$\mathcal{L}_4\{\mathcal{L}_2\{f(x);\ u\};\ y\} = \int_0^\infty xf(x)e^{x^4/4y^4}\left[\int_0^\infty u^3 e^{-y^4\left(u^2 + \frac{x}{2y^4}\right)^2} du\right]dx. \tag{47}$$

Setting

$$t = y^2\left(u^2 + \frac{x^2}{2y^4}\right), \tag{48}$$

on the right-handside of (47), we obtain the following relation:

$$\mathcal{L}_4\{\mathcal{L}_2\{f(x);\ u\};\ y\} = \int_0^\infty xf(x)e^{x^4/4y^4}$$

$$\times\left[\frac{1}{4y^4}e^{-x^4/4y^4} - \frac{\sqrt{\pi}}{8}\frac{x^2}{y^6}\,erfc\left(\frac{x^2}{2y^2}\right)\right]dx. \tag{49}$$

Changing the variable on the left-hand side from x to t according to the transformation $x = \frac{1}{t}$, we obtain the assertion (43).

Example 2.12. We show

$$\int_0^\infty x^{-5}e^{-1/x^4(1-1/4y^4)}erfc\left(\frac{1}{2x^2y^2}\right)dx = \frac{y^2}{2(2y^2+1)}. \tag{50}$$

Proof. If we set

$$f(x) = e^{-x^4}, \tag{51}$$

in the formula (43) of Lemma 2.11, we get

$$\mathcal{L}_4\left\{\mathcal{L}_2\left\{e^{-x^4};\ u\right\};\ y\right\} = \frac{1}{4y^4}\int_0^\infty xe^{-x^4}dx - \frac{\sqrt{\pi}}{8y^6}\int_0^\infty x^{-5}e^{\frac{-1}{x^4}\left(1-\frac{1}{4y^4}\right)}erfc\left(\frac{1}{2x^2y^2}\right)dx. \tag{52}$$

Using the relationship (4) and the well-known formula ([5], p. 177, Entry (10)), the Laplace transforms on the left-hand side of (52) are given by

$$\mathcal{L}\left\{e^{-x^2};\ u^2\right\} = \frac{\sqrt{\pi}}{2}e^{u^4/4}erfc\left(\frac{u^2}{2}\right), \tag{53}$$

$$\mathcal{L}\left\{e^{u/4}erfc\left(\frac{\sqrt{u}}{2}\right);\ y^4\right\} = \frac{y^{-2}}{y^2+\frac{1}{2}}, \tag{54}$$

where $\mathcal{R}(u) > 0$, $\mathcal{R}(y) > 0$. Substituting (53) and (54) into (52), we obtain (50).

Lemma 2.13. *The following identity holds true*:

$$\mathcal{L}_4\left\{f(x^4 - z^4)^{1/4}H(x-z);\ y\right\} = e^{-z^4y^4}\mathcal{L}_4\{f(x);\ y\}, \tag{55}$$

where $H(x))$ *is the Heaviside function.*

Proof. Using the definition (3) of the $\mathcal{L}_4$-transform and the Heaviside function, we get

$$\mathcal{L}_4\left\{f(x^4 - z^4)^{\frac{1}{4}}H(x-z);\ y\right\} = e^{-z^4y^4}\int_0^\infty x^3 f\left((x^4 - z^4)^{\frac{1}{4}}\right)e^{-(x^4-z^4)y^4}dx. \tag{56}$$

The assertion (55) follows from changing the variable of integration to $u = (x^4 - z^4)^{1/4}$ on the right-hand side of (56), and using the definition (3) of the $\mathcal{L}_4$-transform once again.

Theorem 2.14. *The following Parseval-type relationship holds true*:

$$\int_0^\infty y^\nu \mathcal{L}_4\{f(x);\ y\}dx = \frac{1}{4}\Gamma\left(\frac{\nu+1}{4}\right)\int_0^\infty \frac{f(x)}{x^{\nu-2}}dx, \tag{57}$$

provided that the $\mathcal{R}(\nu) > -1$ *and the integrals involved converge absolutely*.

