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DISCRETE AND INTEGRAL VARIANTS OF THE JENSEN TYPE INEQUALITIES

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Abstract

We consider inequalities concerning convex functions and convex combinations with the common center. Including convex combinations to the integral method, the obtained discrete inequalities are transformed to integral inequalities. The results are applied to quasi-arithmetic means and the Hermite-Hadamard inequality.

1. Introduction

Repeat the concept of linear, affine, and convex set. Convenient way of presenting are combinations of vectors and scalars in a real vector (linear) space $\mathcal{X}$. Let $x_i \in \mathcal{X}$ be vectors (points), $\alpha_i \in \mathbb{R}$ be scalars (coefficients), and the sum

$$c = \sum_{i=1}^{n} \alpha_i x_i, \tag{1.1}$$

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be their combination. If there are no conditions on the coefficients α_i, the combination in (1.1) is linear. If the scalar sum $\sum_{i=1}^{n} \alpha_i = 1$, the combination in (1.1) is affine. If $\sum_{i=1}^{n} \alpha_i = 1$ and all $\alpha_i \in [0, 1]$, the combination in (1.1) is convex. The vector c itself is called the center of the observed convex combination.

Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of vectors. The set $\mathcal{S}$ is linear (respectively, affine, convex) if it contains all linear (respectively, affine, convex) combinations of its vectors. The smallest vector subspace (respectively, translated vector subspace, convex vector set) that contains $\mathcal{S}$ is called the linear (respectively, affine, convex) hull of $\mathcal{S}$, and it is denoted with $\operatorname{lin} \mathcal{S}$ (respectively, $\operatorname{aff} \mathcal{S}$, $\operatorname{conv} \mathcal{S}$). The linear (respectively, affine, convex) hull of the vector set $\mathcal{S}$ consists of all linear (respectively, affine, convex) combinations of the vectors from $\mathcal{S}$.

2. Linear, Affine and Convex Function

According to the set names, we have the function names. For the purpose of this paper, we will use real valued functions of one real variable. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. It is said that f is linear (respectively, affine) if the equality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i f(x_i), \tag{2.1}$$

holds for all linear (respectively, affine) combinations $\sum_{i=1}^{n} \alpha_i x_i$, and f is convex if the inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{2.2}$$

holds for all convex combinations $\sum_{i=1}^{n} \alpha_i x_i$. The equality in (2.1) and inequality in (2.2) are usually first defined for binomial combinations

$\alpha_1 x_1 + \alpha_2 x_2$, then using the principle of mathematical induction generalize to all finite combinations. The inequality in (2.2) presents the well-known discrete form of Jensen's inequality (a general overview of Jensen's inequality can be seen in [4]).

Let $[a, b] \subset \mathbb{R}$ be a closed interval, where $a < b$. Then every number $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$x = \frac{b-x}{b-a} a + \frac{x-a}{b-a} b. \tag{2.3}$$

The above binomial combination is convex, if and only if the number x belongs to the interval $[a, b]$. Given function $f : \mathbb{R} \to \mathbb{R}$, let $f_{\{a,b\}}^{\text{line}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f. Using the affinity of $f_{\{a,b\}}^{\text{line}}$, we have

$$f_{\{a,b\}}^{\text{line}}(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b). \tag{2.4}$$

The consequence of the representations in (2.3) and (2.4) is the following characterization of convex functions:

Theorem A (Convex function on the line)**.** *Every convex function* $f : \mathbb{R} \to \mathbb{R}$ *verifies the inequality*

$$f(x) \leq f_{\{a,b\}}^{\text{line}}(x) \quad \text{for } x \in [a, b]; \tag{2.5}$$

and the reverse inequality

$$f(x) \geq f_{\{a,b\}}^{\text{line}}(x) \quad \text{for } x \notin (a, b). \tag{2.6}$$

3. Inequalities with Convex Combinations

The main result in this section is Theorem 3.1, which represents the discrete form of the Jensen type inequality for convex combinations with the same center.

The following theorem deals with two convex combinations of the same center (one convex combination with two sub-combinations was studied in [8], operator combination in [3], similar types of inequalities for Q-class functions in [5]).

