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SIMULTANEOUS METHODS FOR FINDING ALL ZEROS OF A POLYNOMIAL

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Abstract

The purpose of this paper is to present three new methods for finding all simple zeros of polynomials simultaneously. First, we give a new method for finding simultaneously all simple zeros of polynomials constructed by applying the Weierstrass method to the zero in the trapezoidal Newton's method, and prove the convergence of the method. We also present two modified Newton's methods combined with the derivative-free method, which are constructed by applying the derivative-free method to the zero in the trapezoidal Newton's method and the midpoint Newton's method, respectively. Finally, we give a numerical comparison between various simultaneous methods for finding zeros of a polynomial.

1. Introduction

With a typical iteration method such as Newton's method, an initial approximation of a zero converges to a specific zero, but the Weierstrass method (or Durand-Kerner method) approximates all simple (real or complex) zeros of polynomial simultaneously (see [2, 4]).

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Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be a polynomial of degree n having simple zeros with constants $a_1, a_2, \ldots, a_n$. Let $r_1, \ldots, r_n$ be the distinct zeros of $P(z)$ and let distinct complex numbers $z_1, \ldots, z_n$ be their approximations. The *Weierstrass method* (Durand-Kerner method) is defined as

$$z_i^{m+1} = z_i^m - \frac{P(z_i^m)}{\prod_{j \neq i}(z_i^m - z_j^m)}, \tag{1}$$

for $m \geq 0$, and this method is one of the most frequently used iterative methods which give simultaneous computation of all zeros of P. If a function $W_i(z)$ is defined by

$$W_i(z) = \frac{P(z)}{\prod_{j \neq i}(z - z_j)},$$

then $W_i(z)$ has the same zeros as the polynomial P, and so the problem of finding the zeros of P reduces to that of zeros of the function $W_i(z)$. If we denote $W_i = W_i(z_i)$ for $i = 1, 2, \ldots, n$ in the case of $z = z_i$, (1) can be written as

$$\hat{z}_i = z_i - W_i, \tag{2}$$

where z_i is a current approximation and $\hat{z}_i$ is a new approximation to a zero of polynomial $P(z)$. The method constructed by (2) is called the *Weierstrass-like method* (briefly, WLM).

The aim of this paper is to present three new methods for finding all simple zeros of polynomials simultaneously. These new methods are based on the Frontini-Sormani's midpoint Newton's method ([7]) and the Weerakoon's trapezoidal Newton's method ([8]), which were modifications of the Newton's method through iterative approximations.

It is well-known that Newton's method is defined by $x^* = x - \frac{f(x)}{f'(x)}$ with an approximation x and a new approximation x^* of a zero, and is efficient to find a zero of an equation $f(x) = 0$ for a differentiable function f with proper conditions and a sufficiently close initial value (see [8]).

In [8], Weerakoon and Fernando proposed the *trapezoidal Newton's method* defined by

$$\hat{x} = x - \frac{2f(x)}{f'(x) + f'(x^*)}. \tag{3}$$

They applied Newton's method to the x^* of the denominator.

Along with (3), the *midpoint Newton's method* that Frontini-Sormani proposed in [7] is constructed as

$$\hat{x} = x - \frac{f(x)}{f'(x + \frac{1}{2}(x^* - x))}. \tag{4}$$

They also applied Newton's method to the x^* of the denominator, and so set $x^* = x - \frac{f(x)}{f'(x)}$.

Both the trapezoidal Newton's method and the midpoint Newton's method are of cubic order, while the original Newton's method was of quadratic order. A variety of methods can be applied to the x^* in addition to Newton's method. Petković et al. [5] derived the following simultaneous method for finding all simple zeros of polynomials by applying the Weierstrass method to the x^* in the midpoint Newton's method:

$$\hat{z}_i = z_i - \frac{P(z_i)}{P'(z_i - \frac{1}{2}W_i)}, \tag{5}$$

which is called *Newton-Weierstrass method* (or NWM). Also, Petković and Petković [6] found the following *derivative-free method* (or DFM) defined as:

$$\hat{z}_i = z_i - \frac{W_i}{1 - P(z_i - W_i)/P(z_i)}, \tag{6}$$

which has a similar form with the one above and this method is of cubic order.