Proof. Using the definition (3) of the $\mathcal{L}_4$-transform, we get

$$\int_0^\infty y^\nu \mathcal{L}_4\{f(x);\ y\}dy = \int_0^\infty y^\nu\left[\int_0^\infty x^3 e^{-x^4y^4} f(x)dx\right]dy. \tag{58}$$

Changing the order of integration, which is permissible under the assumptions of the theorem, we have

$$\int_0^\infty y^\nu \mathcal{L}_4\{f(x);\ y\}dy = \int_0^\infty x^3 f(x)\left[\int_0^\infty y^\nu e^{-x^4y^4}dy\right]dx. \tag{59}$$

Now, using the relationship (4) and evaluating the inner integral in (59) the assertion (57) is obtained.

Corollary 2.15. *The following identities hold true*, *if the integrals involved converge absolutely*:

$$\int_0^\infty y^\nu \mathcal{L}_4\left\{g(x); (z^4 + y^4)^{1/4}\right\}dy = \frac{1}{4}\Gamma\left(\frac{\nu+1}{4}\right)\mathcal{L}_4\left\{\frac{g(x)}{x^{\nu+1}};\ z\right\}, \tag{60}$$

$$\int_z^\infty u^3(u^4 - z^4)^{(\nu-3)/4}\mathcal{L}_4\{g(x);\ u\}du = \frac{1}{4}\Gamma\left(\frac{\nu+1}{4}\right)\mathcal{L}_4\left\{\frac{g(x)}{x^{\nu+1}};\ z\right\}. \tag{61}$$

Proof. We substitute

$$f(x) = e^{-z^4x^4}g(x), \tag{62}$$

into (57) and then utilize the following identity, which is obtained by using the definition (3) of the $\mathcal{L}_4$-transform:

$$\mathcal{L}_4\left\{e^{-z^4x^4}g(x);\ y\right\} = \mathcal{L}_4\left\{g(x); (z^4 + y^4)^{1/4}\right\}. \tag{63}$$

Thus (60) is obtained. The assertion (61) follows upon changing the variable of the integration in (60) to $u^4 = z^4 + y^4$.

Corollary 2.16. *Under the assumptions of Theorem* 2.14, *we have*

$$\mathcal{L}_4\{y^{\nu-3}\mathcal{L}_4\{g(x);\ y\};\ z\} = \frac{1}{4}\Gamma\left(\frac{\nu+1}{4}\right)\int_z^\infty \frac{g\left((x^4 - z^4)^{1/4}\right)}{x^{\nu-2}}dx. \tag{64}$$

Proof. Substituting

$$f(x) = g\left((x^4 - z^4)^{1/4}\right)H(x - z), \tag{65}$$

into (57) of Theorem 2.14, we get

$$\int_0^\infty y^\nu \mathcal{L}_4\{f(x);\ y\}dy = \int_0^\infty y^\nu \mathcal{L}_4\left\{g(x^4 - z^4)^{1/4}H(x - z);\ y\right\}dy. \tag{66}$$

Then, utilizing the identity (55) of Lemma 2.13, we have

$$\int_0^\infty y^\nu \mathcal{L}_4\{f(x);\ y\}dy = \frac{1}{4}\Gamma\left(\frac{\nu+1}{4}\right)\int_z^\infty \frac{g\left((x^4 - z^4)^{1/4}\right)}{x^{\nu-2}}dx. \tag{67}$$

Using the identity (55) of Lemma 2.13 and the definition (3) of the $\mathcal{L}_4$-transform, we obtain (64).

Remark 2.17. If we change the variable of the integration in (64) to $u^4 = x^4 - z^4$, we obtain the identity

$$\mathcal{L}_4\{y^{\nu-3}\mathcal{L}_4\{g(x);\ y\};\ z\} = \frac{1}{4}\Gamma\left(\frac{\nu+1}{4}\right)\int_0^\infty \frac{u^3 g(u)du}{(u^4 + z^4)^{(\nu+1)/4}}. \tag{68}$$

If we set $\nu = 3$ in (64), we find the known relation (11).

