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VECTOR REPRESENTATION OF QUATERNIONS SOLUTIONS OF QUATERNIONIC QUADRATIC EQUATIONS

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Abstract

The fundamental theorem of algebra over the quaternion skew field has gained little attention. It is of less importance than that over the complex number field, though some problems may reduce to it.

In this work, we use the vector representation of quaternions in order to obtain simplified forms of the solutions of the quaternionic equation $x^2 + bx + c = 0$ with b, c in $\mathcal{H}$. Similarly, we conclude that the following equation $x^2 + xab - bax = 0$ with a, b in $\mathcal{H}$ has the unique solution $x = 0$, although for some special cases we obtain infinite number of solutions.

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1. Introduction

In 1843, Hamilton discovered the quaternions. That discovery was the culmination of a series of 'mathematical acts' that started at least as early as 1831 when Hamilton was seriously seeking a basis for the real numbers in the 'intuition of pure time'. Hamilton's early results were published in his essay "On Algebra as the Science of Pure Time" on the foundation of the negative and the imaginary numbers. That work was followed by Hamilton's search for triplets, the three-dimensional analogon of the complex numbers. This search finally led to the discovery of the quatemions. Several authors have studied the genesis of the quatemions. Early authors like Whittaker and Van der Waerden' concentrated on the role of internal mathematical factors at the moment of the breakthrough. Crowe has also studied the breakthrough but emphasized the connection with Hamilton's earlier foundational work. More recently, Pycior and Hankins have written elaborate studies in which they concentrated on the role of philosophical and esthetical considerations in the development of Hamilton's mathematical work.

The family of quatemions plays a role in quantum physics [1, 2, 3]. It often appears in mathematics as an algebraic system-a skew field or non commutative division algebra [4]. While matrices over commutative rings have gained much attention [5], the literature on matrices with quatemion entries, though dating back to 1936 [6], is fragmentary.

By solving a real linear system, Zhang proposed to compute some roots of a quadratic polynomial in [7]. But, he does not discuss how to find all the roots. In [8], Porter reduced solving a quadratic polynomial to a linear polynomial of the form provided a root of the given quadratic polynomial is already known. However, he did not discuss how to find such root. In [9], given determined how many roots a quadratic polynomial can have, but he did not give the explicit formulas for computing the roots.

The set $\mathcal{H}$ of quaternions is the vector space $\mathcal{R}^4$ with componentwise addition and scalar multiplication. We denote the canonical basis elements by

$$1 = (1, 0, 0, 0), \quad i = (0, 1, 0, 0), \quad j = (0, 0, 1, 0), \quad k = (0, 0, 0, 1).$$

Thus, the general quaternion has the form

$$q = (\alpha, \beta, \gamma, \delta) = \alpha 1 + \beta i + \gamma j + \delta k \equiv \alpha + \beta i + \gamma j + \delta k, \quad \alpha, \beta, \gamma, \delta \in \mathcal{R}.$$

Hamilton [10] was the first who defined multiplication on $\mathcal{H}$ by using the relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

The set $\mathcal{H}$ of quaternions is a skew-field and an associative division algebra with unit. The general quaternion $q = (\alpha, \beta, \gamma, \delta) = \alpha + \beta i + \gamma j + \delta k$, can be represented as a pair

$$q = (\alpha, \underline{u}),$$

where $\alpha \in \mathcal{R}$ and $\underline{u} = (\beta, \gamma, \delta) \in \mathcal{R}^3$. This representation simplifies the formula of multiplication. In fact, by using vector products for the arbitrary quaternions $q_1 = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = (\alpha_0, \underline{u})$, $q_2 = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k = (\beta_0, \underline{v})$, we have

$$q_1 q_2 = (\alpha_0, \underline{u})(\beta_0, \underline{v}) = (\alpha_0\beta_0 - \underline{u} \cdot \underline{v}, \beta_0 \underline{u} + \alpha_0 \underline{v} + \underline{u} \times \underline{v}).$$