Theorem 3.1 (Jensen's type inequality with center)**.** *Let* $\sum_{i=1}^{n}\alpha_i x_i$ *be a convex combination of the points* $x_i \in [a, b]$, *and* $\sum_{j=1}^{m}\beta_j y_j$ *be a convex combination of the points* $y_j \notin (a, b)$.

If the center equality

$$c = \sum_{i=1}^{n}\alpha_i x_i = \sum_{j=1}^{m}\beta_j y_j, \tag{3.1}$$

is valid, then the inequality

$$f(c) \le \sum_{i=1}^{n}\alpha_i f(x_i) \le \sum_{j=1}^{m}\beta_j f(y_j), \tag{3.2}$$

holds for every convex function $f : \mathbb{R} \to \mathbb{R}$.

Proof. The left-hand side of the inequality in (3.2) is the Jensen inequality. The right-hand side follows from the series of inequalities:

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i f(x_i) &\le \sum_{i=1}^{n}\alpha_i f_{\{a,b\}}^{\text{line}}(x_i) = f_{\{a,b\}}^{\text{line}}\left(\sum_{i=1}^{n}\alpha_i a_i\right)\\ &= f_{\{a,b\}}^{\text{line}}\left(\sum_{j=1}^{m}\beta_j y_j\right) = \sum_{j=1}^{m}\beta_j f_{\{a,b\}}^{\text{line}}(y_j)\\ &\le \sum_{j=1}^{m}\beta_j f(y_j),\end{aligned}$$

derived applying the inequality in (2.5) to x_i, and the inequality in (2.6) to y_j. □

The geometric insight to the inequality in (3.2) presented in Figure 1 shows that the point $P_1(c,\ f(c))$ is below the point

$$P_2(c,\ \sum_{i=1}^{n} \alpha_i f(x_i)) \in \mathrm{conv}\{(x_1,\ f(x_1)),\ \dots,\ (x_n,\ f(x_n))\},$$

and the point P_2 is below the point

$$P_3\left(c,\ \sum_{j=1}^{m} \beta_j f(y_j)\right) \in \mathrm{conv}\{(y_1,\ f(y_1)),\ \dots,\ (y_m,\ f(y_m))\}.$$

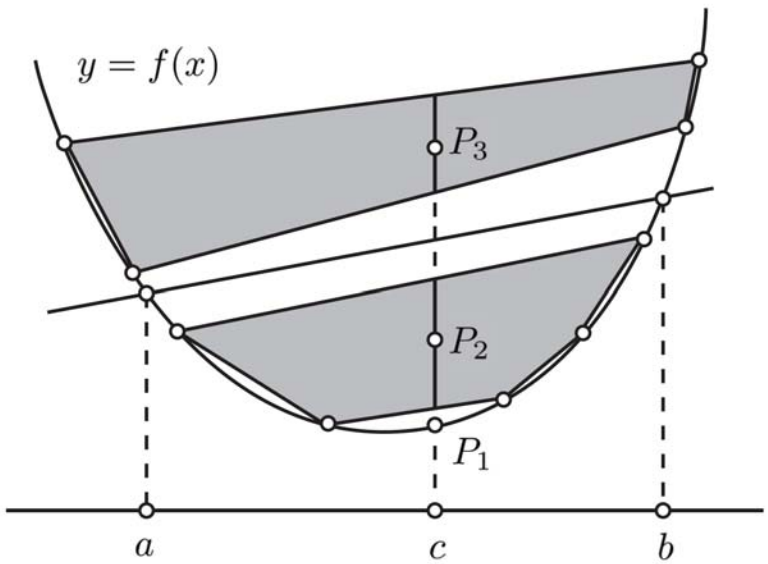

Figure 1. Geometric representation of the inequality in (3.2).

Theorem 3.1 also applies in the case $a = b$. This special case can be proved using the function $f_{\{a\}}^{\mathrm{line}}$ of any support line of the convex function f at the point a. The immediate consequence of this boundary case is:

Corollary 3.2. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is convex, if and only if it satisfies the right-hand side of the inequality in* (3.2) *under the condition in* (3.1).

Proof. The necessity is proved in Theorem 3.1. Let us prove the sufficiency, that is, the Jensen inequality for arbitrary convex combination.

