IJAMML 2:2 (2015) 101-123 June 2015
ISSN: 2394-2258
Available at http://scientificadvances.co.in

EXISTENCE OF POSITIVE SOLUTIONS FOR SOME NONLINEAR SEMIPOSITONE MULTI-POINT BOUNDARY VALUE SYSTEM WITH TWO PARAMETERS

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Abstract

In this paper, the fixed point theorem in cones is used to obtain the existence of multiple positive solutions for systems of nonlinear m-point semipositone boundary value problems:

$$\begin{cases} -(Lu)(t) = \lambda a(t)(f_1(u(t), v(t)) + g_1(u(t), v(t))), \\ -(Lv)(t) = \mu b(t)(f_2(u(t), v(t)) + g_2(u(t), v(t))), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ v(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} \beta_i v(\xi_i), \end{cases}$$

where $(Lu)(t) = (p(t)u'(t))' + q(t)u(t)$ and $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\alpha_i, \beta_i \in R^+$, f_i and g_i are both semipositone.

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The project is supported financially by the Foundations for Jining Medical College Natural Science (No. JYQ14KJ06), Outstanding Middle-Aged and Young Scientists of Shandong Province (BS2010SF004), and the National Natural Science Foundation of China (11371221, 11071141).

2010 Mathematics Subject Classification: 34B10, 34B15, 34B16.
Submitted by Jianqiang Gao.
Received May 15, 2015

Keywords: systems of differential equations, positive solutions, semipositone, cone.

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1. Introduction

In this paper, we consider the existence of positive solutions for the following m-boundary value system (SBVS for short):

$$\begin{cases} -(Lu)(t) = \lambda a(t)(f_1(u(t), v(t)) + g_1(u(t), v(t))), \\ -(Lv)(t) = \mu b(t)(f_2(u(t), v(t)) + g_2(u(t), v(t))), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ v(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} \beta_i v(\xi_i), \end{cases} \tag{1}$$

where λ and μ are positive parameters, $(Lu)(t) = (p(t)u'(t))' + q(t)u(t)$, $a(t), b(t) \in C[J', R^+]$, f_i and $g_i : J \times R^+ \times R^+ \to R (i = 1, 2)$ are continuous, $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\alpha_i, \beta_i \in R^+$, here $J = [0, 1]$, $J' = (0, 1)$, $R^+ = [0, +\infty)$, $a(t), b(t)$ may be singular at $t = 0, 1$. Since the nonlinearity f_i and g_i may change sign, the problem studied in this paper is so-called semipositone problem in the literature which arises naturally in chemical reactor theory. Up to now, much attention has been attached to the existence of positive solutions for semipositone differential equations and system of differential equations, see [1-11] and references therein to name a few. Recently, when nonlinearity is continuous, Su et al. [10] obtained the existence of positive solutions for the following second-order semipositone two-point boundary-value system:

$$\begin{cases} x''(t) + \lambda f_1(t, u(t), v(t)) + \mu g_1(t, u(t), v(t)) = 0, \\ y''(t) + \lambda f_2(t, u(t), v(t)) + \mu g_2(t, u(t), v(t)) = 0, \\ x(0) = x(1) = y(0) = y(1) = 0, \quad t \in J. \end{cases}$$

On the other hand, the multi-point boundary-value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. For the background in this area, we refer the reader to the references [12-14]. The study of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Ilin and Moiseev [14]. Since then, nonlinear multi-point boundary-value problems have been studied by many authors using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory, and fixed point theorem in cones. When f is nonnegative, Zhang and Sun [2] investigated the positive solution for the following m-point boundary value problem under some conditions concerning the first eigenvalue with respect to the relevant linear operators:

$$\begin{cases} -(L\varphi)(x) = h(x)f(\varphi(x)), \\ \varphi(0) = 0, \quad \varphi(1) = \sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i). \end{cases}$$

Motivated by [10] and [2], the purpose of this paper is to consider the existence of multiple positive solutions for SBVS (1) and the following SBVS:

$$\begin{cases} -(Lu)(t) = \lambda a(t) f(u(t), v(t)), \\ -(Lv)(t) = \mu b(t) g(u(t), v(t)), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ v(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} \beta_i v(\xi_i). \end{cases} \tag{2}$$

Compared with [10], since the nonlinearities in question allow to be singular at $t = 0, 1$, the construction of the cone is different from that in [10]. Furthermore, the operator discussed in this paper is the general Sturm-Liouville operator and the boundary value problem is the more complicated one.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and lemmas. The main results are formulated in Section 3 and an example is also given in Section 4.

2. Preliminaries and Several Lemmas

Let $E = C[0, 1]$, $E_+ = \{u \in E : u(t) \geq 0, t \in J\}$, $E \times E = C[0, 1] \times C[0, 1]$, then $(E \times E, \|(\cdot, \cdot))\|$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$, $\|x\| = \max_{t \in J} |x(t)|$. We also introduce the space $L^1(0, 1)$ with norm $\|x\|_1 = \int_0^1 |x(t)| dt$.

