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RELIABILITY FUNCTION OF LINEAR COMBINATION OF MARSHALL AND OLKIN'S BIVARIATE EXPONENTIAL DISTRIBUTION

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Abstract

In this paper, reliability function of $Z_1 = X + Y$ denoted by $\overline{F}_{Z_1}(z_1)$, is derived when (X, Y) follow Marshall and Olkin's bivariate exponential distribution (MOBVE) with dependence between X and Y first. Furthermore, $\overline{F}_Z(z)$, which is the reliability function of $Z = \alpha X + \beta Y$, is also obtained for the same condition (X, Y). Besides, the fact that $\overline{F}_{Z_1}(z_1)$ will become $\overline{F}_Z(z)$ when $\alpha = \beta = 1$ can be proved at last.

Keywords: reliability function, linear combination, bivariate exponential.

1. Introduction

Without a doubt, bivariate exponential distributions are one of the most applied distributions in the area of reliability. When there are two or more variables affecting the system, in most of the cases the analysis

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is carried out by assuming that they are statistically independent. However, the assumption of independence does not hold sometime in practice. Several bivariate models have been introduced in the literature. Some well know bivariate exponential distributions are those by Gumbel [1], Freund [2], Marshall and Olkin [3], Block and Basu [4], Downton [5] and so on. And these distributions attracted many practical applications in reliability problems. Besides, the distribution of $Z = \alpha X + \beta Y$, which is of interest in quality and reliability engineering, has been studied by several authors especially when X and Y are independent random variables. However, there is relatively little work of this kind when X and Y are dependent random variables. Gupta and Nadarajah [6] provides exact and approximate distributions for the combination of inverted Dirichlet components. Guo [7] gives the exact distributions of the linear combination of the bivariate exponential distributions, and Zhang [8] gives some revise to the results of Guo's.

This paper gives the reliability function of $Z_1 = X + Y$ and $Z = \alpha X + \beta Y$ according to the work above. Rest of the paper is organized as follows. In Section 2, the necessary pre-knowledge is prepared for the following work. In Section 3, the reliability function of $Z_1 = X + Y$ and $Z = \alpha X + \beta Y$ has been derived. Conclusion of the paper will be done in Section 4.

2. Pre-knowledge

The bivariate exponential distribution in this paper refers to Marshall and Olkin's bivariate exponential distribution $(\text{MOBVE}(\lambda_1, \lambda_2, \lambda_{12}))$ as follows.

Definition 2.1 ([9]). Marshall and Olkin's bivariate exponential distribution $(\text{MOBVE}(\lambda_1, \lambda_2, \lambda_{12}))$ has the joint pdf specified by

$$f(x, y) = \begin{cases} \lambda_1(\lambda_2 + \lambda_{12})\exp\{-\lambda_1 x - (\lambda_2 + \lambda_{12})y\}, & \text{if } x < y, \\ \lambda_2(\lambda_1 + \lambda_{12})\exp\{-\lambda_2 y - (\lambda_1 + \lambda_{12})x\}, & \text{if } x > y, \\ \lambda_{12}\exp\{-(\lambda_1 + \lambda_2 + \lambda_{12})y\}, & \text{if } x = y, \end{cases} \tag{2.1}$$

for $x > 0$, $y > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_{12} > 0$.

And the following theorem will be used in the Section 3.

