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SOME PARALLEL RESULTS FOR THE HIRST SETS OF REGULAR AND GENERALIZED CONTINUED FRACTION EXPANSIONS

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Abstract

Given any infinite set B of positive integers $b_1 < b_2 < \cdots$, let $\tau(B) = \inf \{\tau > 0 : \sum_{n=1}^{\infty} \frac{1}{b_n^{\tau}} < \infty\}$. For any $x \in (0, 1]$, let $a_n(x)$ and $g_n(x)$ be the n-th partial quotients of regular and generalized continued fraction expansion of x, respectively. Define

$$E(B) = \{x \in (0, 1) : a_n(x) \in B \,\forall\, n \geq 1, \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\},$$

$$G(B) = \{x \in (0, 1) : k_n(x) \in B \,\forall\, n \geq 1, \text{ and } k_n(x) \to \infty \text{ as } n \to \infty\},$$

Hirst [4] proved the inequality $\dim_H E(B) \leq \frac{\tau(B)}{2}$ for any infinite set B and the equality $\dim_H E(B) = \frac{\tau(B)}{2}$ for $B = \{1, 2^t, 3^t, \cdots\}$, $t \in \mathbb{N}$, where $\dim_H$ denotes the Hausdorff dimension. In this note, we show that $G(B)$ is twice the size of $E(B)$ for all the above cases.

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1. Introduction

In 2003, Schweiger [7] introduced a whole class of continued fraction algorithm with increasing digits, called generalized continued fractions (GCF), which is induced by the map $T_\epsilon : (0, 1] \to (0, 1]$

$$T_\epsilon(x) := \frac{-1 + (k+1)x}{1 + \epsilon - k\epsilon x}, \quad x \in I(k) := \left(\frac{1}{k+1}, \frac{1}{k}\right], \tag{1.1}$$

where the parameter $\epsilon : \mathbf{N} \to \mathbf{R}$ satisfies

$$\epsilon(k) + k + 1 > 0. \tag{1.2}$$

For any $x \in (0, 1]$, define its partial quotients $k_1, k_2, \cdots, k_n$ in the GCF_ϵ expansion as

$$k_1 = k_1(x) := \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{and} \quad k_n = k_n(x) := k_1(T_\epsilon^{n-1}(x)).$$

By the algorithm (1.1), it follows [7] that

$$x = \frac{A_n + B_n T_\epsilon^n(x)}{C_n + D_n T_\epsilon^n(x)}, \text{ for all } n \geq 1,$$

where the numbers A_n, B_n, C_n, and D_n are given by the recursive relations

$$\begin{pmatrix} C_n & D_n \\ A_n & B_n \end{pmatrix} = \begin{pmatrix} C_{n-1} & D_{n-1} \\ C_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} k_n + 1 & k_n \epsilon(k_n) \\ 1 & 1 + \epsilon(k_n) \end{pmatrix} n \geq 1. \tag{1.3}$$

For any non-decreasing integer vector $(k_1, \cdots, k_n)$, define the n-th order cylinders as follows:

$$B(k_1, \cdots, k_n) = \{x \in (0, 1] : k_j(x) = k_j, \forall 1 \leq j \leq n\}.$$

Since there is a one-to-one correspondence between $x \in (0, 1]$ and the non-decreasing integer sequence $(k_1, k_2, \cdots)$, we have that

Lemma 1.1 ([7]). *The cylinder* $B(k_1, \cdots, k_n)$ *is just the interval with the end-points* $L_n = \dfrac{A_n}{C_n}$ *and* $R_n = \dfrac{K_n A_n + B_n}{K_n C_n + D_n}$. *As a consequence, the length of* $B(k_1, \cdots, k_n)$ *is*

$$|B(k_1, k_2, \cdots, k_n)| = \frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)}. \tag{1.4}$$

A calculation shows $(k_{n+1} = k)$

$$|B(k_1, k_2, \cdots, k_n, k)| = \frac{B_n C_n - A_n D_n}{(kC_n + D_n)((k+1)C_n + D_n)}. \tag{1.5}$$

Furthermore, the following lemmas hold:

Lemma 1.2 ([9]). *Let the sequences* $(A_n)_{n\geq 1}, (B_n)_{n\geq 1}, (C_n)_{n\geq 1}$, *and* $(D_n)_{n\geq 1}$ *be recursively defined by* (1.3). *Then*

$$B_n C_n - A_n D_n = \prod_{i=1}^{n} (1 + k_i + \epsilon(k_i)). \tag{1.6}$$

Lemma 1.3 ([7]). *Suppose there is a constant* δ *such that* $|\epsilon(k)| \leq k^{\delta}$ *with* $0 \leq \delta < 1$ *for all* $k \geq 1$, *then*

$$\frac{1}{2} \leq \frac{k_n C_n}{k_n C_n + D_n} \leq 2, \tag{1.7}$$

for all $k_n \geq K(\delta)$.

