HOM-LIE ALGEBRAS ON STRICTLY UPPER
TRIANGULAR MATRICES LIE ALGEBRA OVER A
COMMUTATIVE RING: DERIVATION CASE

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Abstract

Let \( n \) be the nilpotent Lie algebra consisting of all strictly upper triangular \( n \times n \) matrices over a commutative ring. In this paper, we characterize the decomposition of the derivations of \( n \) when \( (n, \phi) \) is a hom-Lie algebra. Meanwhile, the isomorphic classes are also determined.

1. Introduction and Preliminaries

The concept of hom-Lie algebra was first introduced by Hartwig et al. in their paper [7], which satisfies the skew symmetry and \( \sigma \)-twisted Jacobi identity, where \( \sigma \) is an algebra homomorphism. The motivations to study hom-Lie structures are related to physics and to deformations of Lie algebras, in particular Lie algebras of vector fields. The paradigmatic
examples are $q$-deformations of Witt and Virasoro algebras constructed in pioneering works (see [1], [2]-[4], [6]). Jin and Li [8] discussed the hom-Lie algebra structures on finite dimensional semisimple Lie algebras. They proved that the hom-Lie algebra structures are almost trivial and determined the isomorphic classes of non-trivial hom-Lie algebras. Firstly, we give some definitions which follows those of [10].

**Definition 1.** (1) A hom-Lie algebra is a triple $(L, [, ], \alpha)$ consisting of a vector space $L$, a skew-symmetric bilinear map (bracket) $[, ,]: L \times L \rightarrow L$ and a linear map $\alpha : L \rightarrow L$ satisfying the following hom-Jacobi identity:

$$[[\alpha(x), [y, z]] + [[\alpha(y), [z, x]] + [[\alpha(z), [x, y]]] = 0. \quad (1.1)$$

(2) A hom-Lie algebra is called a multiplicative hom-Lie algebra, if $\alpha$ is a Lie algebra endomorphism, i.e., for any $x, y \in L$, we have

$$\alpha([x, y]) = [[\alpha(x), \alpha(y)].$$

(3) A hom-Lie algebra is called a regular hom-Lie algebra, if $\alpha$ is a Lie algebra automorphism.

This paper mainly focuses on the case that the linear map $\alpha$ is a derivation, i.e., $\alpha([x, y]) = [[\alpha(x), y] + [x, \alpha(y)]$, for any $x, y \in L$. In the other words, for an arbitrary derivation $\alpha$, when $(\mathfrak{n}, \alpha)$ is a hom-Lie algebra, we get the decomposition of $\alpha$ and determine all the isomorphic classes of hom-Lie algebra $(\mathfrak{n}, \alpha)$.

We state some notation and list five types of automorphisms of $\mathfrak{n}$ as in [5] and three types of derivations as in [11], which will be used later.

Let $R$ be a commutative ring with identity, $R^*$ be the set of invertible elements of $R$, and $M_n(R)$ be the set of $n \times n$ matrices over $R$, where $n$ is a positive integer. Obviously, $M_n(R)$ is a Lie algebra under the Lie product defined as follows:

$$[A, B] = AB - BA, \quad \forall A, B \in M_n(R).$$
Denote by $\mathfrak{n}$ the $n \times n$ strictly upper triangular matrices over $R$, which is a nilpotent Lie subalgebra of $M_n(R)$. Define a lower central series of $\mathfrak{n}$

$$\mathfrak{n} = \mathfrak{n}_1, \mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}_1], \mathfrak{n}_3 = [\mathfrak{n}, \mathfrak{n}_2], \cdots. \quad (1.2)$$

Let $\text{Aut}(\mathfrak{n})$ (resp., $\text{Der}(\mathfrak{n})$) be denoted the set of all the automorphisms (resp., derivations) of $\mathfrak{n}$. Let $E$ be the identity matrix of $M_n(R)$ and $E_{ij}$ be the matrix whose sole nonzero entry is 1 in the $(i, j)$ position. One can check that $\{E_{ij} | 1 \leq i < j \leq n\}$ is a basis of $\mathfrak{n}$ as a vector space and $\{E_{i,i+1} | 1 \leq i \leq n-1\}$ is the generators of $\mathfrak{n}$ as a Lie algebra.

- Five types of automorphisms

  (1) Inner automorphisms

  For any $z \in \mathfrak{n}$, $x = E + z$ is invertible and the map $\sigma_x : y \mapsto xyx^{-1}$ is an automorphism of $\mathfrak{n}$, which is called the inner automorphism.