In this paper, we present three new methods for the simultaneous approximation of all simple zeros of polynomials by applying the Weierstrass-like method and the derivative-free method to x^* in the trapezoidal Newton's method and the midpoint Newton's method.

Throughout this paper, the convergence of zeros will be discussed and the order will be calculated for new constructed methods. We will use the notation $a = O_M(b)$ for two complex numbers a and b, whose moduli are of the same order, that is, $|a| = O(|b|)$. In addition, the error is defined as $|e| = \max_{i=1,\ldots,n}\{|e_i|\}$ with $e_i = z_i - r_i$ for $i = 1, \ldots, n$.

In all discussions, the order related to e_i, which is an error of the previously approximated zeros z_i, is presumed to be the same. After that, we will show that the order related to e_i, which is an error of the approximated zeros concerning each method, is identical. For the same being, the order related to the already approximated zeros $\hat{e}_i$ is hypothesized to be identical as follows:

$$e_i = O_M(e) \text{ for all } i.$$

In Section 2, we give a new method for finding simultaneously all simple zeros of polynomials constructed by applying the Weierstrass method to the x^* in the trapezoidal Newton's method, and prove the convergence of the method. In Section 3, we present two modified

Newton's methods combined with the derivative-free method. They are constructed by applying the derivative-free method to the x^* in the trapezoidal Newton's method and the midpoint Newton's method, respectively. In Section 4, we give a numerical comparison between various simultaneous methods for finding zeros of a polynomial. Finally, we conclude that the convergence of all new constructed methods in this paper are similar or superior than other iterative methods of cubic order.

2. Weierstrass-Like Trapezoidal Newton's Method

In this section, we construct a new method for finding simultaneously all simple zeros of polynomials of cubic order. By applying the Weierstrass method (5) to the x^* in the trapezoidal Newton's method (3), we derive a new method constructed as follows:

$$\hat{z}_i = z_i - \frac{2P(z_i)}{P'(z_i) + P'(z_i - W_i)}. \tag{7}$$

We call (7) the *Weierstrass-like trapezoidal Newton's method*, and from this, simply, call it **Method 1.**

The calculation and discussion of the order of Method 1 are similar to those of the Newton-Weierstrass method, which is an alteration of Petković's midpoint Newton's method (see [5]). From (7), we have the following:

Lemma 1. *For a polynomial* $P(z)$, *we have*

$$\frac{P(z_i)}{P'(z_i)}\left(\frac{P''(z_i)P(z_i)}{2P'(z_i)^2} O_M(e) + O_M(e^2)\right) = O_M(e^3).$$

Proof. By the Taylor's expansion around r_i, we have that

$$P(z_i) = P'(r_i)\left(e_i + \frac{P''(r_i)}{2P'(r_i)} e_i^2 + O_M(e^3)\right),$$

$$P'(z_i) = P'(r_i)\left(1 + \frac{P''(r_i)}{P'(r_i)} e_i + \frac{P'''(r_i)}{2P'(r_i)} e_i^2 + O_M(e^3)\right), \tag{8}$$

$$P''(z_i) = P'(r_i)\left(\frac{P''(r_i)}{P'(r_i)} + \frac{P'''(r_i)}{P'(r_i)} e_i + \frac{P''''(r_i)}{2P'(r_i)} e_i^2 + O_M(e^3)\right).$$