The GCF algorithm extends our knowledge on one-dimensional dynamical systems. The behaviours of the sequence $k_n(x)$ as $n \to \infty$ are of interest and the metric properties of the sequence have been investigated by a number of authors, see [7, 8, 9]. In particular, the second author in [9] proved that for any $-1 < \epsilon \leq 1$,

$$\lim_{n\to\infty} k_n^{1/n}(x) = e; \quad \lim_{n\to\infty} (k_1 k_2 \cdots k_n)^{1/n^2} = \sqrt{e} \ \lambda \text{ a.e..}$$

For a more special case that $\epsilon = 0$, the GCF is just Engel series expansion, which have been studied in a series of papers by Borel [1], Levy [5], Erdos et al. [2], and Renyi [6] etc..

Now, let us introduce the Hirst's results in the regular continued fractions (RCF). For any irrational $x \in (0, 1)$, let $\{a_n\}_{n\geq 1}$ be the sequence of the partial quotients of RCF for x. Let $B = \{b_n\}_{n\geq 1}$ be an infinite subset of natural numbers $\mathbb{N}$. Denote by $\tau(B)$ the convergence exponent of the series $\sum_{n=1}^{\infty} b_n^{-1}$, i.e.,

$$\tau(B) = \inf\left\{\tau > 0 : \sum_{n=1}^{\infty} b_n^{-\tau} < \infty\right\}.$$

Define

$$E(B) := \left\{x \in (0, 1] : a_n(x) \in B \,\forall\, n \geq 1, \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\right\}.$$

Hirst [4] proved the inequality $\dim_H E(B) \leq \frac{\tau(B)}{2}$ for any infinite set B, and the equality $\dim_H R(B) = \frac{\tau(B)}{2}$ for the special $B = \{1, 2^t, 3^t, \cdots\}, t \in \mathbb{N}$. Now, we consider the same question in the setting of generalized continued fractions. If there is a constant δ such that $|\epsilon(k)| \leq k^{\delta}$ with $0 \leq \delta < 1$ for all $k \geq 1$. Let

$$G_{\epsilon}(B) = \{x \in (0, 1) : k_n(x) \in B \,\forall\, n \geq 1, \text{ and } k_n(x) \to \infty \text{ as } n \to \infty\},$$

where $\{k_n(x), n \geq 1\}$ are the sequence of the partial quotients of x in CEF expansion. We get the parallel results with the Hirst sets $E(B)$ in RCF expansions as

Theorem 1.4. *Suppose there is a constant* δ *such that* $|\epsilon(k)| \leq k^{\delta}$ *with* $0 \leq \delta < 1$ *for all* $k \geq 1$, *then* $\dim_H G_{\epsilon}(B) \leq \tau(B)$ *for any infinite set B.*

Theorem 1.5. *Suppose there is a constant* δ *such that* $|\epsilon(k)| \leq k^{\delta}$ *with* $0 \leq \delta < 1$ *for all* $k \geq 1$, *then* $\dim_H G_{\epsilon}(B) = \tau(B)$, *for* $B = \{1, 2^t, 3^t, \cdots\}$ *and* $t \in \mathbb{N}$.

2. Proof of the Main Result

Proof of Theorem 1.4. Fixed $s > \tau(B)$, choose v_0 large enough so that for any $n_j \geq v_0$,

$$\sum_{n=n_j}^{\infty} \left(\frac{2}{b_n}\right)^s \leq 1,$$

since $\sum_{n=1}^{\infty} \frac{1}{b_n^s}$ is convergent. Then for any increasing sequence $\{k_n(x),\ n \geq 1\} \subset B$, there exists a smallest integer v satisfying $e_v \geq b_{v_0}$, and

$$\sum_{\substack{e_n \in B \\ k_n \geq k_{n_j}}} \left(\frac{2}{k_n}\right)^s \leq 1, \quad \forall\, n_j \geq v. \tag{2.1}$$

Now let's estimate the Hausdorff dimension of $G_\epsilon(B)$.