Throughout this paper, we always suppose the following (H_1) holds:

(H_1) $\quad p(t) \in C^1[0, 1], \quad p(t) > 0, \quad q(t) \in C[0, 1], \quad q(t) \leq 0.$

Then we have the following:

Lemma 1 ([2]). *Assume that* (H_1) *holds. Let* $\Phi_1(t)$, $\Phi_2(t)$ *be the unique solution of*

$$\begin{cases} (Lu)(t) = 0, & 0 < t < 1, \\ u(0) = 0, \ u(1) = 1, \end{cases} \tag{3}$$

and

$$\begin{cases} (Lu)(t) = 0, & 0 < t < 1, \\ u(0) = 1, \ u(1) = 0, \end{cases} \tag{4}$$

respectively. Then

(i) $\Phi_1(t)$ *is increasing on* $[0, 1]$, $\Phi_1(t) > 0$, $t \in (0, 1]$;

(ii) $\Phi_2(t)$ *is decreasing on* $[0, 1]$ *and* $\Phi_2(t) > 0$, $t \in [0, 1)$.

It's clear, $\Phi_1(t), \Phi_2(t) \in C([0, 1], [0, 1])$. Let

$$k(t, s) = \begin{cases} \frac{1}{\rho}\Phi_1(t)\Phi_2(s), & 0 \le t \le s \le 1, \\ \frac{1}{\rho}\Phi_1(s)\Phi_2(t), & 0 \le s \le t \le 1, \end{cases} \tag{5}$$

where $\rho = \Phi_1'(0)$ (we know from Lemma 1, $\Phi_1'(0) > 0$) and

$$K_1(t, s) = k(t, s) + D^{-1}\Phi_1(t)\sum_{i=1}^{m-2}\alpha_i k(\xi_i, s), \quad 0 \le t, s \le 1, \tag{6}$$

$$K_2(t, s) = k(t, s) + D^{-1}\Phi_1(t)\sum_{i=1}^{m-2}\beta_i k(\xi_i, s), \quad 0 \le t, s \le 1, \tag{7}$$

where $D = 1 - \sum_{i=1}^{m-2}\alpha_i\Phi_1(\xi_i)$. Then $K_i(t, s)$ is continuous on $[0, 1]\times[0, 1]$ and $K_i(t, s)(i = 0, 1)$ is nonnegative for any $t, s \in [0, 1]$. Obviously, for $e(t) = \frac{1}{\rho}\Phi_1(t)\Phi_2(t)$, by Lemma 1, (5)-(7) we have

$$K_1(t, s) \le \left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)e(s) \text{ or } \left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)e(t), \quad 0 \le t, s \le 1, \tag{8}$$

$$K_2(t, s) \le \left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)e(s) \text{ or } \left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)e(t), \quad 0 \le t, s \le 1. \tag{9}$$

By Lemma 1, we have

$$k(t_0, s)\Phi_1(t)\Phi_2(t) \le \frac{1}{\rho}\Phi_1(t)\Phi_2(t)\Phi_1(s)\Phi_2(s) \le k(t, s)$$

$$\le \frac{1}{\rho}\Phi_1(t)\Phi_2(t), \quad t_0 \in [0, 1]. \tag{10}$$

Lemma 2. *Suppose that* $\overline{\omega}_1 \in C([0, 1], [0, +\infty))$ *satisfy the following boundary value problem*:

$$\begin{cases} -(Lu)(t) = a(t), & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \end{cases}$$

then

$$\overline{\omega}_1 = \int_0^1 K_1(t, s)a(s)ds \le \left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\|a\|_1 e(t), \quad t \in J.$$

Proof. It is easy to be proven by (8), we omit the details.

Similarly, we have

Lemma 3. *Suppose that* $\overline{\omega}_2 \in C([0, 1], [0, +\infty))$ *satisfy the following boundary value problem*:

$$\begin{cases} -(Lu)(t) = b(t), \ 0 < t < 1, \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

then

$$\overline{\omega}_2 = \int_0^1 K_2(t, s)b(s)ds \le \left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\|b\|_1 e(t), \quad t \in J.$$

We make the following assumptions for system (1):

(H_2) $\quad \sum_{i=1}^{m-2}\alpha_i\Phi_1(\xi_i) < 1, \ \sum_{i=1}^{m-2}\beta_i\Phi_1(\xi_i) < 1;$

(S_1) $f_i,\ g_i \in C(R^+ \times R^+,\ R)\,(i = 0,\ 1),\ f_1(u,\ v) \geq 0,\ g_1(u,\ v) \geq 0,$ $\forall(u,\ v) \in J \times R^+;\ f_2(u,\ v) \geq 0,\ g_2(u,\ v) \geq 0,\ \forall(u,\ v) \in R^+ \times J;\ a(t),$ $b(t) \in C[J',\ R^+]$ with

$$0 < \int_0^1 e(s)a(s)ds < +\infty, \quad 0 < \int_0^1 e(s)b(s)ds < +\infty.$$

(S_2) For any $u \geq 0,\ v \geq 0,$ there exist $M_i > 0,\ (i = 1,\ 2)$ such that

$$f_i(u,\ v) \geq -M_1, \quad g_i(u,\ v) \geq -M_2, \quad i = 1,\ 2;$$

(S_3) $f_i^\infty = \infty,\ g_i^\infty = 0,$ where

$$f_i^\infty = \lim_{\substack{u+v\to+\infty \\ u>0,v>0}} \frac{f_i(u,\ v)}{u+v}, \quad g_i^\infty = \lim_{\substack{x+y\to+\infty \\ u>0,v>0}} \frac{g_i(u,\ v)}{u+v}, \quad i = 1,\ 2.$$

For SBVS (2), we adopt the following assumptions:

(S_1') $f,\ g \in C(R^+ \times R^+,\ R),\ f(u,\ v) \geq 0,\ \forall(u,\ v) \in J \times R^+;\ g(u,\ v) \geq 0,$ $\forall(u,\ v) \in R^+ \times J;\ a(t),\ b(t) \in C[J',\ R^+]$ with

$$0 < \int_0^1 e(s)a(s)ds < +\infty, \quad 0 < \int_0^1 e(s)b(s)ds < +\infty.$$

(S_2') For any $u \geq 0,\ v \geq 0,$ there exist $M_i > 0,\ (i = 1,\ 2)$ such that

$$f(u,\ v) \geq -M_1, \quad g(u,\ v) \geq -M_2.$$

(S_3') $f^\infty = \infty,\ g^\infty = 0,$ where

$$f^\infty = \lim_{\substack{x+y\to+\infty \\ u>0,v>0}} \frac{f(u,\ v)}{u+v}, \quad g^\infty = \lim_{\substack{x+y\to+\infty \\ u>0,v>0}} \frac{g(u,\ v)}{u+v}.$$

Remark. From (S_1) and (S_1'), we know there exist t_0 and t_1 such that $a(t_0), b(t_1) > 0$. We can pick out $\theta \in (0, \frac{1}{2})$ such that $t_0, t_1 \in [\theta, 1-\theta]$.

For notational convenience, we denote

$$F_i(u, v) = f_i(u, v) + M_1, \quad G_i(u, v) = g_i(u, v) + M_2, \quad i = 1, 2.$$

$$F = f(u, v) + M_1, \quad G = g(t, x, y) + M_2.$$

For any $L > 0$, denote

$$F_L = \max\left\{\sup_{\substack{u+v\le L\\ u>0, v>0}} |f_1(u, v)|, \sup_{\substack{u+y\le L\\ u>0, v>0}} |f_2(u, v)|\right\},$$

$$G_L = \max\left\{\sup_{\substack{u+v\le L\\ u>0, v>0}} |g_1(u, v)|, \sup_{\substack{u+v\le L\\ u>0, v>0}} |g_2(u, v)|\right\},$$

$$F_L' = \sup_{\substack{u+v\le L\\ u>0, v>0}} |f(u, v)|, \; G_L' = \sup_{\substack{u+v\le L\\ u>0, v>0}} |g(u, v)|.$$

Let

$$\overline{F}_i(u, v) = \begin{cases} F_i(u, v), & u \ge 0, v \ge 0, \\ F_i(u, 0), & u \ge 0, v \le 0, \\ F_i(0, v), & u < 0, v \ge 0, \\ F_i(0, 0), & u < 0, v < 0, \end{cases}$$

$$\overline{G}_i(u, v) = \begin{cases} G_i(u, v), & u \ge 0, v \ge 0, \\ G_i(u, 0), & u \ge 0, v \le 0, \\ G_i(0, v), & u < 0, v \ge 0, \\ G_i(0, 0), & u < 0, v < 0, \end{cases} \qquad i = 1, 2,$$

$$\omega_1(t) = \lambda(M_1 + M_2)\overline{\omega}_1, \quad \omega_2(t) = \mu(M_1 + M_2)\overline{\omega}_2.$$

Define an operator $T : E \times E \to E \times E$ as follows:

$$T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)), \tag{11}$$

where

$$T_1(u, v)(t) = \lambda\int_0^1 K_1(t, s)a(s)[\overline{F}_1(s, u - \omega_1, v - \omega_2) + \overline{G}_1(s, u - \omega_1, v - \omega_2)]ds, \tag{12}$$

$$T_2(u, v)(t) = \mu\int_0^1 K_2(t, s)b(s)[\overline{F}_2(s, u - \omega_1, v - \omega_2) + \overline{G}_2(s, u - \omega_1, v - \omega_2)]ds. \tag{13}$$

Lemma 4. *Let* (H_1) *and* (H_2) *be satisfied, then* (u, v) *is a positive solution for SBVS* (1) *if and only if* $(\tilde{u}, \tilde{v}) = (u + \omega_1, v + \omega_2)$ *is a fixed point of operator T, and* $\tilde{u}(t) > \omega_1(t)$, $\tilde{v}(t) > \omega_2(t)$, $t \in J$.

Let

$$K = \{u \in E_+ : u(t) \geq \|u\|, \forall t \in J\},$$

$$K_\theta = \{u \in E_+ : u(t) \geq M_\theta\|u\|, \forall t \in [\theta, 1 - \theta]\}. \tag{14}$$

Choose $M_\theta = \min_{t\in[\theta,1-\theta]} \Phi_1(t)\Phi_2(t)$. It is easy to verify that K_θ and $K_\theta \times K_\theta$ are cones in E and $E \times E$, respectively.

Lemma 5. *Let* (H_1) *and* (H_2) *be satisfied, then* $T : K_\theta \times K_\theta \to K_\theta \times K_\theta$ *is completely continuous.*