Example 2.18. The following identities hold true:

$$\int_0^\infty \frac{\cos(a^2x^4)}{x^{\nu-2}}dx = \frac{1}{4}a^{(\nu-3)/2}\Gamma\left(\frac{3-\nu}{4}\right)\cos\left[\pi\left(\frac{3-\nu}{8}\right)\right], \tag{69}$$

where $-1 < \mathcal{R}(\nu) < 3$ and

$$\int_0^\infty \frac{\sin(a^2x^4)}{x^{\nu-2}}\,dx = \frac{1}{4}a^{(\nu-3)/2}\Gamma\left(\frac{3-\nu}{4}\right)\sin\left[\pi\left(\frac{3-\nu}{8}\right)\right], \tag{70}$$

where $-1 < \mathcal{R}(\nu) < 7$.

Proof. We set

$$f(x) = \cos(a^2x^4), \tag{71}$$

in the identity (57) of Theorem 2.14. Using the relationship (4) and the formula ([5], p. 154, Entry (43)), then we obtain

$$\mathcal{L}_4\{\cos(a^2x^4);\ y\} = \frac{y^4}{4(a^4 + y^8)}. \tag{72}$$

Substituting (71) and (72) into (57), we find

$$\int_0^\infty \frac{\cos(a^2x^4)}{x^{\nu-2}}\,dx = \frac{1}{\Gamma\left(\frac{\nu+1}{4}\right)}\int_0^\infty \frac{y^{\nu+4}}{a^4 + y^8}\,dy. \tag{73}$$

We utilize the integral representation for the gamma function ([7], p. 7)

$$\int_0^\infty \frac{t^\alpha}{(1+t)^{1+\beta}}\,dt = \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+1)}, \quad (\mathcal{R}(\beta) > \mathcal{R}(\alpha) > -1), \tag{74}$$

and the well known identity

$$\Gamma(z)\Gamma(1-z) = \pi\cos(\pi z), \tag{75}$$

to evaluate the integral on the right- hand side of (73). Using some results on trigonometric functions, we get (69). We obtain (70) in a similar fashion by substituting $f(x) = \sin(a^2x^4)$ into identity (57).

Theorem 2.19. *The following identities hold true*:

$$\int_0^\infty xf(x)erfc(s^2x^2)\,dx = \frac{4}{\sqrt{\pi}}\int_s^\infty y\mathcal{L}_4\{f(x);\ y\}\,dy, \tag{76}$$

$$\int_0^\infty y\mathcal{L}_4\{f(x);\ y\}dy = \frac{\sqrt{\pi}}{4}\int_0^\infty xf(x)dx, \tag{77}$$

$$\int_0^\infty xf(x)erf(s^2x^2)dx = \int_0^\infty xf(x)dx - \frac{4}{\sqrt{\pi}}\int_s^\infty y\mathcal{L}_4\{f(x);\ y\}dy, \tag{78}$$

provided that each of the integrals involved converges absolutely.

Proof. Using the definition (3) of the $\mathcal{L}_4$-transform, we get

$$\int_s^\infty y\mathcal{L}_4\{f(x);\ y\}dy = \int_s^\infty\left[\int_0^\infty yx^3e^{-x^4y^4}f(x)dx\right]dy. \tag{79}$$

Changing the order of integration, which is permissible under the hypothesis of the theorem, the last equation can be written as

$$\int_s^\infty y\mathcal{L}_4\{f(x);\ y\}dy = \int_0^\infty x^3f(x)\left[\int_s^\infty ye^{-x^4y^4}dy\right]dx. \tag{80}$$

Changing the variable of the inner integration on the right-hand side of (82) from y to t, where $t = y^2x^2$ and using definition (9) of the complementary error function, we find that the inner integral on the right-hand side takes the value

$$\int_s^\infty ye^{-x^4y^4}dy = \frac{\sqrt{\pi}}{4}\int_0^\infty xf(x)erfc(s^2x^2)dx. \tag{81}$$

Substituting (81) into (82), we obtain the assertion (76). The identity (77) easily follows when we set $s = 0$ in (76) and use the fact that $erfc(0) = 1$. Using definition (8) of the error function and the identity $erf(x) + erfc(x) = 1$, we obtain the assertion (78).

3. Examples

Example 3.1. We have

$$\int_0^\infty xe^{-a^2x^4}erfc(s^2x^2)dx = \frac{1}{2a\sqrt{\pi}}\left[\frac{\pi}{2} - \arctan\frac{s^2}{a}\right], \tag{82}$$

where $\mathcal{R}(a) > 0$.