From (8), we obtain

$$\frac{P(z_i)}{P'(z_i)}\left(\frac{P''(z_i)P(z_i)}{2P'(z_i)^2} O_M(e) + O_M(e^2)\right)$$

$$= \frac{1}{2}\left(e_i + \frac{P''(r_i)}{2P'(r_i)} e_i^2 + O_M(e^3)\right)^2 \left(1 - \frac{P''(r_i)}{P'(r_i)} e_i + \frac{P'''(r_i)}{2P'(r_i)} e_i^2 + O_M(e^3)\right)^3$$

$$\times \left(\frac{P''(r_i)}{P'(r_i)} + \frac{P'''(r_i)}{P'(r_i)} e_i + \frac{P''''(r_i)}{2P'(r_i)} e_i^2 + O_M(e^3)\right) O_M(e) + O_M(e^3)$$

$$= \frac{1}{2}(O_M(e))^2 (1 + O_M(e))^3 \left(\frac{P''(r_i)}{P'(r_i)} + O_M(e)\right) O_M(e) + O_M(e^2) = O_M(e^3).$$

□

From Lemma 1, we have the following theorem:

Theorem 1. *If the approximate zero* x_i *grounded from Method* 1 *is close enough to* r_i *and the order of* e_i *is the same, then the order of* $\hat{e}_i$ *is identical, and* $|\hat{e}_i| = O_M(|e|^3)$ *is formed.*

Proof. We easily see that the following equation is satisfied:

$$P(z_i) = \prod_{j=1}^{n}(z_i - r_j) = (z_i - r_j)\prod_{j \neq i}(z_i - r_j) = e_i \prod_{j \neq i}(z_i - r_j) = O_M(e).$$

That is,

$$W_i = W_i(z_i) = O_M(P(z_i)) = O_M(e). \tag{9}$$

If $Q(z) = P(z) - \prod_{j=1}^{n}(z - r_j)$, then $Q(z)$ is a polynomial of order $n - 1$, and $Q(z_i) = P(z_i)$ for all i. Therefore, $Q(z)$ is the Lagrange interpolation of points $z_1, z_2, \ldots, z_n$, and so we have

$$Q(z) = \sum_{j=1}^{n}\left(P(z_j)\prod_{k\neq j}\frac{z-z_k}{z_j-z_k}\right) = \sum_{j=1}^{n}\left(\left(W_j\prod_{k\neq j}(z_j-z_k)\right)\prod_{k\neq j}\frac{z-z_k}{z_j-z_k}\right)$$

$$= \sum_{j=1}^{n}\left(W_j\prod_{k\neq j}(z-z_k)\right) = \left(\sum_{j=1}^{n}\frac{W_j}{z-z_j}\right)\prod_{k=1}^{n}(z-z_k).$$

Therefore, we obtain

$$P(z) = \left(1+\sum_{j=1}^{n}\frac{W_j}{z-z_j}\right)\prod_{j=1}^{n}(z-z_j)$$

$$= W_i\prod_{j\neq i}(z-z_j) + \left(1+\sum_{j\neq i}\frac{W_j}{z-z_j}\right)\prod_{j=1}^{n}(z-z_j). \quad (10)$$

From (10), it follows that

$$\frac{P'(z)}{P(z)} = \left(\sum_{j\neq i}\frac{1}{z-z_j}\right) + \frac{1+\sum_{j\neq i}\frac{W_j}{z-z_j} - (z-z_i)\sum_{j\neq i}\frac{W_j}{(z-z_j)^2}}{W_i + (z-z_i)\left(1+\sum_{j\neq i}\frac{W_j}{z-z_j}\right)}. \quad (11)$$

Substituting $z = z_i$ in (11), we obtain

$$\frac{P'(z_i)}{P(z_i)} = \left(\sum_{j\neq i}\frac{1}{z_i-z_j}\right) + \frac{1+\sum_{j\neq i}\frac{W_j}{z_i-z_j}}{W_i}.$$

Therefore, we have

$$W_i = \frac{P(z_i)}{P'(z_i)}\cdot\frac{1+\sum_{j\neq i}\frac{W_j}{z_i-z_j}}{1-\frac{P(z_i)}{P'(z_i)}\sum_{j\neq i}\frac{1}{z_i-z_j}} = \frac{P(z_i)}{P'(z_i)}(1+O_M(e)). \quad (12)$$