Proof. By (8) and (12), for any $(u, v) \in K \times K$ and $t \in J$, we have

$$T_1(u, v)(t) = \lambda\int_0^1 K_1(t, s)a(s)[\overline{F}_1(s, u - \omega_1, v - \omega_2)$$

$$+ \overline{G}_1(s, u - \omega_1, v - \omega_2)]ds$$

$$\leq \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 e(s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2)$$

$$+ \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds. \tag{15}$$

Hence

$$\|T_1(u, v)\| \leq \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 a(s)e(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2)$$

$$+ \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds. \tag{16}$$

Similarly, we get

$$\|T_2(u, v)\| \leq \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\int_0^1 b(s)e(s)[\overline{F}_2(s, u-\omega_1, v-\omega_2)$$

$$+ \overline{G}_2(s, u-\omega_1, v-\omega_2)]ds. \tag{17}$$

Pick out $t_0 \in [0, 1]$ such that $T_1(u, v)(t_0) = \|T_1(u, v)\|$. By (17) and (12), for any $t \in [\theta, 1-\theta]$, we have

$$T_1(u, v)(t)$$

$$= \lambda\int_0^1 K_1 a(s)(t, s)[\overline{F}_1(s, u-\omega_1, v-\omega_2) + \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds,$$

$$= \lambda\int_0^1 k(t, s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2) + \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds$$

$$+ \lambda D^{-1}\Phi_1(t)\sum_{i=1}^{m-2}\alpha_i\int_0^1 k(\xi_i, s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2)$$

$$+ \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds$$

$$\geq \Phi_1(t)\Phi_2(t)\lambda\int_0^1 k(t_0, s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2) + \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds$$

$$+ \lambda D^{-1}\Phi_1(t)\sum_{i=1}^{m-2}\alpha_i\int_0^1 k(\xi_i, s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2) + \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds$$

$$\geq \Phi_1(t)\Phi_2(t)\{\lambda\int_0^1 k(t_0, s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2) + \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds$$

$$+ \lambda D^{-1}\Phi_1(t_0)\sum_{i=1}^{m-2}\alpha_i\int_0^1 k(\xi_i, s)a(s)[\overline{F}_1(s, u-\omega_1, v-\omega_2) + \overline{G}_1(s, u-\omega_1, v-\omega_2)]ds\}$$

$$= \Phi_1(t)\Phi_2(t)T_1(u, v)(t_0) = \Phi_1(t)\Phi_2(t)\|T_1(u, v)\|$$

$$\geq M_\theta\|T_1(u, v)\|. \tag{18}$$

In the same way, we get

$$T_2(u, v)(t) \geq \Phi_1(t)\Phi_2(t)\|T_2(u, v)\|, \quad t \in [\theta, 1-\theta]. \tag{19}$$

It follows from (18) and (19) that $T : K_\theta \times K_\theta \to K_\theta \times K_\theta$. Ascoli-Arzela theorem and the Lebesgue dominated convergence theorem guarantee that $T : K_\theta \times K_\theta \to K_\theta \times K_\theta$ is completely continuous.

To prove the main results of this paper, we need the following well-known fixed point theorem:

Lemma 6 ([12]). *Suppose E is a real Banach space, $P \subset E$ is a cone, let $\Omega_1, \Omega_2, \Omega_3$ be three bounded open set in E such that $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$,*

$\overline{\Omega}_2 \subset \Omega_3$. *Let operator* $A : P \cap \Omega_3 \to P$ *be completely continuous. Suppose that*

$$\|Ax\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_1,$$

$$\|Ax\| \ge \|x\|, \quad Ax \ne x, \ \forall x \in P \cap \partial\Omega_2,$$

$$\|Ax\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_3.$$

Then A have two fixed points x^*, x^{**} *in* $P \cap \overline{\Omega}_3$ *and* $x^* \in P \cap (\Omega_2 \setminus \Omega_1)$, $x^{**} \in P \cap (\overline{\Omega}_3 \setminus \overline{\Omega}_2)$.

3. Main Results

The following two theorems are the main results:

Theorem 1. *Assume that conditions* (H_1), (H_2), (S_1)-(S_3) *are satisfied. Then for sufficiently small* λ *and* μ, *SBVS* (1) *has at least two positive solutions.*

For system (2), we have the following results:

Theorem 2. *Assume that conditions* (H_1), (H_2), (S'_1)-(S'_3) *are satisfied. Then for sufficiently small* λ *and* μ, *SBVS* (2) *has at least two positive solutions.*

Proof of Theorem 1. By Lemma 4, we need only to prove T has at least two fixed points $(\tilde{x}, \tilde{y})$, $(\tilde{x}', \tilde{y}')$ with $\tilde{x}(t), \tilde{x}'(t) \ge \omega_1(t)$, $\tilde{y}(t), \tilde{y}'(t) \ge \omega_2(t)$, $t \in (0, 1)$.

Firstly, by (S_3), we have

$$\lim_{\substack{u+v \to +\infty \\ u>0, v>0}} \frac{\overline{G}_i(u - \omega_1, v - \omega_2)}{u + v} = 0, \quad i = 1, 2.$$

Then, there exist $L_1 > 1$ and $\tilde{N} > 0$ such that

$$\overline{G}_i(u - \omega_1, v - \omega_2) \le \tilde{N}(u + v), \quad u + v \ge L_1, \quad i = 1, 2,$$

and for λ small enough, $\widetilde{N}$ satisfies

$$(F_R + M_1 + M_2\widetilde{N})\left[\lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right) + \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\right]$$

$$\times \int_0^1 e(s)(a(s) + b(s))ds < 1,$$

$$(F_1 + M_1 + G_1 + M_2)\left[\lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right) + \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\right]$$

$$\times \int_0^1 e(s)(a(s) + b(s))ds < 1, \quad (20)$$

where R is given next. Thus,

$$\overline{G}_i(u - \omega_1, v - \omega_2) \le \widetilde{N}(\|u\| + \|v\|), \quad u + v \ge L_1, \quad i = 1, 2. \quad (21)$$