  (2) Diagonal automorphisms

  Let $d_i \in R^*$ and $d = \text{diag}\{d_1, d_2, \cdots, d_n\}$. Define a map $\eta_d : x \mapsto dxd^{-1}$ and it is easy to testify that this is an automorphism of $\mathfrak{n}$, which is called the diagonal automorphism.

  (3) Central automorphisms

  When $n \geq 4$, we define a linear map $\mu_c$ as follows:

  $$\mu_c(E_{i,i+1}) = E_{i,i+1} + c_i E_{in}, \quad 2 \leq i \leq n - 2; \quad \mu_c(E_{ij}) = E_{ij}, \text{ otherwise.}$$

  This definition is called proper central automorphism in [5].

  (4) Graph automorphisms

  For $x \in M_n(R)$, let $x^t$ denote the transpose of $x$. Set

  $$r = E_{1n} + E_{2,n-1} + \cdots + E_{n-1,2} + E_{n1}.$$ 

  It is clear that $r^2 = E$ and $r = r^t$. The map $\omega_0 : x \mapsto -rx^tr$ is an automorphism of $\mathfrak{n}$, which is called the graph automorphism.
(5) Extremal automorphisms

Assume that \( n \geq 3, b = (b_1, b_2) \in \mathbb{R}^2 \), define a linear map
\[
\zeta_b : E_{12} \mapsto E_{12} + b_1 E_{2n}, E_{n-1,n} \mapsto E_{n-1,n} + b_2 E_{1,n-1}, E_{ij} \mapsto E_{ij}, \text{ otherwise;}
\]
which is called extremal automorphism.

By [5], any automorphism \( \varphi \) of \( n \) can be uniquely expressed as
\[
\varphi = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma.
\] (1.3)

- Three types of derivations

(1) Inner derivations

For any \( x \in n \), define \( adx : y \mapsto [x, y] \), which is called the inner derivation of \( n \).

(2) Central derivations

For \( t = (t_2, t_3, \cdots, t_{n-2}) \in \mathbb{R}^{n-3} \), define linear map
\[
\lambda_t : E_{i,i+1} \mapsto t_i E_{1n}, \quad 2 \leq i \leq n-2; \quad E_{ij} \mapsto 0, \text{ otherwise;}
\]
which is called the central derivation.

(3) Extremal derivations

Define linear map \( \rho \) as follows:
\[
\begin{align*}
\rho(E_{12}) &= a_1 E_{2n}; \\
\rho(E_{n-1,n}) &= a_2 E_{1,n-1}; \\
\rho(E_{ij}) &= 0, \text{ otherwise.}
\end{align*}
\] (1.4)

As it is checked in [11], \( \rho \) is a derivation of \( n \) and called the extremal derivation.

**Remark 1.** The notions of diagonal derivation and central derivation are different with those in [11] in order not to make confusion with diagonal automorphism and central automorphism.
2. Hom-Lie Algebra Structures of \( n \) and Isomorphic Classes

In this section, we firstly show that any derivation \( \phi \) can be written as the sum of three inner derivations, the central derivation, and the extremal derivation when \( (n, \phi) \) is a hom-Lie algebra. In addition, all the isomorphic classes of \( (n, \phi) \) are found.

**Lemma 1.** Let \( (n, \phi) \) be a hom-Lie algebra over a commutative ring \( R \) with \( 2 \in R^* \), then \( \phi \) can be written as

\[
\phi = adw + \nu + \rho,
\]

where \( w \in n_{n-3} \) and \( \nu, \rho \) are central, extremal derivation, respectively.

**Remark 2.** By the result in [11], any derivation \( \phi \) can be expressed as \( \phi = adw + \chi_d + \lambda_c + \rho \), where \( adw, \chi_d, \lambda_c, \rho \) are called inner, diagonal, central, extremal derivation, respectively. When \( 2 \) is in \( R^* \) the extremal derivation \( \rho \) is slight different with that in [11].

