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A REMARK ON THE CONTINUANTS OF TRANSCENDENTAL NUMBERS IN CONTINUED FRACTION EXPANSION

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Abstract

For any irrational $\theta \in [0, 1)$, let $q_n(\theta)$ be n-th continuant of θ in its continued fraction expansion. Davenport and Roth showed that if θ satisfies

$$\log \log q_n > \frac{cn}{\sqrt{\log n}}, \text{ for infinitely many } n \in \mathbb{N},$$

for all $c > 0$, then θ must be transcendental. We call a set A purely transcendental set, if all the elements in A are transcendental. In this note, we intend to explain that if a purely transcendental set is determined merely by the properties of the individual continuants, besides algebraic numbers, most transcendental numbers are excluded from this set. Namely, let ϕ be a positive function defined on $\mathbb{N}$ and set

$$A(\phi) = \{x \in [0, 1) : q_n(x) \geq \phi(n), \text{ infinitely many } n\}.$$

If $A(\phi)$ is a purely transcendental set, then the set $A(\phi)$ is of Hausdorff dimension at most one-half.

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1. Introduction

Diophantine approximation is intimated connected with continued fractions in the sense that, for any irrational $\theta \in [0, 1)$, if

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ must be a convergent of θ in its continued fraction expansion.

It seems that the first result concerning the properties of continuants of transcendental number or equivalently algebraic numbers is Liouville inequality [1], which shows that any algebraic number of degree *d* cannot be approximated by rational numbers at an order greater than *d*. Using this result, one has

Theorem 1.1 ([1]). *Let θ be an irrational and $q_n(\theta)$ be the n-th continuants in its continued fraction expansion. If, for any $c > 0$,*

$$\log \log q_n(\theta) \geq cn, \text{ for infinitely many } n \in \mathbb{N}, \tag{1.1}$$

then θ is transcendental.

According to an estimation on the number of solutions to the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\delta}}, \tag{1.2}$$

when ξ is algebraic, Davenport and Roth [2] derived an improvement of (1.1).

Theorem 1.2 ([2]). *Let θ be an irrational and $q_n(\theta)$ be the n-th continuants in its continued fraction expansion. If, for any $c > 0$,*

$$\log \log q_n \geq \frac{cn}{\sqrt{\log n}}, \text{ for infinitely many } n \in \mathbb{N}, \tag{1.3}$$

then θ is transcendental.

An qualitative improvement of the number of solutions to inequality (1.2), which is given by Bombieri and van der Poorten [6], enable Adamczewski and Bugeaud [3] to obtain an improved one.

Theorem 1.3 ([3]). *Let* θ *be an irrational and* $q_n(\theta)$ *be the n-th continuants in its continued fraction expansion. If, for any* $c > 0$,

$$\log\log q_n > cn^{\frac{2}{3}}(\log n)^{\frac{2}{3}} \log\log n, \text{ for infinitely many } n \in \mathbb{N}, \quad (1.4)$$

then θ *must be transcendental.*

We call a set A purely transcendental set, if all the elements in A are transcendental. In this note, we intend to explain that if a purely transcendental set is characterized merely by precise properties of the individual continuants, besides algebraic numbers, most transcendental numbers are also excluded from this set. Namely, let ϕ be a positive function defined on $\mathbb{N}$ and set

$$A(\phi) = \{x \in [0, 1) : q_n(x) \geq \phi(n), \text{ infinitely many } n\}.$$

We show that

Theorem 1.4. *If* $A(\phi)$ *is a purely transcendental set, then the set* $A(\phi)$ *is of Hausdorff dimension at most one-half.*

2. Preliminaries

We begin with some notations firstly. Let $x \in [0, 1)$ be an irrational number and $[a_1(x), a_2(x), \cdots]$ be its regular continued fraction expansion. For any $n \geq 1$, denote by

$$p_n(x) / q_n(x) := [a_1(x), a_2(x), \cdots, a_n(x)],$$

the n-th convergent of x. With the conventions that $p_{-1}(x) = 1$, $q_{-1}(x) = 0$, $p_0(x) = 0$, $q_0(x) = 1$, one has [4]

$$p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x), \quad n \geq 0,$$

$$q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x), \quad n \geq 0, \tag{2.1}$$

where $\{q_n\}_{n\geq 1}$ are commonly called the *continuants*.

For any $a_1, \cdots, a_n \in \mathbb{N}$, denote by $I_n(a_1, \cdots, a_n)$ the n-th cylinder

$$I_n(a_1, \cdots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, \cdots, a_n(x) = a_n\}.$$

Lemma 2.1 ([4]). *For any* $a_1, \cdots, a_n \in \mathbb{N}$,

$$|I_n(a_1, \cdots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}, \quad \prod_{j=1}^{n} a_j < q_n < \prod_{j=1}^{n}(a_j + 1),$$

where $|\cdot|$ *denotes the length of a subset in* $[0, 1)$ *and* q_n, q_{n-1} *are recursively defined by* (2.1).