It is clear that

$$G_\epsilon(B) \subset \bigcap_{n=1}^{\infty} \bigcup_{\substack{k_1, \cdots, k_n \in B \\ k_1 \leq k_2 \leq \cdots \leq k_n}} I_n(k_1, \cdots, k_n).$$

So, the s-dimensional Hausdorff measure of $G_\epsilon(B)$ can be estimated as

$$\begin{aligned}
\mathcal{H}^s(G_\epsilon(B)) &\leq \liminf_{n\to\infty} \sum_{\substack{k_1, \cdots, k_n \in B \\ k_1 \leq k_2 \leq \cdots \leq k_n}} |I_n(k_1, \cdots, k_n)|^s \\
&= \liminf_{n\to\infty} \sum_{\substack{k_1, \cdots, k_{v-1} \in B \\ k_1 \leq k_2 \leq \cdots \leq k_{v-1}}} |I_{v-1}(k_1, \cdots, k_{v-1})|^s \\
&\quad \times \sum_{\substack{k_v \in B \\ k_v \geq k_{v-1}}} \left(\frac{|I_v(k_1, \cdots, k_v)|}{|I_{v-1}(k_1, \cdots, k_{v-1})|}\right)^s \cdots \sum_{\substack{k_n \in B \\ k_n \geq k_{n-1}}} \left(\frac{|I_n(k_1, \cdots, k_n)|}{|I_{n-1}(k_1, \cdots, k_{n-1})|}\right)^s.
\end{aligned}$$

Thus, the desired result follows if we can show that

$$W_n = \sum_{\substack{k_n \in B \\ k_n \geq k_{n-1}}} \left(\frac{|I_n(k_1, \cdots, k_n)|}{|I_{n-1}(k_1, \cdots, k_{n-1})|} \right)^s \leq 1, \text{ for all } k_n \geq k_v \geq b_{v_0}.$$

By (1.4) and (1.5), for $k_1, k_2, \cdots, k_n \in B$ and $k_1 \leq k_2 \leq, \cdots \leq k_n$ with $k_n \geq k_v$, we have

$$W_{n+1} = \sum_{\substack{b \in B \\ b \geq k_n}} \left(\frac{|I_{n+1}(k_1, \cdots, k_n, b)|}{|I_n(k_1, \cdots, k_n)|} \right)^s$$

$$= \sum_{\substack{b \in B \\ b \geq k_n}} \left(\frac{C_n(k_n C_n + D_n)}{((b+1)C_n + D_n)(bC_n + D_n)} \right)^s.$$

From (1.7), we get that

$$D_n \geq -\frac{k_n C_n}{2}. \tag{2.2}$$

Then by (2.1) and notice that $k_n \leq b$, we have

$$W_{n+1} \geq \sum_{\substack{b \in B \\ b \geq k_n}} \left(\frac{C_n}{(b+1)C_n - \frac{k_n C_n}{2}} \right)^s$$

$$\leq \sum_{\substack{b \in B \\ b \geq k_n}} \left(\frac{2}{b} \right)^s \leq 1,$$

which gives that $\dim_H G(B) \leq s$. Since this is true for all $s > \tau(B)$, we obtain $\dim_H G(B) \leq \tau(B)$ when $|\epsilon(k)| \leq k^\delta$ with $0 \leq \delta < 1$ for all $k \geq 1$.

3. Proof of Theorem 1.2

In this section, the infinite subset B of integers is specified as

$$B = \left\{n^t,\ n \geq 1\right\}, \text{ for some } t \in \mathbb{N},$$

$G_\epsilon(B)$ and $\tau(B)$ are defined before, we will prove that

$$\dim_H G_\epsilon(B) = \tau(B).$$

By Theorem 1.4, we only need to prove $\dim_H G_\epsilon(B) \geq \tau(B)$. It is obvious that $\tau(B) = \frac{1}{t}$ for $B = \{1, 2^t, 3^t, \cdots\}$.