Theorem 2.2. *If X and Y have the joint probability density function* (2.1), *pdf of* $Z = \alpha X + \beta Y$ *is*:

$$f_z(z) = \begin{cases} \frac{\lambda_1(\lambda_2+\lambda_{12})}{\alpha(\lambda_2+\lambda_{12})-\beta\lambda_1}\left[\exp\left(-\frac{\lambda_1+\lambda_2+\lambda_{12}}{\alpha+\beta}z\right)-\exp\left(-\frac{\lambda_2+\lambda_{12}}{\beta}z\right)\right]+ \\ \frac{\lambda_2(\lambda_1+\lambda_{12})}{\alpha\lambda_2-\beta(\lambda_1+\lambda_{12})}\left[\exp\left(-\frac{\lambda_1+\lambda_{12}}{\alpha}z\right)-\exp\left(-\frac{\lambda_1+\lambda_2+\lambda_{12}}{\alpha+\beta}z\right)\right], z>0, \\ 0, z \le 0 \end{cases} \tag{2.2}$$

where $\alpha > 0, \beta > 0$

3. Reliability Function of $Z_1 = X + Y$ and $Z = \alpha X + \beta Y$

Theorem 3.1 derives the reliability function $\overline{F}_{Z_1}(z_1)$ when X and Y are distributed according to (2.2).

Theorem 3.1. *If X and Y are jointly distributed according to* (2.2), *then*

$$\overline{F}_{Z_1}(z_1) = \frac{\tilde{\lambda}_2\lambda_1}{\tilde{\lambda}_2-\lambda_1}\left[\frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right)-\frac{1}{\tilde{\lambda}_2}\exp\left(-\tilde{\lambda}_2 z_1\right)\right]$$

$$+\frac{\lambda_2\tilde{\lambda}_1}{\lambda_2-\tilde{\lambda}_1}\left[\frac{1}{\tilde{\lambda}}\exp\left(-\tilde{\lambda}_1 z_1\right)-\frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right)\right], \tag{3.1}$$

for $0 < z_1 < \infty$, *where* $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}, \tilde{\lambda}_1 = \lambda_1 + \lambda_{12}$, *and* $\tilde{\lambda}_2 = \lambda_2 + \lambda_{12}$.

Proof. According to (2.2), we can readily see that $\overline{F}_{Z_1}(z_1) = \int_{z_1}^{\infty} f_{Z_1}(z_1)$ $dz_1 = A + B$,

$$A = \frac{\tilde{\lambda}_2\lambda_1}{\tilde{\lambda}_2-\lambda_1}\int_{z_1}^{\infty}\left[\exp\left(-\frac{\lambda}{2}z_1\right)-\exp\left(-\tilde{\lambda}_2 z_1\right)\right]dz_1$$

$$= \frac{\tilde{\lambda}_2\lambda_1}{\tilde{\lambda}_2-\lambda_1}\left[-\frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right)+\frac{1}{\tilde{\lambda}_2}\exp\left(-\tilde{\lambda}_2 z_1\right)\right]\Big|_{z_1}^{\infty}$$

$$= \frac{\tilde{\lambda}_2\lambda_1}{\tilde{\lambda}_2 - \lambda_1}\left[\frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right) - \frac{1}{\tilde{\lambda}_2}\exp\left(-\tilde{\lambda}_2 z_1\right)\right],$$

and

$$B = \frac{\lambda_2\tilde{\lambda}_1}{\lambda_2 - \tilde{\lambda}_1}\int_{z_1}^{\infty}\left[\exp(-\tilde{\lambda}_1 z_1) - \exp\left(-\frac{\lambda}{2}z_1\right)\right]dz_1,$$

$$= \frac{\lambda_2\tilde{\lambda}_1}{\lambda_2 - \tilde{\lambda}_1}\left[-\frac{1}{\tilde{\lambda}}\exp(-\tilde{\lambda}_1 z_1) + \frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right)\right]\Big|_{z_1}^{\infty}$$

$$= \frac{\lambda_2\tilde{\lambda}_1}{\lambda_2 - \tilde{\lambda}_1}\left[\frac{1}{\tilde{\lambda}}\exp(-\tilde{\lambda}_1 z_1) - \frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right)\right].$$