Set $R = L_1 + 1$ and $\Omega_1 = \{(u, v) \in E \times E : \|(u, v)\| < R\}$. For any $(u, v) \in (K_\theta \times K_\theta) \cap \partial\Omega_1$ and $t \in [\theta, 1 - \theta]$, by (21), we have

$$T_1(u, v)(t) = \lambda\int_0^1 K_1(t, s)a(s)[\overline{F}_1(s, u - \omega_1, v - \omega_2) + \overline{G}_1(s, u - \omega_1, v - \omega_2)]ds$$

$$\le \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 e(s)a(s)[\overline{F}_1(s, u - \omega_1, v - \omega_2)$$

$$+ \overline{G}_1(s, u - \omega_1, v - \omega_2)]ds$$

$$\le \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 e(s)(a(s) + b(s))[(F_R + M_1 + M_2)$$

$$+ \widetilde{N}(\|u\| + \|v\|)]ds$$

$$\leq \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)[(F_R + M_1 + M_2) + \widetilde{N}]$$

$$\times \int_0^1 e(s)(a(s) + b(s))ds(\|u\| + \|v\|). \quad (22)$$

Similarly, we get

$$T_2(u, v)(t) \leq \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)[(F_R + M_1 + M_2) + \widetilde{N}]$$

$$\times \int_0^1 e(s)(a(s) + b(s))ds(\|u\| + \|v\|). \quad (23)$$

It follows from (22) and (23) that

$$\|T(u, v)\| \leq (F_R + M_1 + M_2 + \widetilde{N})\left[\lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right) + \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\right]$$

$$\times \int_0^1 e(s)ds(\|u\| + \|v\|).$$

Then, by (20), we have

$$\|T(u, v)\| \leq \|(x, y)\|, \quad \forall (u, v) \in (K_\theta \times K_\theta) \cap \partial\Omega_1. \quad (24)$$

Secondly, set $\Omega_2 = \{(u, v) \in E \times E : \|(u, v)\| < 1\}$. For any $(u, v) \in (K_\theta \times K_\theta) \cap \partial\Omega_2$ and $t \in [\theta, 1 - \theta]$, we have

$$T_1(u, v)(t) = \lambda\int_0^1 K_1(t, s)a(s)[\overline{F}_1(s, u - \omega_1, v - \omega_2) + \overline{G}_1(s, u - \omega_1, v - \omega_2)]ds$$

$$\leq \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 e(s)a(s)[\overline{F}_1(s, u - \omega_1, v - \omega_2)$$

$$+ \overline{G}_1(s, u - \omega_1, v - \omega_2)]ds$$

$$\leq \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 e(s)(a(s) + b(s))[(F_1 + M_1) + (G_1 + M_2)]ds$$

$$\leq \lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)(F_1 + M_1 + G_1 + M_2) \times \int_0^1 e(s)(a(s) + b(s))ds. \tag{25}$$

Similarly, we have

$$T_2(u, v)(t) \leq \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)(F_1 + M_1 + G_1 + M_2)\int_0^1 e(s)(a(s) + b(s))ds. \tag{26}$$

By (25), (26), and (20), we get

$$T(u,v)(t) \leq (F_1 + M_1 + G_1 + M_2)\left[\lambda\left(1 + D^{-1}\sum_{i=1}^{m-2}\alpha_i\right) + \mu\left(1 + D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\right] \times \int_0^1 e(s)(a(s) + b(s))ds.$$

Thus,

$$\|T(u, v)\| \leq \|(u, v)\|, \quad \forall (u, v) \in (K_\theta \times K_\theta) \cap \partial\Omega_2. \tag{27}$$

On the other hand, choose $\widetilde{M} > 0$ big enough such that for some $t_* \in [\theta, 1 - \theta]$

$$(\lambda + \mu)\widetilde{M}\min\left\{\int_\theta^{1-\theta} k(t^*, s)a(s)ds, \int_\theta^{1-\theta} k(t^*, s)b(s)ds\right\} \geq 1.$$

By (S_3), we get

$$\lim_{\substack{u+v\to+\infty \\ u>0, v>0}} \frac{\overline{F}_i(u - \omega_1, v - \omega_2)}{u + v} = \infty, \quad i = 1, 2.$$

Therefore, there exists $1 < L_2$ such that for any $u + v \geq L_2$ and $t \in J$

$$\overline{F}_i\,(\,u-\omega_1,\,v-\omega_2\,) \geq \widetilde{M}\,(\,\max_{t\in J} u\,(t) + \max_{t\in J} v\,(t)\,),\quad i=1,\,2. \tag{28}$$

Choose $\dfrac{L_2}{M_\theta} < r < R$ and $\Omega_3 = \{(u,\,v) \in E \times E : \|(u,\,v)\| < r\}$. For any $(u,\,v) \in (K_\theta \times K_\theta) \cap \partial\Omega_3$, we have by the construction of cone K_θ that $u(s) \geq M_\theta \|u\|$, $s \in [\theta,\,1-\theta]$. Therefore, $u(s) + v(s) \geq M_\theta(\|u\| + \|v\|) \geq L_2$. Thus, for $t \in [\theta,\,1-\theta]$, by the definition of $\overline{F}_i$ and (28), we have

$$\overline{F}_i\,(\,u-\omega_1,\,v-\omega_2\,) \geq \widetilde{M}\,(\|u\|+\|v\|),\quad u+v \geq L_2,\quad i=1,\,2. \tag{29}$$