Now we will find the order of Method 1. If the Taylor's expansion is applied to $P'(z_i - W_i)$, then we have

$$\begin{aligned}
\hat{z}_i &= z_i - \frac{2P(z_i)}{P'(z_i) + P'(z_i - W_i)} \\
&= z_i - \frac{2P(z_i)}{P'(z_i) + (P'(z_i) - P''(z_i)W_i + \frac{1}{2}P'''(z_i)W_i^2 + \cdots)} \\
&= z_i - \frac{P(z_i)}{P'(z_i) - \frac{1}{2}P''(z_i)W_i + \frac{1}{4}P'''(z_i)W_i^2 + \cdots}.
\end{aligned}$$

By (9), (12), and Lemma 1, we have that

$$\begin{aligned}
\hat{z}_i &= z_i - \frac{P(z_i)}{P'(z_i)\left(1 - \frac{P''(z_i)}{2P'(z_i)}W_i + O_M(e^2)\right)} \\
&= z_i - \frac{P(z_i)}{P'(z_i)} \cdot \left(1 + \frac{P''(z_i)}{2P'(z_i)}W_i + O_M(e^2)\right) \\
&= z_i - \frac{P(z_i)}{P'(z_i)} \cdot \left(1 + \frac{P''(z_i)P(z_i)}{2P'(z_i)^2}(1 + O_M(e)) + O_M(e^2)\right) \\
&= z_i - \frac{P(z_i)}{P'(z_i)} - \frac{P''(z_i)P(z_i)^2}{2P'(z_i)^3} - \frac{P(z_i)}{P'(z_i)}\left(\frac{P''(z_i)P(z_i)}{2P'(z_i)^2}O_M(e) + O_M(e^2)\right) \\
&= z_i - \frac{P(z_i)}{P'(z_i)} - \frac{P''(z_i)P(z_i)^2}{2P'(z_i)^3} - O_M(e^3).
\end{aligned}$$

The Chebyshev's method is defined by

$$\hat{x} = x - \left(1 + \frac{f''(x)f(x)}{2f'(x)^2}\right)\frac{f(x)}{f'(x)},$$

and of cubic order (see [7, Subsection 5.2]). According to the Chebyshev's method, we see that

$$z_i - \frac{P(z_i)}{P'(z_i)} - \frac{P''(z_i)P(z_i)^2}{2P'(z_i)^3} - r_i = O_M(e^3).$$

Therefore, the order of $\hat{e}_i$ is calculated as follows:

$$|\hat{e}_i| = |\hat{z}_i - r_i| = \left| z_i - \frac{P(z_i)}{P'(z_i)} - \frac{P''(z_i)P(z_i)^2}{2P'(z_i)^3} - r_i + O_M(e^3) \right| = O_M(|e|^3).$$

$\square$

3. Modified Newton's Methods Combined with Derivative-Free Method

In this section, we present two modified Newton's methods combined with the derivative-free method (6) for finding all simple zeros of a polynomials simultaneously. The one is a form that the derivative-free method is applied to the x^* in the trapezoidal Newton's method (3) as follows:

$$\hat{z}_i = z_i - \frac{2P(z_i)}{P'(z_i) + P'\left(z_i - \dfrac{W_i}{1 - P(z_i - W_i)/P(z_i)}\right)}, \tag{13}$$

which is called the *derivative-free trapezoidal Newton's method*, or simply, **Method 2**.

From (13), we have the following theorem:

Theorem 2. *If the approximate zero x_i grounded from Method 2 is close enough to r_i and the order of e_i is the same, then the order of $\hat{e}_i$ is identical, and $|\hat{e}_i| = O_M(|e|^3)$ is formed.*

Proof. Since Petković's derivative-free method (6) is of cubic order

$$z_i - \frac{W_i}{1 - P(z_i - W_i)/P(z_i)} - r_i = O_M(e^3), \tag{14}$$

(see [6]). Using (14) and the Taylor's expansion, $\hat{z}_i$ is calculated as follows. (In this case, $C_j = \frac{1}{j!} \cdot \frac{P^{(j)}(r_i)}{P'(r_i)}$).