For $q_1 = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = (\alpha_0, \underline{u})$, we define its *conjugate* by $\overline{q_1} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k = (\alpha_0, -\underline{u})$, its *real part* by $Req_1 = \frac{1}{2}(q_1 + q_2) = \alpha_0$, its *imaginary part* by $Imq_1 = q_1 - Req_1 = \underline{u}$ and its *norm* or *modulus* by $\|q_1\| = \sqrt{q_1\overline{q_1}} = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. The *inverse* of q_1 exists when $\|q_1\| \neq 0$ and it is equal to $q^{-1} = \frac{1}{\|q_1\|^2}\overline{q_1}$. We say that $q_1, q_2 \in \mathcal{H}$ are *similar* if there exists an invertible $q \in \mathcal{H}$ such that $q_1 = q^{-1}q_2 q$ or equivalently, if $Req_1 = Req_2$ and $\|q_1\| = \|q_2\|$, [7].

We consider the monic quadratic polynomial equation:

$$x^2 + bx + c = 0, \tag{1.1}$$

where $b, c \in \mathcal{H}$. It is equivalent to a real system of four nonlinear equations.

Zhang and Mu [11] worked on quadratic equation (1.1) and tried to compute some roots by solving a real linear system. They did not give a complete solution of (1.1). Porter [8] reduced the problem of solving Equation (1.1) to the problem of solving a linear polynomial equation of the form $px + xq + r = 0$, provided that a root of (1.1) is given. Niven [9] determined the number of roots of Equation (1.1), but he did not give any formula for the computation of the roots. Finally, Huang and So [12] used an algebraic approach to solve the Equation (1.1) or equivalently the nonlinear system.

In this paper, we use the vector representation of quaternions in order to give simplified explicit formulas for the roots of the quadratic equation (1.1).

2. The Equation $x^2 + bx + c = 0, \quad b, c \in \mathcal{H}$

We use the transformation $x = X - \frac{b}{2}$ in order to obtain the equivalent equation

$$X^2 + \alpha X - X\alpha + \beta = 0, \tag{2.1}$$

where $\alpha = \frac{b}{2}$ and $\beta = c - \frac{b^2}{4} = \frac{4c - b^2}{4}$. Since, $\alpha = Re\alpha + Im\alpha$ and $(Re\alpha)X = X(Re\alpha)$ Equation (2.1) is equivalent to the equation

$$X^2 + \alpha X - X\alpha + \beta = 0, \tag{2.2}$$

where $\alpha = Im\frac{b}{2}$ with $Re\alpha = 0$ and $\beta = \frac{4c - b^2}{4}$.

Let $X = X_0 + \underline{w}$, where $X_0 \in \mathcal{R}$ and $\underline{w} = (X_1, X_2, X_3) = X_1 i + X_2 j + X_3 k$, $\alpha = \underline{u} = \left(\frac{b_1}{2}, \frac{b_2}{2}, \frac{b_3}{2}\right)$ and $\beta = \beta_0 + \underline{v}$. Then, we have $X^2 = X_0^2 - \underline{w}^2 + 2X_0\underline{w}$, $\alpha X = -\underline{u}\cdot\underline{w} + X_0\underline{u} + \underline{u}\times\underline{w}$, $X\alpha = -\underline{u}\cdot\underline{w} + X_0\underline{u} - \underline{u}\times\underline{w}$ and Equation (2.2) becomes

$$(X_0^2 - \underline{w}^2 + \beta_0) + 2X_0\underline{w} + 2(\underline{u}\times\underline{w}) + \underline{v} = 0. \tag{2.3}$$

Equivalently, we have the equations

$$X_0^2 - \underline{w}^2 + \beta_0 = 0, \tag{2.4}$$

$$2X_0\underline{w} + 2(\underline{u}\times\underline{w}) + \underline{v} = \underline{0}. \tag{2.5}$$

We now distinguish the following cases:

(1) $\underline{u}, \underline{v}$ are collinear, i.e., $\underline{u}\times\underline{v} = \underline{0} \Leftrightarrow \alpha\beta = \beta\alpha$.

In this case, we consider two different subcases.

(a) $\underline{v} = \underline{0}$ and $\underline{u}$ arbitrary.