**Proof.** (1) If \( \phi = adw, w = \sum_{i<j} a_{ij} E_{ij} \). In (1.1), by taking

\[
x = E_{i,i+1}, \quad y = E_{i+1,i+2}, \quad z = E_{i+2,i+3}, \quad 1 \leq i \leq n - 3,
\]

we can figure out that all \( a_{ij} = 0 \) but \( a_{1,n-2}, a_{1,n-1}, a_{1n}, a_{2,n-1}, a_{2n}, a_{3,n-1} \) and \( a_{3n} \). That

\[
a_{3,n-1} = 0,
\]

follows from

\[
[adw(E_{12}), [E_{23}, E_{n-1,n}]] + [adw(E_{23}), [E_{n-1,n}, E_{12}]]
\]

\[
+ [adw(E_{n-1,n}), [E_{12}, E_{23}]] = 0.
\]

So if \( (n, adw) \) is a hom-Lie algebra, \( w \in n_{n-3} \).
(2) If \( \phi = \chi_d \), where \( d = (d_1, d_2, \cdots, d_n) \). In (1.1), by taking

\[
x = E_{ij}, \quad y = E_{jk}, \quad z = E_{kp}, \quad 1 \leq i < j < k < p \leq n,
\]

it is not difficult to check that \( d_1 = d_2 = \cdots = d_n \). In view of the construction of diagonal derivation in [11], we can get \( d_i = 0, 1 \leq i \leq n \).

(3) If \( \phi = \lambda_c \) or \( \rho \), it satisfies (1.1).

So \( \phi = adw + \lambda_c + \rho \), which completes the lemma. \( \square \)

**Definition 2.** Let \( (L_1, \sigma) \) and \( (L_2, \tau) \) be both hom-Lie algebras, \( \varphi : L_1 \to L_2 \) an algebra isomorphism, we call \( (L_1, \sigma) \) isomorphic to \( (L_2, \tau) \), if \( \varphi \circ \sigma = \tau \circ \varphi \). In the other words, the following diagram commutes:

\[
\begin{array}{ccc}
L_1 & \xrightarrow{\varphi} & L_1 \\
\downarrow{\sigma} & & \uparrow{\tau} \\
L_2 & \xrightarrow{\varphi} & L_2
\end{array}
\]

Suppose \( \sigma, \tau \in \text{Der}(n) \), if \( (n, \sigma) \) and \( (n, \tau) \) are isomorphic as hom-Lie algebra, then there exists a Lie algebra automorphism \( \varphi \) such that \( \varphi \circ \sigma = \tau \circ \varphi \). In the other words, \( \varphi \circ \sigma \circ \varphi^{-1} = \tau \).

**Theorem 1.** Let \( n \) be defined an above, then two hom-Lie algebra structures \( (n, \sigma) \) and \( (n, \tau) \) are isomorphic if and only if the two Lie derivations \( \sigma \) and \( \tau \) are conjugate.

**Lemma 2.** \( \text{ad}(RE_{1n}) \) is an isomorphic class.

**Proof.** Notice that \( E_{1n} \) is in the center of \( n \), \([E_{1n}, E_{ij}] = 0\), we therefore have, for any \( \varphi \in \text{Aut}(n) \),

\[
\varphi \text{ad}E_{1n}\varphi^{-1} = \text{ad}E_{1n}.
\]

\( \square \)
Lemma 3. $ad(RE_{1,n-1} + RE_{2n})$ is an isomorphic class.

Proof. We divide it into five steps to prove this lemma.

**Case 1.** Suppose $\phi$ is an inner automorphism $\sigma_x$.

\[
\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n}) \sigma_x^{-1}(E_{12}) = -a_2E_{1n} = ad(a_1E_{1,n-1} + a_2E_{2n})(E_{12});
\]

\[
\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n}) \sigma_x^{-1}(E_{n-1,n}) = a_iE_{1n} = ad(a_1E_{1,n-1} + a_2E_{2n})(E_{n-1,n});
\]

\[
\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n}) \sigma_x^{-1}(E_{i,i+1}) = 0 = ad(a_1E_{1,n-1} + a_2E_{2n})(E_{i,i+1});
\]

$(2 \leq i \leq n - 2)$.

So $\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n}) \sigma_x^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

**Case 2.** Suppose $\phi$ is a diagonal automorphism $\eta_d$.

\[
\eta_d ad(a_1E_{1,n-1} + a_2E_{2n}) \eta_d^{-1}(E_{12}) = ad(a_1d_1d_{n-1}^{-1}E_{1,n-1} - a_2d_2d_{n-1}^{-1}E_{2n})(E_{12});
\]

\[
\eta_d ad(a_1E_{1,n-1} + a_2E_{2n}) \eta_d^{-1}(E_{n-1,n}) = ad(a_1d_1d_{n-1}^{-1}E_{1,n-1} - a_2d_2d_{n-1}^{-1}E_{2n})(E_{n-1,n});
\]

\[
\eta_d ad(a_1E_{1,n-1} + a_2E_{2n}) \eta_d^{-1}(E_{i,i+1}) = 0 = ad(a_1d_1d_{n-1}^{-1}E_{1,n-1} - a_2d_2d_{n-1}^{-1}E_{2n})
\]

\[
\times (E_{i,i+1}); (2 \leq i \leq n - 2).
\]