So

$$\overline{F}_{Z_1}(z_1) = A + B = \frac{\tilde{\lambda}_2\lambda_1}{\tilde{\lambda}_2 - \lambda_1}\left[\frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right) - \frac{1}{\tilde{\lambda}_2}\exp\left(-\tilde{\lambda}_2 z_1\right)\right]$$

$$+ \frac{\lambda_2\tilde{\lambda}_1}{\lambda_2 - \tilde{\lambda}_1}\left[\frac{1}{\tilde{\lambda}}\exp(-\tilde{\lambda}_1 z_1) - \frac{2}{\lambda}\exp\left(-\frac{\lambda}{2}z_1\right)\right].$$

In fact, reliability function $\overline{F}_{Z_1}(z_1)$ refers the reliability of cold standby system for two different components which can be found in Cheng's paper [10]. Corollary 3.2 shows that reliability of cold standby system with two different independent components is the special case of Theorem 3.1.

Corollary 3.2. *If* $\lambda_{12} = 0$, *that is*, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12} = \lambda_1 + \lambda_2$, $\tilde{\lambda}_1 = \lambda_1 + \lambda_{12} = \lambda_1$ *and* $\tilde{\lambda}_2 = \lambda_2 + \lambda_{12} = \lambda_2$, *then*

$$\overline{F}_{Z_1}(z_1) = \frac{\lambda_2}{\lambda_2 - \lambda_1}\exp(-\lambda_1 z_1) - \frac{\lambda_1}{\lambda_2 - \lambda_1}\exp(-\lambda_2 z_1). \tag{3.2}$$

Proof.

$$\overline{F}_{Z_1}(z_1) = \frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1}\left[\frac{2}{\lambda_1 + \lambda_2}\exp\left(-\frac{\lambda_1 + \lambda_2}{2}z_1\right) - \frac{1}{\lambda_2}\exp(-\lambda_2 z_1)\right]$$

$$+ \frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1}\left[\frac{1}{\lambda_1}\exp(-\lambda_1 z_1) - \frac{2}{\lambda_1 + \lambda_2}\exp\left(-\frac{\lambda_1 + \lambda_2}{2}z_1\right)\right]$$

$$= \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left[\frac{1}{\lambda_1} \exp(-\lambda_1 z_1) - \frac{1}{\lambda_2} \exp(-\lambda_2 z_1) \right]$$

$$= \frac{\lambda_2}{\lambda_2 - \lambda_1} \exp(-\lambda_1 z_1) - \frac{\lambda_1}{\lambda_2 - \lambda_1} \exp(-\lambda_2 z_1).$$

More generally, $\overline{F}_Z(z)$, which is the reliability function of $Z = \alpha X + \beta Y$, has been obtained for the same condition (X, Y) from the following Theorem 3.3:

Theorem 3.3. *If X and Y are jointly distributed according to* (2.2), *then*

$$\overline{F}_Z(z) = \frac{\tilde{\lambda}_2 \lambda_1}{\alpha \tilde{\lambda}_2 - \beta \lambda_1} \left[\frac{\alpha + \beta}{\lambda} \exp(-\frac{\lambda}{\alpha + \beta} z) - \frac{\beta}{\tilde{\lambda}_2} \exp\left(-\frac{\tilde{\lambda}_2}{\beta} z\right) \right]$$

$$+ \frac{\lambda_2 \tilde{\lambda}_1}{\alpha \lambda_2 - \beta \tilde{\lambda}_1} \left[\frac{\alpha}{\tilde{\lambda}_1} \exp(-\frac{\tilde{\lambda}_1}{\alpha} z) - \frac{\alpha + \beta}{\lambda} \exp\left(-\frac{\lambda}{\alpha + \beta} z)\right) \right]. \quad (3.3)$$

Proof.