Then the Equation (2.5) becomes $X_0\underline{w} + \underline{u}\times\underline{w} = \underline{0}$ and gives $X_0\underline{w}^2 + (\underline{u}\times\underline{w})\cdot\underline{w} = 0$ or $X_0\underline{w}^2 = 0 \Leftrightarrow X_0 = 0$ or $\underline{w} = \underline{0}$.

If $X_0 = 0$, then from Equation (2.4), we get

$$\underline{w}^2 = \beta_0 \Leftrightarrow \underline{w} = (X_1, X_2, X_3),$$

with $X_1^2 + X_2^2 + X_3^2 = \beta_0$, provided that $\beta_0 > 0$.

If $\underline{w} = \underline{0}$, then Equation (2.5) becomes

$$X_0^2 = -\beta_0 \Leftrightarrow X_0 = \pm\sqrt{-\beta_0}, \quad \beta_0 \leq 0.$$

(b) $\underline{u} = \lambda\underline{v}$, $\lambda \in \mathcal{R}$, $\underline{v} \neq \underline{0}$.

We observe that it is impossible to have a solution with $X_0 = 0$. In fact, if $X_0 = 0$, then the Equation (2.5) becomes

$$2\lambda(\underline{v} \times \underline{w}) + \underline{v} = \underline{0},$$

which gives $2\lambda(\underline{v} \times \underline{w}) \cdot \underline{v} + \underline{v}^2 = 0$ or $\underline{v} = \underline{0}$ (absurd).

Thus, we look for solutions $X = X_0 + \underline{w}$ with $X_0 \neq 0$. From Equation (2.5), we derive a pair of equations

$$2X_0\underline{w}^2 + \underline{v} \cdot \underline{w} = 0, \tag{2.6}$$

$$2X_0\underline{w} \cdot \underline{v} + \underline{v}^2 = 0, \tag{2.7}$$

from which, using Equation (2.4), we conclude that X_0 is a root of the biquadratic equation

$$4X_0^4 + \beta_0 X_0^2 - \underline{v}^2 = 0. \tag{2.8}$$

Hence, we have the solutions $X_0 = \pm\sqrt{\dfrac{-\beta_0 + \sqrt{\beta_0^2 + \underline{v}^2}}{2}}$.

Moreover, from Equation (2.5), we get

$$2X_0(\underline{w} \times \underline{v}) + 2\lambda[\underline{v}^2\underline{w} - (\underline{w} \cdot \underline{v})\underline{v}] = \underline{0},$$

from which by using equations $2X_0\underline{w} + 2\lambda(\underline{v} \times \underline{w}) + \underline{v} = \underline{0}$ and (2.7), in either case $\lambda \neq 0$ or $\lambda = 0$, we obtain $\underline{w} = -\dfrac{1}{2X_0}\underline{v}$.

Hence, we have two solutions

$$X = X_0 - \frac{1}{2X_0}\underline{v}, \quad \text{where} \quad X_0 = \pm\sqrt{\frac{-\beta_0 + \sqrt{\beta_0^2 + \underline{v}^2}}{2}}.$$

(2) $\underline{u}$, $\underline{v}$ are not collinear, i.e., $\underline{u} \times \underline{v} \neq \underline{0} \Leftrightarrow \alpha\beta \neq \beta\alpha$.

In this case, we consider two different subcases.

(a) If $X_0 = 0$, then the Equations (2.4) and (2.5) give the system

$$\underline{w}^2 = \beta_0, \tag{2.9}$$

$$\underline{v} \times \underline{w} = -\frac{1}{2}\underline{v}, \tag{2.10}$$

which is compatible when $\beta_0 \geq 0$ and $\underline{u} \cdot \underline{v} = 0$. Since $\underline{u}$, $\underline{w}$ and $\underline{u} \times \underline{v}$ are coplanar, $\underline{w}$ can be written as

$$\underline{w} = \lambda\underline{u} + \mu(\underline{u} \times \underline{v}), \quad \lambda, \mu \in \mathcal{R},$$

and it must satisfy Equations (2.9) and (2.10). Thus, from Equation (2.10), we find that

$$\underline{w} = \lambda\underline{u} + \frac{1}{2\underline{u}^2}(\underline{u} \times \underline{v}), \quad \lambda \in \mathcal{R},$$

and from Equation (2.9), we find that

$$\lambda = \pm\frac{\sqrt{4\beta_0\underline{u}^2 - \underline{v}^2}}{2\underline{u}^2}, \text{ provided that } 4\beta_0\underline{u}^2 - \underline{v}^2 \geq 0.$$