So $\eta_d ad(a_1E_{1,n-1} + a_2E_{2n}) \eta_d^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

**Case 3.** Suppose $\phi$ is a central automorphism $\mu_c$.

\[
\mu_c ad(a_1E_{1,n-1} + a_2E_{2n}) \mu_c^{-1}(E_{12}) = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{12});
\]

\[
\mu_c ad(a_1E_{1,n-1} + a_2E_{2n}) \mu_c^{-1}(E_{n-1,n}) = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{n-1,n});
\]

\[
\mu_c ad(a_1E_{1,n-1} + a_2E_{2n}) \mu_c^{-1}(E_{i,i+1}) = 0 = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{i,i+1});
\]

$(2 \leq i \leq n - 2)$.

So $\mu_c ad(a_1E_{1,n-1} + a_2E_{2n}) \mu_c^{-1} \in ad(RE_{1,n-1} + RE_{2n})$. 


**Case 4.** Suppose \( \varphi \) is a graph automorphism \( \omega_0 \).

\[
\omega_0 \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \omega_0^{-1}(E_{12}) = \text{ad}(a_2E_{1,n-1} - a_1E_{2n})(E_{12});
\]

\[
\omega_0 \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \omega_0^{-1}(E_{n-1,n}) = \text{ad}(a_2E_{1,n-1} - a_1E_{2n})(E_{n-1,n});
\]

\[
\omega_0 \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \omega_0^{-1}(E_{i,i+1}) = 0 = \text{ad}(a_2E_{1,n-1} - a_1E_{2n})(E_{i,i+1});
\]

\( (2 \leq i \leq n-2). \)

So \( \omega_0 \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \omega_0^{-1} \in \text{ad}(RE_{1,n-1} + RE_{2n}). \)

**Case 5.** Suppose \( \varphi \) is an extremal automorphism \( \xi_b \).

\[
\xi_b \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \xi_b^{-1}(E_{12}) = \text{ad}(a_1E_{1,n-1} - a_2E_{2n})(E_{12});
\]

\[
\xi_b \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \xi_b^{-1}(E_{n-1,n}) = \text{ad}(a_1E_{1,n-1} - a_2E_{2n})(E_{n-1,n});
\]

\[
\xi_b \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \xi_b^{-1}(E_{i,i+1}) = 0 = \text{ad}(a_1E_{1,n-1} - a_2E_{2n})(E_{i,i+1});
\]

\( (2 \leq i \leq n-2). \)

So \( \xi_b \text{ad}(a_1E_{1,n-1} + a_2E_{2n}) \xi_b^{-1} \in \text{ad}(RE_{1,n-1} + RE_{2n}). \)

We therefore can conclude that for any \( \varphi \in \text{Aut}(n), \)

\[
\varphi \text{ad}(RE_{1,n-1} + RE_{2n}) \varphi^{-1} \in \text{ad}(RE_{1,n-1} + RE_{2n}),
\]

which completes the lemma.

\[\square\]

**Lemma 4.** \( \text{ad}(RE_{1,n-2} + RE_{2,n-1} + RE_{3n}) \) is an isomorphic class.

**Proof.** This process is similar to that of Lemma 3. \[\square\]

**Lemma 5.** \( \lambda_t \) is an isomorphic class.
**Proof.** On one hand, for any $\varphi \in Aut(n)$ and $2 \leq i \leq n - 2,$

$$\varphi \lambda_t \varphi^{-1}(E_{i,i+1}) = \varphi \lambda_t(E_{i,i+1}) \mod n_2 = \varphi(t_i E_{i,n}) = \lambda_t(E_{i,i+1}).$$

On the other hand, $\lambda_t$ vanishes on other generators of $n$ and then

$$\varphi \lambda_t \varphi^{-1} = \lambda_t.$$

\[ \square \]

**Lemma 6.** $\rho$ is in the isomorphic class of $\rho + ad(RE_{1,n-1} + RE_{2n}).$

**Proof.** (i) For any inner automorphism $\sigma_x$, $x = E + \sum a_{ij} E_{ij}$, assume

$$x^{-1} = E + \sum_{i<j} b_{ij} E_{ij},$$

one can figure out that

$$\sigma_x \rho \sigma_x^{-1}(E_{12}) = (\rho + ad(a_2 b_{n-1,n} E_{1,n-1} - a_1 a_{12} E_{2n}))(E_{12});$$

$$\sigma_x \rho \sigma_x^{-1}(E_{n-1,n}) = (\rho + ad(a_2 b_{n-1,n} E_{1,n-1} - a_1 a_{12} E_{2n}))(E_{n-1,n}).$$