Let $\sum_{i=1}^{n} \alpha_i x_i$ be a convex combination and c be its center. Then, from the convex combinations equality

$$\sum_{i=1}^{n} \frac{1}{n} c = \sum_{i=1}^{n} \alpha_i x_i,$$

it follows the inequality

$$\sum_{i=1}^{n} \frac{1}{n} f(c) \leq \sum_{i=1}^{n} \alpha_i f(x_i). \tag{3.3}$$

Since

$$\sum_{i=1}^{n} \frac{1}{n} f(c) = f(c) = f\left(\sum_{i=1}^{n} \alpha_i x_i\right),$$

the inequality in (3.3) presents the Jensen inequality. □

3.1. Application to discrete quasi-arithmetic means

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. In applications of convexity, we often use strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that ψ is convex with respect to φ (ψ is φ-convex), that is, $f = \psi \circ \varphi^{-1}$ is convex on $\varphi(\mathcal{I})$ (this terminology is taken from [9, Definition 1.19]). A similar notation is used for concavity.

Let $\sum_{i=1}^{n} \alpha_i x_i$ be a convex combination from $\mathcal{I}$, that is, all $x_i \in \mathcal{I}$. The discrete φ-quasi-arithmetic mean of the points (particles) x_i with the coefficients (weights) α_i (of the sum equal to 1) is the point

$$M_\varphi(x_i, \alpha_i) = \varphi^{-1}\left(\sum_{i=1}^{n} \alpha_i \varphi(x_i)\right), \tag{3.4}$$

which belongs to $\mathcal{I}$ because the convex combination $\sum_{i=1}^{n} \alpha_i \varphi(x_i)$ belongs to $\varphi(\mathcal{I})$.

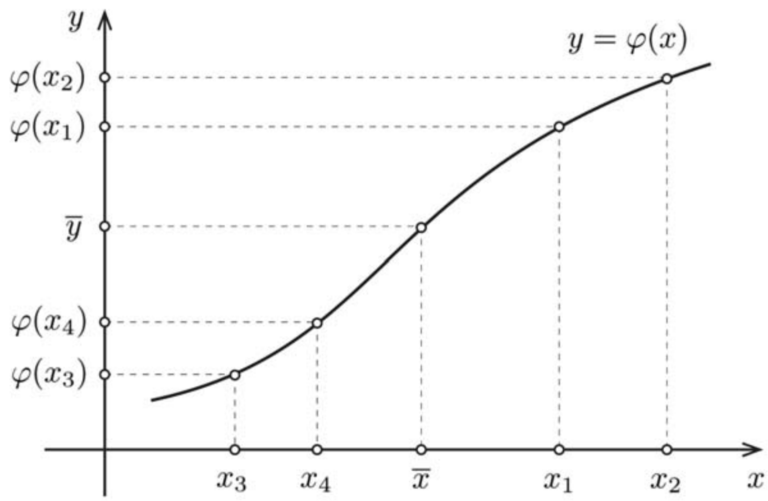

Figure 2. Representation of the quasi-arithmetic mean.

The quasi-arithmetic mean of four points is shown in Figure 2. The presented φ-quasi-arithmetic mean of the points x_1, x_2, x_3, and x_4 with the coefficients $\alpha_1, \alpha_2, \alpha_3$, and α_4 (of the sum equal to 1) designated as

$$\overline{x} = \varphi^{-1}(\overline{y}) = \varphi^{-1}\left(\sum_{i=1}^{4} \alpha_i \varphi(x_i)\right),$$

results by φ^{-1}-mapping of the convex combination $\overline{y} = \sum_{i=1}^{4} \alpha_i \varphi(x_i)$.

Let us apply Theorem 3.1 to the quasi-arithmetic means (more on quasi-arithmetic and power means, see [1] and [4]).