So, for some $t^* \in [\theta,\,1-\theta]$ and $(u,\,v) \in (K \times K) \cap \partial\Omega_3$, by (12) and (13), we have

$$\begin{aligned} T_1(u,\,v)(t^*) &= \lambda\int_0^1 K_1(t^*,\,s)a(s)[\overline{F}_1(u-\omega_1,\,v-\omega_2) + \overline{G}_1(u-\omega_1,\,v-\omega_2)]ds \\ &\geq \lambda\int_0^1 k(t^*,\,s)a(s)\overline{F}_1(u-\omega_1,\,v-\omega_2)ds \\ &\geq \lambda\widetilde{M}\int_\theta^{1-\theta} k\,(\,t^*,\,s\,)\,a\,(s)\,(\|x\|+\|y\|)\,ds. \end{aligned} \tag{30}$$

Similarly, we get

$$T_2\,(u,\,v)\,(\,t^*\,) \geq \mu\widetilde{M}\int_\theta^{1-\theta} k\,(\,t^*,\,s\,)\,b\,(s)\,(\|x\|+\|y\|)\,ds.$$

As a consequence,

$$\begin{aligned} \|T\,(u,\,v)\,(t)\| \geq (\lambda+\mu)\,\widetilde{M}\,\min\left\{\int_\theta^{1-\theta} k\,(\,t^*,\,s\,)\,a\,(s)\,ds,\ \int_\theta^{1-\theta} k\,(\,t^*,\,s\,)\,b\,(s)\,ds\right\} \\ \times(\|x\|+\|y\|)\,ds \geq \|x\|+\|y\|. \end{aligned}$$

So, we have

$$\|T(u, v)\| \geq \|(u, v)\|, \quad \forall (u, v) \in (K_\theta \times K_\theta) \cap \partial\Omega_3. \tag{31}$$

It follows from (24), (27), (31), and Lemma 6 that T have two fixed points $(\tilde{u}, \tilde{v}), (\tilde{u}', \tilde{v}')$ on $K \times K$, and $1 < \|(\tilde{u}, \tilde{v})\| < r < \|(\tilde{u}', \tilde{v}')\| < R$. Finally, we are in position to show that $\tilde{u}(t) > \omega_1(t), \tilde{v}(t) > \omega_2(t), t \in (0, 1)$. For $\|(u, v)\| = (\|u\| + \|v\| > 1)$, we shall give the proof for three cases: (1) $\|u\| > 1$; (2) $\|v\| > 1$; (3) $\|u\| \leq 1, \|v\| \leq 1$ and $\|u\| + \|v\| > 1$.

Case 1. If $\|u\| > 1$, then for λ small enough, by Lemma 2, $u(t) \geq \|u\|\Phi_1(t)\Phi_2(t) > \Phi_1(t)\Phi_2(t) > \lambda(M_1 + M_2)\overline{\omega}_1(t) = \omega_1(t)$. Similarly, we have $\tilde{v}(t) > \omega_2(t), t \in (0, 1)$ when $\|\tilde{v}\| > 1$. On the other hand, if $\|v\| \leq 1$, set $J_1 = \{t \in [0, 1] : \tilde{v}(t) > \omega_2(t)\}, J_2 = \{t \in [0, 1] : \tilde{v}(t) \leq \omega_2(t)\}$. It is clear, $J_1 \cup J_2 = [0, 1]$. Since $(\tilde{u}, \tilde{v})$ is a solution of (11), we have

$$\begin{aligned}\tilde{v}(t) &= \mu\int_0 K_2(t, s)b(s)[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(s, \tilde{u} - \omega_1, \tilde{v} - \omega_2)]ds \\ &= \left(\int_{J_1} + \int_{J_2}\right)\mu K_2(t, s)b(s)[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) \\ &\qquad + \overline{G}_2(s, \tilde{u} - \omega_1, \tilde{v} - \omega_2)]ds.\end{aligned}$$

For $s \in J_1, \tilde{u}(s) - \omega_1(s) > 0, \tilde{v}(s) - \omega_2(s) > 0$, then by the definition of $\overline{F}_2$ and $\overline{G}_2$, we have

$$\begin{aligned}&\mu[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)] \\ &\quad = \mu[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)] \\ &\quad = \mu[(f_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + M_1) + (g_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + M_2)].\end{aligned}$$

Therefore,

$$\int_{J_1} K_2(t, s)b(s)\mu[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)]$$

$$= \int_{J_1} K_2(t, s)b(s)\mu[f_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + g_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)]$$

$$+ \mu(M_1 + M_2)\int_{J_1} K_2(t, s)b(s)ds.$$

For $s \in J_2$, $\tilde{u}(s) - \omega_1(s) > 0$, $\tilde{v}(s) - \omega_2(s) \leq 0$, then by the definition of $\overline{F}_2$ and $\overline{G}_2$, we have

$$\mu[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)]$$

$$= \mu[\overline{F}_2(\tilde{u} - \omega_1, 0) + G_2(\tilde{u} - \omega_1, 0)]$$

$$= \mu[(f_2(\tilde{u} - \omega_1, 0) + M_1) + (g_2(\tilde{u} - \omega_1, 0) + M_2)].$$

Thus,

$$\int_{J_2} K_2(t, s)b(s)\mu[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)]$$

$$= \int_{J_2} K_2(t, s)b(s)\mu[f_2(\tilde{u} - \omega_1, 0) + g_2(\tilde{u} - \omega_1, 0)]$$

$$+ \mu(M_1 + M_2)\int_{J_2} K_2(t, s)b(s)ds.$$

By (S_1), we have

$$f_2(u, v) \geq 0, \quad g_2(u, v) \geq 0, \quad \forall (u, v) \in R^+ \times J.$$

Then, we obtain

$$\tilde{v}(t) = \left(\int_{J_1} + \int_{J_2}\right)\mu K_2(t, s)b(s)[\overline{F}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2) + \overline{G}_2(\tilde{u} - \omega_1, \tilde{v} - \omega_2)]ds$$

$$\geq \mu(M_1 + M_2)\int_0^1 K_2(t, s)b(s)ds = \omega_2(t), \quad t \in (0, 1).$$

So, $\tilde{v}(t) > \omega_2(t)$, $t \in (0, 1)$.