(b) If $X_0 \neq 0$, then the Equation (2.5), gives

$$2X_0(\underline{w} \cdot \underline{u}) + \underline{v} \cdot \underline{u} = 0 \Leftrightarrow \underline{w} \cdot \underline{u} = -\frac{\underline{v} \cdot \underline{u}}{2X_0}. \tag{2.11}$$

Moreover, from some equation, we get

$$2(X_0^2 + \underline{u}^2)\underline{w} = 2(\underline{w} \cdot \underline{u})\underline{u} - X_0\underline{u} - \underline{v} \times \underline{u}.$$

By using (2.11), we finally find

$$\underline{w} = -\frac{1}{2X_0(X_0^2 + \underline{u}^2)}[(\underline{v} \cdot \underline{u})\underline{u} + X_0^2\underline{v} + X_0(\underline{v} \times \underline{u})]. \tag{2.12}$$

It is to verify that $\underline{w}$ given by (2.12), is a solution of the Equation (2.5). Moreover, this is the unique solution of Equation (2.5). In fact, if we suppose that there exists a second solution $\underline{w}_1 \neq \underline{w}$, then from equations

$$2X_0\underline{w}_1 + 2(\underline{u} \times \underline{w}_1) + \underline{v} = \underline{0},$$

$$2X_0\underline{w} + 2(\underline{u} \times \underline{w}) + \underline{v} = \underline{0},$$

by substraction, we obtain

$$2X_0(\underline{w}_1 - \underline{w})^2 = 0 \text{ or } \underline{w}_1 = \underline{w}.$$

Moreover, from Equations (2.4) and (2.5), we obtain

$$\underline{v} \cdot \underline{w} = -2X_0 \underline{w}^2 = -2X_0(X_0^2 + \beta_0). \tag{2.13}$$

Also, from Equation (2.12), we obtain

$$\underline{v} \cdot \underline{w} = -\frac{(\underline{u} \cdot \underline{v})^2 + X_0^2 \underline{v}^2}{2X_0(X_0^2 + \underline{u}^2)}. \tag{2.14}$$

Thus, from (2.13) and (2.14), we find that X_0^2 must satisfy the equation

$$4X_0^6 + 4(\underline{u}^2 + \beta_0)X_0^4 + (4\beta_0 \underline{u}^2 - \underline{v}^2)X_0^2 - (\underline{v} \cdot \underline{u})^2 = 0. \tag{2.15}$$

Putting $X_0^2 = Y,$ we have that Y must be a positive root of the equation $f(Y) = 0,$ where

$$f(Y) = 4Y^3 + 4(\underline{u}^2 + \beta_0)Y^2 + (4\beta_0 \underline{u}^2 - \underline{v}^2)Y - (\underline{v} \cdot \underline{u})^2 = 0.$$

Since $f(0) = -(\underline{u} \cdot \underline{v})^2 < 0$ and $\lim_{Y \to +\infty} f(Y) = +\infty,$ we conclude that the polynomial $f(Y)$ has at least one positive root. Moreover, looking at the sequence of coefficience

$$-(\underline{u} \cdot \underline{v})^2 < 0, \quad 4\beta_0 \underline{u}^2 - \underline{v}^2,\ 4(\underline{u}^2 + \beta_0), \quad 4 > 0,$$

we observe that we can have only one change of sign of the coefficients. It is due to that the system

$$4\beta_0 \underline{u}^2 - \underline{v}^2 > 0, \quad 4(\underline{u}^2 + \beta_0) < 0,$$

is impossible.