Meanwhile, both of them vanish on the other elements of $n$, so

$$\sigma_x \rho \sigma_x^{-1} = \rho + ad(a_2 b_{n-1,n} E_{1,n-1} - a_1 a_{12} E_{2n}).$$

(ii) For any diagonal automorphism $\eta_d$, it acts on $E_{ij}$ as a constant, so $\eta_d \rho \eta_d^{-1} = \rho'$ for some extremal derivation $\rho'$.

(iii) For any central automorphism $\mu_c$, it acts on $E_{12}$ and $E_{n-1,n}$ as identity and $\rho$ vanishes at other elements, which follows that

$$\mu_c \rho \mu_c^{-1} = \rho.$$

(iv) For any graph automorphism $\omega_0$, define an extremal derivation

$$\hat{\rho} : E_{12} \mapsto a_2 E_{2n}; E_{1,n-1} \mapsto a_1 E_{1,n-1}; E_{ij} \mapsto 0,$

otherwise.
It is easy to show that $\omega^{-1}_0 \rho \omega_0 = \hat{\rho}$.

(v) For any extremal automorphism $\xi_b$, it is not difficult to verify that $\xi_b \rho \xi_b^{-1} = \rho$. Hence, for any $\phi \in Aut(n)$, $\phi \rho \phi^{-1} \in \rho + ad(RE_{1,n-1} + RE_{2n})$.

\begin{proof}
\end{proof}

**Theorem 2.** Suppose that $R$ is a commutative ring such that $2 \in R^*$, $\phi$ is a derivation of $n$ and $(n, \phi)$ a hom-Lie algebra. Set $x = s_{1}E_{1,n}$, $y = s_{2}E_{1,n-1} + s_{3}E_{2n}$, $z = s_{4}E_{1,n-2} + s_{5}E_{2,n-1} + s_{6}E_{3n}$, $s_{i} \in R$.

If $n \geq 4$, each hom-Lie algebra is isomorphic to one of the following 24 types:

\begin{equation}
\{(n, adx + ady + adz + \lambda_t + \rho)\};
\end{equation}

\begin{equation}
\{(n, adx + ady + adz + \lambda_t), (n, adx + ady + adz + \rho), (n, adx + ady + \lambda_t + \rho), (n, ady + adz + \lambda_t + \rho)\};
\end{equation}

\begin{equation}
\{(n, adx + ady + adz), (n, adx + ady + \lambda_t), (n, adx + ady + \rho), (n, adx + adz + \lambda_t), (n, ady + adz + \lambda_t), (n, ady + adz + \rho),
\end{equation}

\begin{equation}
(n, ady + \lambda_t + \rho)\};
\end{equation}

\begin{equation}
\{(n, adx + ady), (n, adx + adz), (n, adx + \lambda_t), (n, ady + adz), (n, ady + \lambda t), (n, ady + \rho), (n, adx + \lambda t), (n, ady + \lambda t)\};
\end{equation}

\begin{equation}
\{(n, adx), (n, ady), (n, adz), (n, \lambda_t)\};
\end{equation}

\begin{equation}
\{(n, 0)\}.
\end{equation}
Proof: Claim 1. Suppose $D_1$ is isomorphic to $D_2$, $D_3$ is isomorphic to itself, then $D_1 + D_3$ is isomorphic to $D_2 + D_3$.

If there is an automorphism $\varphi$ of $n$ such that $\varphi D_1 \varphi^{-1} = D_2$ and $\varphi D_3 \varphi^{-1} = D_3$, then

$$\varphi (D_1 + D_3) \varphi^{-1} = D_2 + D_3.$$ 

Claim 2. Suppose $D_i$ is isomorphic to $D_i$, $i = 1, 2$ and $D_1$ is not isomorphic to $D_2$, then $D_1 + D_2$ is isomorphic to itself.

It follows from

$$\varphi (D_1 + D_2) \varphi^{-1} = \varphi D_1 \varphi^{-1} + \varphi D_2 \varphi^{-1} = D_1 + D_2.$$ 

In view of Claim 1 and Claim 2 and Lemma 6, we can conclude that there are 24 isomorphic classes of hom-Lie algebra $(n, \varphi)$ stated as above.

\[ \Box \]

References