Case 2. If $\|\tilde{v}\| > 1$. In a similar way to Case 1, we get $\tilde{u}(t) > \omega_1(t)$, $\tilde{v}(t) > \omega_2(t)$, $t \in (0, 1)$.

Case 3. If $\|\tilde{u}\| \leq 1$, $\|\tilde{v}\| \leq 1$ and $\|\tilde{u}\| + \|\tilde{v}\| > 1$. We set

$$J_1 = \{t \in [0, 1] : \tilde{u}(t) > \omega_1(t)\}, \quad J_2 = \{t \in [0, 1] : \tilde{u}(t) \leq \omega_1(t)\};$$

$$B_1 = \{t \in [0, 1] : \tilde{v}(t) > \omega_2(t)\}, \quad B_2 = \{t \in [0, 1] : \tilde{v}(t) \leq \omega_2(t)\}.$$

Similar to the proof of Case 1, we have $J_1 = (0, 1)$, $B_1 = (0, 1)$, i.e., $\tilde{u}(t) > \omega_1(t)$, $\tilde{v}(t) > \omega_2(t)$, $t \in (0, 1)$. Thus, we have proved that $(u, v) = (\tilde{u} - \omega_1, \tilde{v} - \omega_2)$ is a positive solution for the system (1).

Proof of Theorem 2. The proof is similar to that of Theorem 1. Define operator $T^* : E \times E \to E \times E$ as follows:

$$T(u, v)(t) = (T_1^*(u, v)(t), T_2^*(u, v)(t)),$$

where

$$T_1^*(u, v)(t) = \lambda \int_0^1 K_1(t, s) a(s) \overline{F}(u - \omega_1, v - \omega_2) ds,$$

$$T_2^*(u, v)(t) = \mu \int_0^1 K_2(t, s) b(s) \overline{G}(u - \omega_1, v - \omega_2) ds.$$

Firstly, by (S_3'), we have

$$\lim_{\substack{u+v \to +\infty \\ u>0, v>0}} \frac{\overline{G}(u - \omega_1, v - \omega_2)}{u + v} = 0.$$

Then, there exist $L_1^* > 1$ and $\tilde{N}^* > 0$ such that

$$\overline{G}(u - \omega_1, v - \omega_2) \leq \tilde{N}^*(u + v), \quad u + v \geq L_1^*,$$

and for λ, μ small enough, $\widetilde{N}^*$ satisfies

$$\left[\lambda(F'_{R^*}+M_1)\left(1+D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)+\mu\widetilde{N}\left(1+D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\right]\int_0^1 e(s)(a(s)+b(s))ds<1,$$

$$\left[\lambda(F_1+M_1)\left(1+D^{-1}\sum_{i=1}^{m-2}\alpha_i\right)+\mu(G'_1+M_2)\left(1+D^{-1}\sum_{i=1}^{m-2}\beta_i\right)\right]$$

$$\times\int_0^1 e(s)(a(s)+b(s))ds<1.$$

where R^* is given next. Thus,

$$\overline{G}_i(u-\omega_1,\, v-\omega_2)\le \widetilde{N}(\|u\|+\|v\|),\quad u+v\ge L_1,\quad i=1,\,2. \tag{32}$$

Set $R^*=L_1^*+1$ and $\Omega_1^*=\{(u,\,v)\in E\times E:\|(u,\,v)\|<R^*\}$. By the same discussion of Theorem 1, we get

$$\|T(u,\,v)\|\le\|(u,\,v)\|,\quad \forall(u,\,v)\in(K\times K)\cap\partial\Omega_1^*.$$

Obviously, (27) is also valid for Theorem 2. On the other hand, choose $\widetilde{M}^*>0$ big enough such that for some $t_*\in[\theta,\,1-\theta]$

$$(\lambda+\mu)\,\widetilde{M}\,\min\left\{\int_\theta^{1-\theta}k(t^*,\,s)\,a(s)\,ds,\ \int_\theta^{1-\theta}k(t^*,\,s)\,b(s)\,ds\right\}\ge 1.$$

By (S_3), we get

$$\lim_{\substack{u+v\to+\infty\\ u>0,v>0}}\frac{\overline{F}(u-\omega_1,\,v-\omega_2)}{u+v}=\infty,$$

$$\lim_{\substack{u+v\to+\infty\\ u>0,v>0}}\frac{\overline{G}(u-\omega_1,\,v-\omega_2)}{u+v}=\infty,$$

Therefore, there exists $1 < L_2^*$ such that for any $u + v \geq L_2^*$ and

$$\bar{F}(u-\omega_1, v-\omega_2) \geq \widetilde{M}(\max_{t\in J} u(t) + \max_{t\in J} v(t)),\ \bar{G}(u-\omega_1, v-\omega_2)$$

$$\geq \widetilde{M}(\max_{t\in J} u(t) + \max_{t\in J} v(t)).$$

Choose $\frac{L_2^*}{M_\theta} < r^* < R^*$ and $\Omega_3^* = \{(u, v) \in E \times E : \|(u, v)\| < r^*\}$. For any $(u, v) \in (K \times K) \cap \partial\Omega_3^*$, similar to the same method of Theorem 1, we have

$$\|T(u, v)\| \geq \|(u, v)\|, \quad \forall (u, v) \in (K \times K) \cap \partial\Omega_3^*.$$

Then Theorem 2 follows from Lemma 6.