Therefore, according to the Harriot-Descartes rule [13], the polynomial $f(Y)$ has exactly one positive root. Thus, we have proved the following:

Theorem 2.1. *Let*

$$X^2 + \alpha X - X\alpha + \beta = 0, \tag{2.16}$$

be a quadratic equation with $\alpha, \beta \in \mathcal{H}$ *and* $Re\alpha = 0$. *Let also* $X = X_0 + \underline{w}$, $\alpha = \underline{u}$ *and* $\beta = \beta_0 + \underline{v}$, *where* $X_0, \beta_0 \in \mathcal{R}$ *and* $\underline{w}, \underline{u}, \underline{v} \in \mathcal{R}^3$. *Then we distinguish the following cases*:

(1) *Let* $\underline{u}, \underline{v}$ *are collinear* (*i.e.*, $\underline{u} \times \underline{v} = \underline{0} \Leftrightarrow \alpha\beta = \beta\alpha$).

(a) *We have the subcases*:

(i) *If* $\underline{v} = \underline{0}$ *and* $\beta_0 < 0$, *then* $X = \pm\sqrt{-\beta_0}$ (*two solutions*).

(ii) *If* $\underline{v} = \underline{0}$ *and* $\beta_0 = 0$, *then* $X = 0$ (*one solution*).

(iii) *If* $\underline{v} = \underline{0}$ *and* $\beta_0 > 0$, *then* $X = \underline{w}$, *with* $\underline{w}^2 = \beta_0$ (*infinite solutions*).

(b) *If* $\underline{u} = \lambda\underline{v}$, $\lambda \in \mathcal{R}$ *and* $\underline{v} \neq \underline{0}$, *then* $X = X_0 - \frac{1}{2X_0}\underline{v}$, *where* X_0^2 *is the positive root of the equation*

$$4X_0^4 + 4\beta_0 X_0^2 - \underline{v}^2 = 0,$$

$$i.e., \; X_0 = \pm\sqrt{\frac{-\beta_0 + \sqrt{\beta_0^2 + \underline{v}^2}}{2}} \; (\textit{two solutions}).$$

(2) *Let* $\underline{u}, \underline{v}$ *be non collinear* (*i.e.*, $\underline{u} \times \underline{v} \neq \underline{0} \Leftrightarrow \alpha\beta \neq \beta\alpha$). *Then the Equation* (2.16) *has a unique solution* $X = X_0 + \underline{w}$, *with* $X_0 \neq 0$ *and* $\underline{w} = -\frac{1}{2X_0(X_0^2 + \underline{u}^2)}[(\underline{u} \cdot \underline{v})\underline{u} + X_0^2\underline{v} + X_0(\underline{v} \times \underline{u})]$, *where* X_0^2 *is the unique positive root of the polynomial equation*

$$4X_0^6 + 4(\underline{u}^2 + \beta_0)X_0^4 + (4\beta_0\underline{u}^2 - \underline{v}^2)X_0^2 - (\underline{u} \cdot \underline{v})^2 = 0.$$

Moreover, when $\underline{u} \cdot \underline{v} = 0$ *and* $4\beta_0\underline{u}^2 - \underline{v}^2 \geq 0$, *then the Equation* (2.16) *has one or two pure imaginary solutions*

$$X = \lambda \underline{u} + \frac{1}{2\underline{u}^2}(\underline{u} \times \underline{v}), \text{ where } \lambda = \pm \frac{\sqrt{4\beta_0 \underline{u}^2 - \underline{v}^2}}{2\underline{u}^2}.$$

3. The Equation $x^2 + xab - bax = 0, \quad a, b \in \mathcal{H}$

We consider the quaternionic equation

$$x^2 + xab - bax = 0, \quad a, b \in \mathcal{H}, \tag{3.1}$$

arising in the study of the spectrum of 2×2-quaternionic matrices [14]. We use the vector representation of quaternions in order to determine the explicit solutions of the Equation (3.1).