Proof. If we set $f(x) = e^{-a^2x^4}$ in (76) of Theorem 2.19 and use the relation,

$$\mathcal{L}_4\left\{e^{-a^2x^4};\ y\right\} = \frac{1}{4}\frac{1}{y^4 + a^2}, \tag{83}$$

we obtain the assertion (82).

Remark 3.2. Setting $s = 0$ and $x^2 = u$ in (82), we obtain the well-known result

$$\int_0^\infty e^{-a^2u^2} du = \frac{\sqrt{\pi}}{2a}. \tag{84}$$

If $s > 0$, then we have

$$\arctan\left(\frac{a}{s}\right) = \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right). \tag{85}$$

Using the relations (83) and (85), we obtain

$$\int_0^\infty xe^{-a^2x^4} erfc\,(s^2x^2)dx = \frac{1}{2a\sqrt{\pi}}\arctan\left(\frac{a}{s^2}\right). \tag{86}$$

Setting $s = 1$ and $x^2 = u$ in (86) gives the following well-known formula ([7], p. 649, 6.285 (1)):

$$\int_0^\infty e^{-a^2u^4} erfc\,(u)du = a^{-1}\pi^{-1/2}\arctan(a). \tag{87}$$

Example 3.3. We have

$$\int_0^\infty x^3e^{-a^2x^4} erf\,(s^2x^2)dx = \frac{1}{4a^2}s^2(s^4 + a^2)^{-1/2}, \tag{88}$$

where $\mathcal{R}(a^2 + s^4) > 0$.

Proof. If we set $f(x) = x^2 e^{-a^2x^4}$ in the relation (78) and use the formula ([8], p. 144, Entry 5(3)), we obtain

$$\int_0^\infty x^3 e^{-a^2x^4} erf\,(s^2x^2)dx = \frac{1}{4a^2} - \frac{1}{2}\int_s^\infty \frac{ydy}{(y^4 + a^2)^{3/2}}. \tag{89}$$

The integral on the right-hand side of (89) may be evaluated by setting $v = y^2$ and then changing the variable of the integration from v to u, where $v = a\tan u$, we get

$$\int_0^\infty \frac{ydy}{(y^4 + a^2)^{3/2}} = \frac{1}{2a^2}\int_{\arctan(s^2/a)}^{\pi/2} \cos\,udu. \tag{90}$$

Using the known relation

$$\sin(\arctan x) = \frac{x}{(1 + x^2)^{1/2}}, \tag{91}$$

and setting the expression (90) into (89), we obtain the assertion (88).

Example 3.4. We show for $\mathcal{R}(\nu) > -1$

$$\int_0^\infty x^{2\nu+1} J_\nu(a^2x^2) e^{\frac{y^4}{4x^4}} erfc\left(\frac{y^2}{4x^2}\right)dx = \frac{2}{\sqrt{\pi}}\, a^{\nu-1} y^{\nu+1} K_\nu(a^2y), \tag{92}$$

where J_ν and K_ν are the Bessel functions.

Proof. Setting $f(x) = x^{2\nu} J_\nu(a^2x^2)$ in the identity (34) of Lemma 2.9, we get

$$\mathcal{L}_2\{\mathcal{L}_4\{f(x);\, u\};\, y\} = \frac{\sqrt{\pi}}{4}\int_0^\infty x^{2\nu+1} J_\nu(a^2x^2) e^{y^4/4x^4} erfc\left(\frac{y^2}{4x^2}\right)dx. \tag{93}$$

Using the relation (4) and the known identities ([5], p. 185, Entry (30); p. 146, Entry (29)), we have for $\mathcal{R}(\nu) > -1$, $\mathcal{R}(u) > 0$, $\mathcal{R}(y) > 0$,

$$\mathcal{L}_4\{x^{2\nu} J_\nu(a^2x^2);\, u\} = \frac{a^{2\nu}}{2^{\nu+2}}\, e^{-a^4/4u^4} u^{-4(\nu+1)}, \tag{94}$$

and for $\mathcal{R}(a) > 0$, $\mathcal{R}(y) > 0$,

$$\mathcal{L}_2\left\{u^{-4(\nu+1)}e^{-a^4/4u^4};\ y\right\} = \left(\frac{2}{a}\right)^{\nu+1} y^{\nu+1}K_\nu(a^2y). \tag{95}$$

Substituting (94) and (95) into (93), we obtain the assertion (92).