Let φ be a positive function on $\mathbb{N}$. Set

$$E(\varphi) = \{x \in [0, 1) : a_n(x) \geq \varphi(n), \text{ for infinitely many } n\}.$$

A complete result on the Hausdorff dimension of $E(\varphi)$ was given in [5], but only the needed part is cited here.

Lemma 2.2 ([5]). *Write* $b = \exp\left\{\liminf_{n\to\infty} \frac{\log\log \varphi(n)}{n}\right\}$. *If*

$$\liminf_{n\to\infty} \frac{\log \varphi(n)}{n} = \infty,$$

then $\dim_H E(\varphi) = \frac{1}{b+1}$.

Recall that

$$A(\phi) = \{x \in [0, 1) : q_n(x) \geq \phi(n), \text{ for infinitely many } n\}.$$

Lemma 2.3. *If* $A(\phi)$ *is a purely transcendental set, then*

$$\liminf_{n\to\infty} \frac{\log \phi(n)}{n} = \infty.$$

Proof. For any integer $B \geq 1$, let $x_B = [B, B, \cdots,]$. Lagrange's theorem asserts that x_B is quadratic irrational. Whence,

$$\phi(n) \geq q_n(x_B) \geq B^n, \text{ for } n \text{ ultimately.}$$

3. Proof of Main Result

In this section, we give the exact Hausdorff dimension of the purely transcendental set $A(\phi)$.

Lemma 3.1. *If* $A(\phi)$ *is a purely transcendental set, then* $\dim_H A(\phi) = \frac{1}{1+b}$, *where* $b = \exp\left\{\liminf_{n\to\infty} \frac{\log\log \varphi(n)}{n}\right\}$.

Proof. In the light of Lemma 2.3, the lower bound of $\dim_H A(\phi)$ is a direct consequence of Lemma 2.2.

Now we turn to the upper bound. Two cases will be distinguished according as $b = 1$ or $b > 1$.

(i) $b = 1$. For any $t > 1$, we introduce a family of measures μ_t:

$$\mu_t(I_n(a_1, \cdots, a_n)) = e^{-np(t) - t\sum_{j=1}^{n} \log a_j(x)}, \tag{3.1}$$

where $p(t) = \log \sum_{n=1}^{\infty} \frac{1}{n^t}$.

Now, let $\mathcal{I}(n)$ be the family of all n-th order cylinders $I(a_1, \cdots, a_n)$, which satisfies $q_n \geq \phi(n)$. Then,

$$A(\phi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{I_n(a_1,\cdots,a_n)\in\mathcal{I}(n)} I_n(a_1, \cdots, a_n).$$

For each $N \geq 1$, we select all those cylinders in $\bigcup_{n=N}^{\infty} \mathcal{I}(n)$, which are maximal ($I \in \bigcup_{n=N}^{\infty} \mathcal{I}(n)$ is maximal if there is no other I' in $\bigcup_{n=N}^{\infty} \mathcal{I}(n)$ such that $I \subset I'$ and $I \neq I'$). We denote by $\mathcal{J}(N)$ the set of all maximal cylinders in $\bigcup_{n=N}^{\infty} \mathcal{I}(n)$. It is evident that $\mathcal{J}(N)$ is a cover of $A(\phi)$ for any $N \geq 1$.

Fix $t > 1$ and $\epsilon > 0$. Choose N_0 large enough such that for any $n \geq N_0$, $\epsilon \log \phi(n) \geq np(t)$.

Fix $N \geq N_0$. Then for any $I_n(a_1, \cdots, a_n) \in \mathcal{J}(N)$, we have

$$|I_n(a_1, \cdots, a_n)|^{\frac{t+\epsilon}{2}} \leq e^{-(t+\epsilon)\log q_n} \leq e^{-t\sum_{j=1}^{n} \log a_j - np(t)} = \mu_t(I_n(a_1, \cdots, a_n)).$$

So,

$$\sum_{I_n(a_1, \cdots, a_n) \in \mathcal{J}(N)} |I_n(a_1, \cdots, a_n)| \leq \sum_{I_n(a_1, \cdots, a_n) \in \mathcal{J}(N)} \mu_t(I_n(a_1, \cdots, a_n)) \leq 1.$$

This implies $\dim A(\phi) \leq 1/2 = \dfrac{1}{b+1}$.

(ii) $b > 1$. By the definition of b, one has, for any $\epsilon > 0$,

$$\prod_{j=1}^{n} e^{(b-\epsilon)^{j-1}(b-\epsilon-1)} \leq \phi(n), \text{ for } n \text{ ultimately.} \tag{3.2}$$

So, it follows

$$A(\phi) \subset \{x \in [0, 1) : a_n(x) \geq \left[e^{(b-\epsilon)^{n-1}(b-\epsilon-1)}\right] - 1, \textit{ i.o., } n\}.$$

As a consequence of Lemma 2.2, one gets $\dim_H A(\phi) \leq \dfrac{1}{1+b}$. □

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