$$\hat{z}_i = z_i - \frac{2P(z_i)}{P'(z_i) + P'\left(z_i - \frac{W_i}{1 - P(z_i - W_i)/P(z_i)}\right)}$$

$$= z_i - \frac{2P(r_i + e_i)}{P'(r_i + e_i) + P'(r_i + O_M(e^3))}$$

$$= z_i - \frac{2(P(r_i) + P'(r_i)e_i + \frac{1}{2}P''(r_i)e_i^2 + \frac{1}{6}P'''(r_i)e_i^3 + \cdots)}{(P'(r_i) + P''(r_i)e_i + \frac{1}{2}P'''(r_i)e_i^2 + \cdots) + (P'(r_i) + P''(r_i)O_M(e^3) + \cdots)}$$

$$= z_i - \frac{2P'(r_i)(e_i + C_2e_i^2 + C_3e_i^3 + O_M(e^4))}{2P'(r_i)(1 + C_2e_i + \frac{3}{2}C_3e_i^2 + O_M(e^3))}$$

$$= z_i - e_i(1 + C_2e_i + C_3e_i^2 + O_M(e^3))\left(1 - C_2e_i + \frac{3}{2}C_3e_i^2 + O_M(e^3)\right)$$

$$= z_i - e_i + O_M(e^3).$$

Therefore, the order of $\hat{e}_i$ is calculated as follows:

$$|\hat{e}_i| = |\hat{z}_i - r_i| = \left|z_i - e_i + O_M(e^3) - r_i\right| = O_M(|e|^3).$$

□

Now we apply the derivative-free method to the x^* in the midpoint Newton's method (4) and construct the iteration as follows:

$$\hat{z}_i = z_i - \frac{P(z_i)}{P'\left(z_i - \frac{1}{2} \cdot \frac{W_i}{1 - P(z_i - W_i)/P(z_i)}\right)}, \tag{15}$$

which is called the *derivative-free midpoint Newton's method*. From this, we call it **Method 3** simply. From (15), we have the following theorem:

Theorem 3. *If the approximate zero* x_i *grounded from Method* 3 *is close enough to* r_i *and the order of* e_i *is the same, then the order of* $\hat{e}_i$ *is identical, and* $|\hat{e}_i| = O_M(|e|^3)$ *is formed.*

Proof. By using (14) and Taylor's expansion, $\hat{z}_i$ is calculated as follows. (In this case, $C_j = \frac{1}{j!} \cdot \frac{P^{(j)}(r_i)}{P'(r_i)}$).

$$\hat{z}_i = z_i - \frac{P(z_i)}{P'\left(z_i - \frac{1}{2} \cdot \frac{W_i}{1 - P(z_i - W_i)/P(z_i)}\right)}$$

$$= z_i - \frac{P(z_i)}{P'\left(\frac{1}{2} z_i + \frac{1}{2} r_i + O_M(e^3)\right)}$$

$$= z_i - \frac{P(r_i + e_i)}{P'\left(r_i + \frac{1}{2} e_i + O_M(e^3)\right)}$$

$$= z_i - \frac{P(r_i) + P'(r_i)e_i + \frac{1}{2} P''(r_i)e_i^2 + \frac{1}{6} P'''(r_i)e_i^3 + \cdots}{P'(r_i) + P''(r_i)(\frac{1}{2} e_i + O_M(e^3)) + \frac{1}{2} P'''(r_i)(\frac{1}{2} e_i + O_M(e^3))^2 + \cdots}$$

$$= z_i - \frac{P'(r_i)(e_i + C_2 e_i^2 + C_3 e_i^3 + O_M(e^4))}{P'(r_i)(1 + C_2 e_i + \frac{3}{4} C_3 e_i^2 + O_M(e^3))}$$

$$= z_i - e_i(1 + C_2 e_i + C_3 e_i^2 + O_M(e^3))\left(1 - C_2 e_i + \frac{3}{4} C_3 e_i^2 + O_M(e^3)\right)$$

$$= z_i - e_i + O_M(e^3).$$

Therefore, the order of $\hat{e}_i$ can be calculated as follows:

$$|\hat{e}_i| = |\hat{z}_i - r_i| = \left|z_i - e_i + O_M(e^3) - r_i\right| = O_M(|e|^3).$$