Example 3.5. We have

$$\int_0^\infty y^\nu e^{a^4y^4} erfc(a^2y^2)dy = \frac{a^{-\nu-1}}{4}\Gamma\left(\frac{\nu+1}{4}\right)\csc\left[\pi\left(\frac{\nu+3}{4}\right)\right], \tag{96}$$

where $-3 < \mathcal{R}(\nu) < 1$.

Proof. We set $f(x) = x^{-2}(x^4 + a^4)^{-1}$ in the identity (57) of Theorem 2.14. Using the relationship (4) and the formulas ([9], p. 16, Entry (3)), ([2], p. 399, formula (17)), ([6], p. 216, Entry (5)), we get

$$\int_0^\infty y^\nu e^{a^4y^4} erfc\,(a^2y^2)dy = \frac{a^2}{\pi}\Gamma\left(\frac{\nu+1}{4}\right)\mathcal{P}_4\{x^{-\nu-3};\ a\}, \tag{97}$$

where

$$\mathcal{P}_4\{x^{-\nu-3};\ a\} = \frac{\pi}{4}a^{-\nu-3}\csc\left[\pi\left(\frac{\nu+3}{4}\right)\right]. \tag{98}$$

Substituting (98) into (97), we obtain the assertion (96).

Example 3.6. We have

$$\int_0^\infty x^{(\nu-3)/2}e^{-a^2x^2}dx = \frac{1}{2}a^{-(\nu-1)/2}\Gamma\left(\frac{\nu+1}{4}\right), \tag{99}$$

where $\mathcal{R}(\nu) > 1$, $\mathcal{R}(a) > 0$.

Proof. We set $f(x) = x^{-2}e^{-a^2/4x^4}$ in the identity (57) of Theorem 2.14. Using the relation (4) and setting $z = (\nu-1)/4$ in the formula for the Γ function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \tag{100}$$

we get

$$\int_0^\infty x^{-\nu} e^{-a^2/4x^4} dx = 2^{-2}\left(\frac{2}{a}\right)^{(\nu-1)/2} \Gamma\left(\frac{\nu - 1}{4}\right). \tag{101}$$

The assertion (99) follows, when we change the variable of the integration on the left- hand side of (101) to $u = 1/2x^2$.

Remark 3.7. If we put $\nu = 3$ in (99), we obtain the well-known formula (84). If we put $\mu = (\nu - 3)/2$ in (99), we deduce

$$\mathcal{L}_2\{x^{\mu-1};\ y\} = \frac{1}{2} y^{-(\mu+1)} \Gamma\left(\frac{\mu + 1}{2}\right), \tag{102}$$

where $\mathcal{R}(\mu) > 1$.

Remark 3.8. We show

$$\int_0^\infty \frac{y^\nu}{(y^4 + z^4)^2 + a^4} dy = \frac{\pi[z^8 + a^4]^{(\nu-3)/8}}{4a^2 \sin\left(\pi\left(\frac{\nu+1}{4}\right)\right)} \sin\left[\frac{3-\nu}{8} \arctan\left(\frac{a^2}{z^4}\right)\right], \tag{103}$$

where $-1 < \mathcal{R}(\mu) < 3$.

Proof. Setting $g(x) = \sin(a^2 x^4)$ in the identity (60) of Corollary 2.15 and using the relation (4), the formula ([5], p. 152, Entry (15)) and the relation

$$\Gamma\left(\frac{\nu+1}{4}\right)\Gamma\left(\frac{3-\nu}{4}\right) = \pi \csc\left[\pi\left(\frac{\nu+1}{4}\right)\right], \tag{104}$$

we obtain the assertion (103).

Example 3.9. We have

$$\int_0^\infty \frac{erf(a^2x^2)}{x^{\nu-2}}\,dx = \frac{\sqrt{\pi}a^{\nu-3}}{(\nu-3)\,\Gamma\left(\frac{\nu-1}{4}\right)\sin\left[\pi\left(\frac{\nu-1}{4}\right)\right]}, \tag{105}$$

where $3 < \mathcal{R}(\nu) < 5$.