Corollary 3.3. *Let* $\mathcal{I} \subseteq \mathbb{R}$ *be an interval, and* $[a, b] \subseteq \mathcal{I}$ *be a bounded closed subinterval. Let* $\sum_{i=1}^{n} \alpha_i x_i$ *and* $\sum_{j=1}^{m} \beta_j y_j$ *be convex combinations of the same center, where* $x_i \in [a, b]$ *and* $y_j \in \mathcal{I} \setminus \langle a, b \rangle$. *Let* $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ *be strictly monotone continuous functions.*

If ψ *is either* φ*-convex and increasing or* φ*-concave and decreasing, and if the equality*

$$M_{\varphi}(x_i, \alpha_i) = M_{\varphi}(y_j, \beta_j), \tag{3.5}$$

is valid, then the inequality

$$M_{\psi}(x_i, \alpha_i) \leq M_{\psi}(y_j, \beta_j) \tag{3.6}$$

holds.

If ψ *is either* φ*-convex and decreasing or* φ*-concave and increasing, then the reverse inequality is valid in* (3.6).

Proof. Take $\mathcal{J} = \varphi(\mathcal{I})$, $[c, d] = \varphi([a, b])$, and $f = \psi \circ \varphi^{-1} : \mathcal{J} \to \mathbb{R}$. We apply Theorem 3.1 to the points $u_i = \varphi(x_i) \in \mathcal{J}$ and $v_j = \varphi(y_j) \in \mathcal{J} \setminus \langle c, d \rangle$. Prove the case where the function ψ is φ-convex and increasing.

Starting with the equality $\varphi(M_{\varphi}(x_i, \alpha_i)) = \varphi(M_{\varphi}(y_j, \beta_j))$, that is,

$$\sum_{i=1}^{n} \alpha_i u_i = \sum_{j=1}^{m} \beta_j v_j,$$

and relying on Theorem 3.1, we get

$$\sum_{i=1}^{n} \alpha_i f(u_i) \leq \sum_{j=1}^{m} \beta_j f(v_j).$$

Applying the increasing function ψ^{-1} to the above inequality, it follows the inequality in (3.6) because $f(u_i) = \psi(x_i)$ and $f(v_j) = \psi(y_j)$. □

As a special case of the quasi-arithmetic mean in (3.4) with $\mathcal{I} = \langle 0, +\infty\rangle$, $\varphi_r(x) = x^r$ for $r \neq 0$ and $\varphi_0(x) = \ln x$, we can observe the discrete power mean

$$M_n^{[r]}(x_i, \alpha_i) = \begin{cases} \left(\sum_{i=1}^{n} \alpha_i x_i^r\right)^{\frac{1}{r}} & \text{for } r \neq 0, \\ \exp\left(\sum_{i=1}^{n} \alpha_i \ln x_i\right) & \text{for } r = 0. \end{cases} \tag{3.7}$$

Note that

$$M_n^{[1]}(x_i, \alpha_i) = \sum_{i=1}^{n} \alpha_i x_i.$$

Corollary 3.4. *Let* $[a, b] \subset \langle 0, +\infty\rangle$ *be a bounded closed subinterval. Let* $\sum_{i=1}^{n} \alpha_i x_i$ *and* $\sum_{j=1}^{m} \beta_j y_j$ *be convex combinations of the same center, where* $x_i \in [a, b]$ *and* $y_j \in \langle 0, +\infty\rangle \setminus \langle a, b\rangle$.

If the equality

$$M_n^{[1]}(x_i, \alpha_i) = M_m^{[1]}(y_j, \beta_j), \tag{3.8}$$

is valid, then the inequality

$$M_n^{[r]}(x_i, \alpha_i) \leq M_m^{[r]}(y_j, \beta_j), \tag{3.9}$$

holds for $r \geq 1$, *at the same time as the inequality*

$$M_n^{[r]}(x_i, \alpha_i) \geq M_m^{[r]}(y_j, \beta_j), \tag{3.10}$$

holds for $r \leq 1$.

Proof. The proof follows from Corollary 3.3 using the functions $\varphi(x) = x$, and $\psi(x) = x^r$ for $r \neq 0$ or $\psi(x) = \ln x$ for $r = 0$. □

4. Inequalities with Barycenters

The main result in this section is Theorem 4.1, which represents the integral form of the Jensen type inequality for the sets with the same barycenter.