$$\overline{F}_Z(z) = \int_z^\infty f_z(z) dz = A + B$$

$$A = \frac{\tilde{\lambda}_2 \lambda_1}{\alpha \tilde{\lambda}_2 - \beta \lambda_1} \int_z^\infty \left[\exp(-\frac{\lambda}{\alpha + \beta} z) - \exp\left(-\frac{\tilde{\lambda}_2}{\beta} z\right) \right] dz$$

$$= \frac{\tilde{\lambda}_2 \lambda_1}{\alpha \tilde{\lambda}_2 - \beta \lambda_1} \left[-\frac{\alpha + \beta}{\lambda} \exp(-\frac{\lambda}{\alpha + \beta} z) + \frac{\beta}{\tilde{\lambda}_2} \exp\left(-\frac{\tilde{\lambda}_2}{\beta} z\right) \right] \Big|_z^\infty$$

$$= \frac{\tilde{\lambda}_2 \lambda_1}{\alpha \tilde{\lambda}_2 - \beta \lambda_1} \left[\frac{\alpha + \beta}{\lambda} \exp(-\frac{\lambda}{\alpha + \beta} z) - \frac{\beta}{\tilde{\lambda}_2} \exp\left(-\frac{\tilde{\lambda}_2}{\beta} z\right) \right],$$

and

$$B = \frac{\lambda_2 \tilde{\lambda}_1}{\alpha \lambda_2 - \beta \tilde{\lambda}_1} \int_z^\infty \left[\exp(-\frac{\tilde{\lambda}_1}{\alpha} z) - \exp\left(-\frac{\lambda}{\alpha + \beta} z\right) \right] dz$$

$$= \frac{\lambda_2 \tilde{\lambda}_1}{\alpha \lambda_2 - \beta \tilde{\lambda}_1} \left[-\frac{\alpha}{\tilde{\lambda}_1} \exp(-\frac{\tilde{\lambda}_1}{\alpha} z) + \frac{\alpha + \beta}{\lambda} \exp\left(-\frac{\lambda}{\alpha + \beta} z)\right) \right] \Big|_z^\infty$$

$$= \frac{\lambda_2\tilde{\lambda}_1}{\alpha\lambda_2 - \beta\tilde{\lambda}_1}\left[\frac{\alpha}{\tilde{\lambda}_1}\exp\left(-\frac{\tilde{\lambda}_1}{\alpha}z\right) - \frac{\alpha+\beta}{\lambda}\exp\left(-\frac{\lambda}{\alpha+\beta}z\right)\right].$$

So

$$\overline{F}_Z(z) = A + B = \frac{\tilde{\lambda}_2\lambda_1}{\alpha\tilde{\lambda}_2 - \beta\lambda_1}\left[\frac{\alpha+\beta}{\lambda}\exp\left(-\frac{\lambda}{\alpha+\beta}z\right) - \frac{\beta}{\tilde{\lambda}_2}\exp\left(-\frac{\tilde{\lambda}_2}{\beta}z\right)\right]$$

$$+ \frac{\lambda_2\tilde{\lambda}_1}{\alpha\lambda_2 - \beta\tilde{\lambda}_1}\left[\frac{\alpha}{\tilde{\lambda}_1}\exp\left(-\frac{\tilde{\lambda}_1}{\alpha}z\right) - \frac{\alpha+\beta}{\lambda}\exp\left(-\frac{\lambda}{\alpha+\beta}z\right)\right].$$

Corollary 3.4. *When* $\alpha = \beta = 1$, *Theorem* 3.3 *becomes to Corollary* 3.2.

Proof. When $\alpha = \beta = 1$, the results is obviously true from (3.3) and (3.2).

4. Conclusion

Specific expressions of reliability function of $Z_1 = X + Y$ denoted by $\overline{F}_{Z_1}(z_1)$, is derived when (X, Y) follow Marshall and Olkin's bivariate exponential distribution (MOBVE) with dependence between X and Y. Besides, $\overline{F}_Z(z)$ is also obtained for the same condition (X, Y). Finally, $\overline{F}_{Z_1}(z_1)$ will become $\overline{F}_Z(z)$ when $\alpha = \beta = 1$ has been proved. Reliability of linear combination of other bivariate distributions will be studied in future.

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