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WHEN ARE THE ZERO-ENERGY SOLUTIONS TO THE SCHRÖDINGER EQUATION BOUNDED AT INFINITY?

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Abstract

Conditions for the boundedness of the solutions to equation $-u'' + q(x)u = 0$, $x \in \mathbb{R}$, are given. The function $q(x)$ is assumed compactly supported real-valued bounded and locally integrable.

1. Introduction

Consider the one-dimensional equation

$$-u'' + q(x)u = 0, \quad x \in \mathbb{R}, \quad \mathbb{R} := \{x : -\infty < x < \infty\}. \tag{1.1}$$

Assume that

$$\max_{x \in \mathbb{R}} |q(x)| \leq Q, \quad q(x) = 0 \text{ for } |x| \geq R. \tag{1.2}$$

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The questions discussed in this paper are:

(a) When are $u(\pm\infty)$ bounded?

(b) When do the limits $u(\pm\infty)$ exist?

These questions are of practical interest, but, it seems, were not considered earlier. A closely related question is:

Under what condition on a potential the Schrödinger operator in $\mathbb{R}^3$ does not have zero as an eigenvalue?

This question was studied and answered in papers [1], [2].

2. Approach

Equation (1.1) is equivalent to the integral equation

$$u(x) = a + bx + \int_{-R}^{x} (x - s)q(s)u(s)ds := a + bx + Vu, \tag{2.1}$$

where a, b are constants. Thus, $u(-\infty)$ exists and is bounded if and only if $b = 0$, and $u(\infty)$ exists and is bounded if and only if the following relation holds:

$$\int_{-R}^{R} q(s)u(s)ds = 0. \tag{2.2}$$

If relation (2.2) holds and $b = 0$, then

$$u(-\infty) = a, \quad u(\infty) = a - \int_{-R}^{R} sq(s)u(s)ds. \tag{2.3}$$

Given a, one can uniquely solve the Volterra integral equation (2.1) with $b = 0$ and then check if condition (2.2) holds for u. The solution to the Volterra equation (2.1) with $b = 0$ is

$$u = a\sum_{j=0}^{\infty} V^j 1. \tag{2.4}$$

Thus, condition (2.2) can be written as

$$\int_{-R}^{R} q(s) \sum_{j=0}^{\infty} V^{j} 1 ds = \sum_{j=0}^{\infty} \int_{-R}^{R} q(s) V^{j} 1 ds = 0. \tag{2.5}$$

Let us formulate the result.

Theorem 1. *Let assumption* (1.2) *hold. Then the limits* $u(\pm\infty)$ *of any solutions to Equation* (1.1) *exist and are bounded if and only if* $b = 0$ *and condition* (2.5) *holds.*

Remark 1. Condition $b = 0$ is not difficult to check numerically: if $x < -R$, then $u = a + bx$, so $b = 0$ if and only if $u(x)$ is a constant when x changes in the region $x < -R$.

Condition (2.5) is difficult to check numerically: since computers have finite precision it is difficult (theoretically impossible) to check numerically if an integral is equal to zero exactly. Practically, it means that an arbitrarily small variation of q may lead to solutions unbounded at infinity.

3. Problem on Half-Axis

In scattering theory on half-axis $\mathbb{R}_{+} = [0, \infty)$, one has Equation (1.1) on $\mathbb{R}_{+}$, the solution satisfies the boundary condition

$$u = 0, \tag{3.1}$$

and the questions are:

(c) When does $u(\infty)$ exist?

(d) When is $u(\infty) < \infty$?

To answer these questions, let us write an integral equation

$$u(x) = bx + \int_{0}^{x} (x - s) qu ds, \quad b = \text{const}, \tag{3.2}$$

equivalent to differential equation (1.1) on $\mathbb{R}_+$ with the boundary condition (3.1). A necessary and sufficient condition for $u(\infty)$ to exist and be finite is

$$b + \int_0^R quds = 0. \tag{3.3}$$

If condition (3.3) holds, then

$$u(\infty) = -\int_0^R squds. \tag{3.4}$$

One can write the unique solution to (3.2) as

$$u = b\sum_{j=0}^{\infty} V^j x, \quad Vf := \int_0^x (x-s)\, q(s) f(s) ds, \tag{3.5}$$

and condition (3.3) as

$$1 + \sum_{j=0}^{\infty} \int_0^R q(s)\,(V^j s) ds = 0. \tag{3.6}$$

This is a necessary and sufficient condition for $u(\infty)$ to exist and be finite. To check if condition (3.3) holds exactly is numerically impossible. Therefore, this condition shows that an arbitrarily small perturbation of q may lead to a solution u such that $|u(\infty)| = \infty$.

The two references deal with the conditions under which zero is not an eigenvalue of a Schrödinger operator.

References

[1] A. G. Ramm, Sufficient conditions for zero not to be an eigenvalue of the Schrödinger operator, Journal of Mathematical Physics 28(6) (1987), 1341-1343.

[2] A. G. Ramm, Conditions for zero not to be an eigenvalue of the Schrödinger operator, II, Journal of Mathematical Physics 29(6) (1988), 1431-1432.

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