The integral analogy of the concept of convex combination is the concept of barycenter. Let μ be a positive measure (according to [10, page 16], in fact, nonnegative measure) on $\mathbb{R}$. Let $\mathcal{A} \subseteq \mathbb{R}$ be a μ-measurable set with $\mu(\mathcal{A}) > 0$. Given the positive integer n, let $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_{ni}$ be the partition of pair wise disjoint μ-measurable sets $\mathcal{A}_{ni}$, and $x_{ni} \in \mathcal{A}_{ni}$ be points. Then we have the convex combination

$$\sum_{i=1}^{n} \frac{\mu(\mathcal{A}_{ni})}{\mu(\mathcal{A})} x_{ni} = c_n,$$

whose center c_n belongs to $\operatorname{conv} \mathcal{A}$. If the sequence $(c_n)_n$ converges, then the μ-barycenter of $\mathcal{A}$ can be defined by

$$B(\mathcal{A}, \mu) = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{\mu(\mathcal{A}_{ni})}{\mu(\mathcal{A})} x_{ni} \right) = \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} x d\mu(x). \tag{4.1}$$

The integral arithmetic mean of some integrable function can also be introduced using the convex combinations. If $f : \mathcal{A} \to \mathbb{R}$ is a μ-integrable function, then the μ-arithmetic mean of f on $\mathcal{A}$ is defined by

$$M(f, \mathcal{A}, \mu) = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{\mu(\mathcal{A}_{ni})}{\mu(\mathcal{A})} f(x_{ni}) \right) = \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(x) d\mu(x). \tag{4.2}$$

If $\mathcal{A}$ is the interval, then its μ-barycenter $B(\mathcal{A}, \mu)$ belongs to $\mathcal{A}$, and if f is continuous on $\mathcal{A}$, then its μ-arithmetic mean on $\mathcal{A}$ belongs to $f(\mathcal{A})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and $\mathcal{A} \subseteq \mathbb{R}$ be a μ-measurable set with $\mu(\mathcal{A}) > 0$. If f is linear, then the equality

$$f\left(\int_{\mathcal{A}} x d\mu(x)\right) = \int_{\mathcal{A}} f(x) d\mu(x) \tag{4.3}$$

holds. If f is affine, then we have the equality

$$f\left(\frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} x d\mu(x)\right) = \frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} f(x) d\mu(x), \tag{4.4}$$

or otherwise stated, $f(B(\mathcal{A}, \mu)) = M(f, \mathcal{A}, \mu)$. If f is μ-integrable and convex, then the inequality

$$f\left(\frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} x d\mu(x)\right) \le \frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} f(x) d\mu(x) \tag{4.5}$$

holds, that is, $f(B(\mathcal{A}, \mu)) \le M(f, \mathcal{A}, \mu)$. The inequality in (4.5) presents the important integral form of Jensen's inequality.

Theorem 4.1 (Jensen's type inequality with barycenter)**.** *Let* μ *be a measure on* $\mathbb{R}$. *Let* $\mathcal{A} \subseteq [a, b]$ *and* $\mathcal{B} \subseteq \mathbb{R} \setminus \langle a, b\rangle$ *be sets with positive measures.*

If the barycenter equality

$$c = \frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} x d\mu(x) = \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}} x d\mu(x), \tag{4.6}$$

is valid, then the inequality

$$f(c) \le \frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} f(x) d\mu(x) \le \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}} f(x) d\mu(x), \tag{4.7}$$

holds for every convex μ-*integrable function* $f : \mathbb{R} \to \mathbb{R}$.

Proof. The left-hand side of the inequality in (4.7) is the Jensen inequality. The right-hand side follows from the series of inequalities

$$\frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} f(x)d\mu(x) \le \frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} f_{\{a,b\}}^{\text{line}}(x)d\mu(x) = f_{\{a,b\}}^{\text{line}}\left(\frac{1}{\mu(\mathcal{A})}\int_{\mathcal{A}} xd\mu(x)\right)$$

$$= f_{\{a,b\}}^{\text{line}}\left(\frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}} xd\mu(x)\right) = \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}} f_{\{a,b\}}^{\text{line}}(x)d\mu(x)$$

$$\le \frac{1}{\mu(\mathcal{B})}\int_{\mathcal{B}} f(x)d\mu(x),$$

obtained using the affinity of the function $f_{\{a,b\}}^{\text{line}}$, and applying the inequalities $f(x) \le f_{\{a,b\}}^{\text{line}}(x)$ for $x \in \mathcal{A}$, and $f_{\{a,b\}}^{\text{line}}(x) \le f(x)$ for $x \in \mathcal{B}$. □

5. Application to the Hermite-Hadamard Inequality

Using the Jensen type inequality, we briefly demonstrate the derivation of the Hermite-Hadamard inequality (for essentials on this inequality, see [6], [7] or [2]).

Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the inequality

$$f(c) \le \sum_{i=1}^{n} \alpha_i f(a_i) \le f_{\{a,b\}}^{\text{line}}(c), \tag{5.1}$$

where $c = \sum_{i=1}^{n} \alpha_i a_i$, follows from Theorem 3.1 and its proof. Given the positive integer n, take the convex combination equality

$$\frac{1}{2}a + \frac{1}{2}b = \sum_{i=1}^{n} \frac{\Delta\bar{x}_i}{b-a}\bar{x}_i,$$

where $\Delta\bar{x}_i = (b-a)/n = x_{i+1} - x_i$ and $\bar{x}_i = (x_{i+1} + x_i)/2$. Using the inequality in (5.1) with $c = \frac{1}{2}a + \frac{1}{2}b$, we have

$$f\left(\frac{1}{2}a+\frac{1}{2}b\right) \le \sum_{i=1}^{n} \frac{\Delta \overline{x}_i}{b-a} f(\overline{x}_i) \le f_{\{a,b\}}^{\text{line}}\left(\frac{1}{2}a+\frac{1}{2}b\right) = \frac{1}{2}f(a)+\frac{1}{2}f(b),$$

and letting n to infinity, we obtain the classic Hermite-Hadamard inequality

$$f\left(\frac{1}{2}a+\frac{1}{2}b\right) \le \frac{1}{b-a}\int_{[a,b]} f(x)dx \le \frac{1}{2}f(a)+\frac{1}{2}f(b). \qquad (5.2)$$

Putting $c = \frac{1}{2}a+\frac{1}{2}b$, the coordinates of the points P_1, P_2, and P_3 in Figure 3 are as follows:

$$P_1(c, f(c)), \quad P_2\left(c, \frac{1}{b-a}\int_{[a,b]} f(x)dx\right), \quad P_3(c, \frac{1}{2}f(a)+\frac{1}{2}f(b)).$$

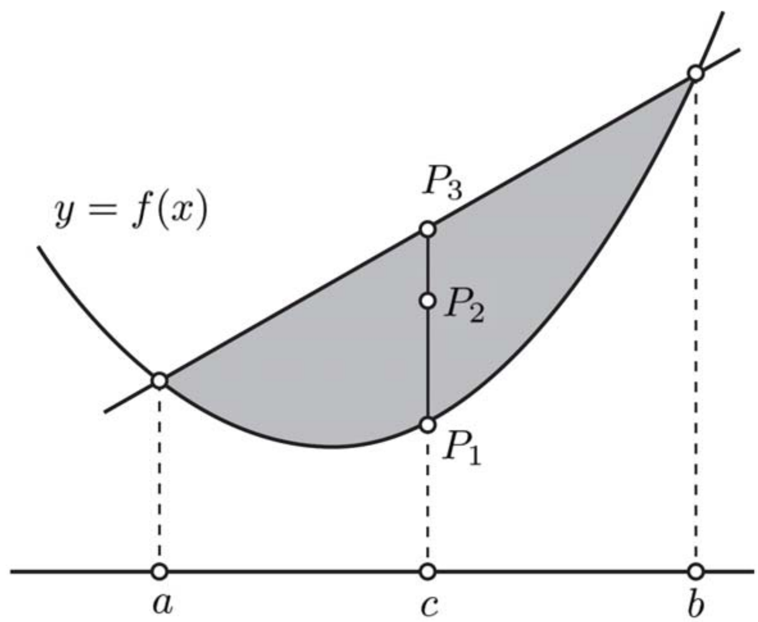

Figure 3. Graphical representation of the Hermite-Hadamard inequality.

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