4. An Example

Consider the singular semipositone three-point boundary-value problem system:

$$\begin{cases} -u''(t) = \lambda a(t)(f_1(u(t), v(t)) + g_1(u(t), v(t))), \\ -v''(t) = \mu b(t)(f_2(u(t), v(t)) + g_2(u(t), v(t))), \\ u(0) = 0, \quad u(1) = \frac{1}{2}u(\frac{1}{2}), \\ v(0) = 0, \quad v(1) = \frac{1}{4}u(\frac{1}{2}), \quad 0 < t < 1, \end{cases} \tag{33}$$

where λ and μ are positive parameters and

$$\alpha_1 = \frac{1}{2},\ \beta_1 = \frac{1}{4},\ \xi_1 = \frac{1}{2},\ a(t) = \frac{1}{\sqrt{t(1-t)}},\ b(t) = \frac{1}{3\sqrt{t}},\ \Phi_1(x) = x,\ \Phi_1'(0) = 1,$$

$$f_1(u, v) = \begin{cases} u^{\frac{5}{2}} + 3v^6, & (u, v) \in [0, 1] \times [0, +\infty), \\ u^3 + 3v^6 - 4(u-1)e^{-2u}, & (u, v) \in [1, +\infty) \times [0, +\infty), \end{cases}$$

$$f_2(u, v) = u^{\frac{4}{3}}v^5, \quad g_1(u, v) = \sqrt{u} + \cos^2(7v - 3),$$

$$g_2(u, v) = \begin{cases} e^{-u}\sqrt[3]{u} + 3e^{-2v}\sqrt[3]{v^2}, & (u, v) \in [0, +\infty) \times [0, 1], \\ e^{-u}\sqrt[3]{u} + 3e^{-2v}\sqrt[4]{v^2} - 2(v-1)e^{-v}, & (u, v) \in [0, +\infty) \times [1, +\infty). \end{cases}$$

It is easy to check that $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{S}_1), (\mathrm{S}_3)$ hold. For any $u \geq 0$, $v \geq 0$, we have

$$f_1(u, v) \geq -4, \quad f_2(u, v) \geq 0, \quad g_1(u, v) \geq 0, \quad g_2(u, v) \geq -2,$$

so, condition (S_2) also holds. Then Theorem 1 shows that for $\lambda > 0$ and $\mu > 0$ small enough, system (33) has at least two positive solutions.

References

[1] V. Anuradha, D. D. Hai and R. Shivaji, Existence results for superlinear semipositone BVP's, Pro. Amer. Math. Soc. 124 (1996), 747-763.

[2] G. Zhang and J. Sun, Existence of positive solutions for singular second-order m-point boundary value problems, Acta Math. Appl. Sinica 20(4) (2004), 655-664.

[3] J. Yang and Z. Wei, On existence of positive solutions of Sturm-Liouville boundary value problems for a nonlinear singular differential system, Appl. Math. Comput. 217 (2011), 6097-6104.

[4] X. Xu, D. Jiang and C. Yuan, Multiple positive solutions to singular positone and semipositone Dirichlet-type boundary value problems of nonlinear fractional differential equations, Nonlinear Anal. 74 (2011), 5685-5696.

[5] X. Zhang and L. Liu, Positive solutions of superlinear semipositone singular Dirichlet boundary value problems, J. Math. Anal. Appl. 316 (2006), 535-537.

[6] J. Jiang, L. Liu and Y. Wu, Positive solutions to singular fractional differential system with coupled boundary conditions, Commun. Nonlinear Sci. Numer. Simulat. 18 (2013), 3061-3074.

[7] R. Ma and Q. Ma, Positive solutions for semipositone m-point boundary-value problems, Acta Mathematica Sinica 20(2) (2004), 273-282.

[8] C. Yuan, Two positive solutions for $(n - 11, 1)$-type semipositone integral boundary value problems for coupled systems of nonlinear fractional differential equations, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012), 930-942.

[9] X. Zhang and L. Liu, Existence of positive solutions for a singular semipositone differential system, Math. Comput. Model. 47 (2008), 115-126.

[10] H. Su, L. Liu and Y. Wu, Positive solutions for a nonlinear second-order semipositone boundary value system, Nonlinear Anal. 71 (2009), 3240-3248.

[11] L. Liu, X. Zhang and Y. Wu, On existence of positive solutions of a two-point boundary value problem for a nonlinear singular semipositone system, Appl. Math. Comput. 192 (2007), 223-232.

[12] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.

[13] S. Timoshenko, Theory of Elastic Stability, McGraw-Hill, New York, 1961.

[14] V. A. Ilin and E. I. Moiseev, Nonlical boundary value problem of the second kind for a Sturm-Liouville operator, Differential Equations 23(8) (1987), 979-987.

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