$\square$

4. Numerical Comparison

In this section, we give numerical experiments and comparisons between various simultaneous methods for finding zeros of a polynomial. These methods are all of cubic order. They include Method 1, Method 2, Method 3, the derivative-free method (DFM), the Petković's Newton-Weierstrass method (NWM), and the Weierstrass-like method (WLM).

For a polynomial $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, we choose initial approximations as Aberth's approach (see [1]):

$$z_k^{(0)} = -\frac{a_1}{n} + R \exp\left(\frac{i\pi}{n}\left(2k - \frac{3}{2}\right)\right).$$

In this case, R is a radius of a circle, where the initial zeros by Aberth's approach are located in complex number plane. We use the following Henrici's fomula to select R (see [3]):

$$R = 2 \max_{1 \le k \le n} |a_k|^{1/k}.$$

According to Henrici's formula, a disk $\{z : |z| < R\}$ centered at the origin contains all zeros of polynomial $P(z)$.

The polynomials that we used on numerical comparison are as follows:

$$\begin{aligned} P_1 &= (x-1)(x-2)(x-3)(x-4), \\ P_2 &= (x-1)(x-2)(x-3)(x-4)(x-5), \qquad (16) \\ P_3 &= (x-1)(x-2)(x-3)(x-4)(x-5)(x-6), \\ P_4 &= x^8 + 5x^7 + 3x^6 + 7x^5 + 6x^4 + 8x^3 + x^2 + 3x + 7. \end{aligned}$$

Here $P_1(x)$, $P_2(x)$, and $P_3(x)$ are Wilkinson's polynomials when $n = 4, 5, 6$, respectively.

We approximated the zeros until it satisfy the following condition:

$$\max_{1 \le k \le n} \left| P(z_k^{(m)}) \right| < 10^{-10}. \tag{17}$$

In Table 1, we give a numerical comparison between several methods to find all zeros of those polynomials (16). It contains the iteration number m and the value $\max_{1 \le k \le n} \left| P(z_k^{(m)}) \right|$ of iterative methods, after we approximated (17) to a satisfying label. The smaller the m, the faster approximated on the zeros. When m is the same, it can be interpreted that a smaller $\max_{1 \le k \le n} \left| P(z_k^{(m)}) \right|$ leads to a higher accuracy of approximation. All computations have been done using MATLAB.

Table 1. The number of iterations (the error) of iterative methods

Poly. (16)	Method 1 (7)	Method 2 (13)	Method 3 (15)	DFM (6)	NWM (5)	WLM (2)
P_1	9(8e-14)	8(9e-14)	7(1e-14)	9(9e-14)	8(1e-10)	13(3e-12)
P_2	12(7e-13)	11(4e-13)	9(4e-13)	11(2e-13)	11(1e-12)	17(2e-12)
P_3	14(2e-11)	13(5e-12)	11(8e-12)	13(8e-11)	13(7e-12)	21(2e-11)
P_4	14(2e-11)	13(2e-11)	10(2e-11)	14(2e-11)	13(2e-11)	21(2e-11)

5. Conclusion

In this paper, three new methods for the simultaneous approximation of all simple zeros of polynomials by utilizing the trapezoidal Newton's method and the midpoint Newton's method were proposed. It was proven that each method was of third order. By simultaneously approximating all simple zeros of polynomials and by comparing numerical experiments with various methods that are of third order, we obtained that the results of Method 1 and Method 2 are similar with that of previous methods. But, we found out that the result of Method 3 are superior than that of any other methods. All methods we constructed in this paper are new and creative. It seems that these methods can be applied to various fields, and the study on the applications of Method 3 is now in progress.

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