Now, we cite a dimensional result concerning a specially defined Cantor set. Let $I = E_0 \supset E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of sets, with each E_n a union of a finite number of disjoint closed intervals. For every $n \geq 1$, each interval of E_n contains at least two intervals of E_{n+1}. In addition, the maximum length of intervals in E_n tends to 0 as $n \to \infty$. Then the set $E = \bigcap_{k=1}^{\infty} E_n$ is a totally disconnected subset of I, and is called a general Cantor set. If each interval of E_{n-1} contains at least m_n intervals of $E_n (n = 1, 2, \cdots)$, which are separated by gaps of at least θ_n, where $0 < \theta_{n+1} < \theta_n$ for each n. Then the lower bounded of the Hausdorff dimension of E can be given by the following lemma:

Lemma 3.1 ([3]).

$$\dim_H E \geq \liminf_{n\to\infty} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \theta_n)}.$$

Proof of Theorem 1.5. For each $n \geq 1$, let

$$G_n = \{I_n(k_1, \cdots, k_n) : k_i(x) \in B,\ 2^{it} \leq k_i(x) < 2^{(i+1)t},\ 1 \leq i \leq n\}.$$

Recall that $B = \{n^t : n \in \mathbb{N}\}$, then G_n can be explicitly expressed as

$$G_n = \{I_n(k_1, \cdots, k_n) : k_i(x) = k_i^t \text{ with } 2^i \le k_i < 2^{i+1}, \text{ for all } 1 \le i \le n\}. \tag{3.1}$$

Let

$$G_B = \bigcap_{n=1}^{\infty} E_n.$$

Firstly, we note that G_B is just the set

$$\{x \in (0, 1] : k_n(x) \in B,\ 2^{nt} \le k_n(x) < 2^{(n+1)t}, \quad \forall n \ge 1\}.$$

Clearly, it is a subset of $G_\epsilon(B)$.

Secondly, it fulfills the construction of the Cantor set in Lemma 3.1. Now, we specify the integers $\{m_n\}_{n\ge1}$ and the real numbers $\{\theta_n\}_{n\ge1}$.

Due to the definition of G_n, each interval of G_{n-1} contains $m_n = 2^n$ intervals of G_n, and any interval of G_k does not contain the interval $I_{n+1}(k_1, \cdots, k_n, 2^{(n+2)t})$ of G_{n+1}. So we have

$$m_1 m_2 \cdots m_{n-1} = 2^{1+2+\cdots+(n-1)} = 2^{\frac{n(n-1)}{2}}; \tag{3.2}$$

and any two of intervals in k_n are separated by at least one interval $I_{n+1}(k_1, \cdots, k_n, 2^{(n+2)t})$.

By (1.5), we have

$$|I_{n+1}(k_1, \cdots, k_n, 2^{(n+2)t})| = \frac{B_n C_n - A_n D_n}{(2^{(n+2)t} C_n + D_n)((2^{(n+2)t} + 1)C_n + D_n)}. \tag{3.3}$$

From (1.7), we get

$$D_n \le k_n C_n. \tag{3.4}$$

Thus and by (1.3), we get that

$$\begin{aligned} C_n &= (k_n + 1)C_{n-1} + D_{n-1} \\ &\leq (k_n + 1)C_{n-1} + k_{n-1}C_{n-1} \\ &\leq 3k_n C_{n-1} \leq \cdots \\ &\leq 3^n k_n k_{n-1} \cdots k_1. \end{aligned} \tag{3.5}$$

Combine (1.6) with the above three results to get that

$$\begin{aligned} &|I_{n+1}(k_1, \cdots, k_n, 2^{(n+2)t})| \\ &\geq \frac{\prod_{i=1}^{n}(1 + k_i + \epsilon(k_i))}{9 \cdot (2^{(n+2)t} C_n)^2} \\ &\geq \frac{k_1 k_2 \cdots k_n}{9(2^{(n+2)t} 3^n k_1 k_2 \cdots k_n)^2} \\ &= \frac{1}{9^{n+1} 4^{(n+2)t} k_1 k_2 \cdots k_n}. \end{aligned} \tag{3.6}$$

Substituting $k_i \leq 2^{(i+1)t}$ of (3.1) into (3.6), one has

$$|I_{n+1}(k_1, \cdots, k_n, 2^{(n+2)t})| \geq \frac{1}{9^{n+1} 4^{(n+2)t} \cdot 2^{\frac{n(n+2)}{2}t}} =: \theta_n. \tag{3.7}$$

Combining with (3.2), (3.7) and Lemma 3.1, we obtain

$$\dim_H G(B) \geq \liminf_{k \to \infty} \frac{\log 2^{\frac{n(n-1)}{2}}}{-\log\left(2^n \cdot \frac{1}{9^{n+1} 4^{(n+2)t} \cdot 2^{\frac{n(n+2)}{2}t}}\right)} = \frac{1}{t}.$$

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