Proof. Setting $f(x) = erf(a^2x^4)$ in the identity (57) of Theorem 2.14 and using the relation (4), the formula ([5], p. 176, Entry (4)), the relation (75), respectively, we obtain the assertion (105).

Example 3.10. We have

$$\int_0^\infty \frac{y^\nu}{(y^4+z^4)^{1/2}}\,e^{-a(y^4+z^4)^{1/2}}\,dy = \frac{\left(\frac{2z^2}{a}\right)^{\frac{\nu-1}{4}}}{2\sqrt{\pi}}\,\Gamma\left(\frac{\nu+1}{4}\right)K_{\frac{1-\nu}{4}}(az^2), \tag{106}$$

and

$$\int_z^\infty u(u^4-z^4)^{\frac{\nu-3}{4}}e^{-au^2}\,du = \frac{1}{2\sqrt{\pi}}\left(\frac{2z^2}{a}\right)^{\frac{\nu-1}{4}}\Gamma\left(\frac{\nu+1}{4}\right)K_{\frac{1-\nu}{4}}(az^2), \tag{107}$$

where $-1 < \mathcal{R}(\nu) < 1$.

Proof. We set $g(x) = x^{-2}e^{-a^2/4x^4}$ into the identity (60) of Corollary 2.15. Using the formulas (4), ([5], p. 146, Entries (27) and (29)), we obtain

$$\mathcal{L}_4\left\{x^{-2}e^{-a^2/4x^4};\,(y^4+z^4)^{1/4}\right\} = \frac{\sqrt{\pi}e^{-a(y^4+z^4)^{1/2}}}{4(y^4+z^4)^{1/2}}, \tag{108}$$

$$\mathcal{L}_4\left\{x^{-\nu-3}e^{-a^2/4x^4};\,z\right\} = \frac{1}{2}\left(\frac{2z^2}{a}\right)^{(\nu-1)/4}K_{\frac{1-\nu}{4}}(az^2), \tag{109}$$

respectively. Now, the assertion (106) follows when we substitute the results (108) and (109) into (60) of Corollary 2.15. Similarly, when we substitute the results (108), (109), and $g(x) = x^{-2}e^{-a^2/4x^4}$ into (61) of

Corallary 2.15 and we use the formula ([5], p. 146, Entry (27)), the assertion (107) is obtained. Remark 3.11. If we put $\mu = (1-\nu)/4$ in (106), we deduce

$$K_{\mu}(az^2) = \frac{2\sqrt{\pi}\left(\frac{2z^2}{a}\right)^{\mu}}{\Gamma\left(\frac{1}{2}-\mu\right)} \int_0^{\infty} y^{1-4\mu}(y^4 + z^4)^{-1/2} e^{-a(y^4+z^4)^{-1/2}} dy, \quad (110)$$

where $\mathcal{R}(\mu) < 1/2$, $-1 < \mathcal{R}(\nu) < 1$.

Setting $az^2 = x$ and changing the variable of integration on the right of (110) to $t = yz$, we obtain the following integral representation for the modified Bessel function of the second kind $K_{\mu}(x)$:

$$K_{\mu}(x) = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\mu\right)} \left(\frac{2}{x}\right)^{\mu} \int_0^{\infty} t^{1-4\mu}(t^4+1)^{-1/2} e^{-x(t^4+1)} dt, \quad (111)$$

where $\mathcal{R}(\mu) < 1/2$.

Similarly, we deduce another integral representation for the function $K_{\mu}(x)$ from the result (107) of Example 3.10:

$$K_{\mu}(x) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\mu\right)} \left(\frac{2}{x}\right)^{\mu} \int_1^{\infty} t(t^4-1)^{-\mu-1/2} e^{-xt^2} dt, \quad (112)$$

where $\mathcal{R}(\mu) < 1/2$. Changing the variable of integration on the right-hand side of (112) to $w = t^2$, we get the well known formula:

$$K_{-\mu}(x) = K_{\mu}(x) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\mu\right)} \left(\frac{x}{2}\right)^{\mu} \int_1^{\infty} (w^2-1)^{\mu-1/2} e^{-xw} dw, \quad (113)$$

where $\mathcal{R}(\mu) > -1/2$.

We conclude this investigation by remarking that many other infinite integrals can be evaluated in this manner by applying the above theorems and lemmas.

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