Since $Re(ab) = Re(ba)$ and $xRe(ab) = Re(ba)x$, without loss of generality, we suppose that $Re(ab) = Re(ba) = 0$. Let $a = a_0 + \underline{a}, b = b_0 + \underline{b}$, $x = x_0 + \underline{w} = x_0 + (x_1, x_2, x_3)$. Then Equation (3.1), is equivalent to the system

$$x_0^2 - \underline{w}^2 - 2\underline{w} \cdot (\underline{a} \times \underline{b}) = 0, \tag{3.2}$$

$$x_0^2 \underline{w} + x_0(\underline{a} \times \underline{b}) + \underline{w} \cdot (a_0 \underline{b} + b_0 \underline{a}) = \underline{0}. \tag{3.3}$$

We distinguish the following cases:

(1) Let $(a_0, b_0) \neq (0, 0)$. We have the subcases:

(a) If $x_0 = 0$, then from (3.3), we get that $\underline{w} = \lambda(a_0 \underline{b} + b_0 \underline{a}), \lambda \in \mathcal{R}$ and from Equation (3.2), we conclude that $\underline{w} = \underline{0}$.

(b) If $x_0 \neq 0$, then considering the scalar product of both parts of (3.3) succesively by $\underline{w}$ and $\underline{a} \times \underline{b}$, we derive the equations

$$\underline{w}^2 + \underline{w} \cdot (\underline{a} \times \underline{b}) = 0, \tag{3.4}$$

$$\underline{w} \cdot (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{b})^2 = 0, \tag{3.5}$$

from which by addition we find $(\underline{w} + \underline{a} \times \underline{b})^2 = 0$ or $\underline{w} = -\underline{a} \times \underline{b}$. It is easy to prove that $\underline{w} = -\underline{a} \times \underline{b}$ is the unique solution of (3.3). However, from (3.2), we have

$$x_0^2 - \underline{w}^2 - 2\underline{w} \cdot (\underline{a} \times \underline{b}) = 0 \text{ or } x^2 + (\underline{a} \times \underline{b})^2 = 0,$$

which is impossible for $x_0 \neq 0$.

Therefore, for $(a_0, b_0) \neq (0, 0)$, we have the unique solution $x = 0$.

(2) Let $(a_0, b_0) = (0, 0)$. Then we have two subcases:

(a) If $x_0 = 0$, then Equation (3.2) gives

$$\underline{w}^2 + 2\underline{w} \cdot (\underline{a} \times \underline{b}) = 0 \text{ or } (\underline{w} + \underline{a} \times \underline{b})^2 = (\underline{a} \times \underline{b})^2.$$

Thus, we have $\|\underline{w} + \underline{a} \times \underline{b}\| = \|\underline{a} \times \underline{b}\|$ and since $Re(\underline{w} + \underline{a} \times \underline{b}) = Re(\underline{a} \times \underline{b})$ we conclude that $\underline{w} + \underline{a} \times \underline{b}$ is similar to $\underline{a} \times b$, that is,

$$\underline{w} = -\underline{a} \times \underline{b} + q(\underline{a} \times \underline{b})q^{-1} = 0, \quad q \in \mathcal{H}^*. \tag{3.6}$$

Therefore, when $\underline{a} \times \underline{b} \neq \underline{0}$, Equation (3.1) has an infinity of solutions given by (3.6). When $\underline{a} \times \underline{b} = \underline{0}$, Equation (3.1) has the unique solution $x = 0$.

(b) If $x_0 \neq 0$, then from Equation (3.3), we find $\underline{w} = -\underline{a} \times \underline{b}$. However, from Equation (3.2), we have $x_0^2 + (\underline{a} \times \underline{b})^2 = 0$, which is impossible.

Thus, we have proved the following:

Theorem 3.1. *Let*

$$x^2 + xab - bax = 0, \tag{3.7}$$

where $x = x_0 + \underline{w}$, $a = a_0 + \underline{a}$, $b = b_0 + \underline{b}$, *with* $a_0 b_0 = \underline{a} \cdot \underline{b}$. *Then we have the following cases*:

(1) *When* $(a_0, b_0) \neq (0, 0)$, *or* $(a_0, b_0) = (0, 0)$ *and* $\underline{a} \times \underline{b} = \underline{0}$, *then Equation* (3.7) *has the unique solution* $x = 0$.

(2) *When* $(a_0, b_0) = (0, 0)$ *and* $\underline{a} \times \underline{b} \neq \underline{0}$, *then Equation* (3.7) *has an infinite numbers of solutions* $x = -\underline{a} \times \underline{b} + q(\underline{a} \times \underline{b})q^{-1}$, $q \in \mathcal{H}^*$.

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