

DYNAMIC RESPONSE OF A FLEXIBLE CANTILEVER BEAM UNDER MOVING DECENTRALIZED MASS WITH CONSTANT SPEED

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Abstract

In the present paper, dynamic and vibration behaviour of a flexible cantilever Euler-Bernoulli beam under moving decentralized mass with constant velocity has been studied that loads locating on the beam during the specified time. First, extract the Lagrangian law by using potential energy and kinetic equations, and using the Hamilton equations to obtain motion equation of a system in form of partial differential equations. These equations are coupled because of the bound mass transverse with beam vibration. The mass assumed decentralized and rigid body. According to the boundary conditions for a cantilever beam and using none dimension defined parameters; gain the dimensionless equations of motion of the system. Thus, to solve the integral

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form of equation used numerical integration method (Simpson's three-point). Finally, the dimensionless system of equations using numerical methods Rayleigh-Ritz approximated by ordinary differential equations and finite difference methods to solve them are discussed. To validate the survey results of the proposed model, compared with the motion of Euler-Bernoulli beam model by centralized mass movement. To more understanding this model, solve an example by MATLAB Software.

1. Introduction

There are many systems in mechanical and civil engineering that need to be considered as continuous systems and resolved in the next step, like a flexible beam carrying a moving mass to which many practical examples can be attributed: cars moving on the bridge; transportation of cranes while loading, robotic arms or using the moving mass that can be used as vibration controller of the beam (vibration stabilizer) or even under gyroscope [1, 2].

The main problem in the study of such systems, given the simplest model for beam (like Euler-Bernoulli beam), is the dependence between the moving mass and the beam that makes it difficulty and complicated to solve the equations of motion in the system.

This issue was first proposed in the design of railway lines, later it was considered in other transportation engineering structures. The first published researches were done in this case [3, 4], then some articles related to this topic were published [5, 6]. There are new books on analyzing in different conditions [7]. Some studies examined the effects of mass's high-speed motion on the beam [8, 9]. In the past studies, moment of inertia effect of the moving mass has been ignored and the model considered as a mass or a concentrated load that moves along the beam.

In transportation engineering applications, a mass motion is usually considered with constant speed on a continuous beam that results in partial differential equations with coupled parameters related to the mass location. Due to the coupled parameters in the equations, they cannot be solved as eigenvalues with independent variables. Lateral

integration method has been presented to obtain the shape of modes related to the mass motion [10]. Although it is too difficult to obtain numerical results using this method, many researches have been published in the field [11-15].

The effect of moving load weight to cantilevered beam weigh that been studied given the moving mass load as constant, and using an infinite series, an exact solution presented for the model [16]. Dynamic stability of the Euler-Bernoulli beam's lateral response, which is continuously changing with a concentrated moving mass, has been examined [17]. To describe the dynamic response of a beam under a number of moving masses, a Fourier sinusoid presented using the changes [18]. In another research, considering the constant moving mass, other algorithm is offered to solve the dynamic response of an elastic beam element [19].

Then, given shear deformation and rotary inertia moment, vibration response of a beam with a mass moving at a constant speed has been studied by using the beam theories [20]. A method presented for direct and accurate modelling based on Green's functions for structures consisting of a beam related to the mass motion at a constant speed [21].

Numerical integration has been used to solve equations of motion for an Euler-Bernoulli cantilevered beam with respect to the mass location [23]. The results compared with the available experimental data and published reports indicate that the results are in good agreement with experimental data [13, 19, 22].

Vibration response of a beam element has also been examined in relation to the motion of the moving load with non-uniform velocities as acceleration [24]. Vibration analysis of a beam under the forces of a train has been done with simulating the issue as a number of concentrated loads and loading with constant speed [25].

Other researchers studied the effect of mass motion speed on the beam and after extracting equations of motion for the system, solved them in analytical-numerical method [26]. Others examined dynamic and modal behaviour of an Euler-Bernoulli beam under a moving mass. Dirac delta function has been used to describe the location of the moving mass and its moment of inertia effect along the beam.

In the following, to control the beam's response, control algorithms presented with recursive parameters of displacement and speed [27].

2. Geometric Description of the Model

The system consists of an Euler-Bernoulli cantilevered beam with following characteristics:

Beam density (ρ), beam length (L), beam cross section (A), beam moment of inertia (I), beam modulus of elasticity (E), and rotational inertia of the beam (J_b).

The decentralized mass is also shown with (M) as a small beam having a cross section (A_m), length (L_m), and rotational inertia (J_m) that has same width and thickness (h) as shown in Figure 1.

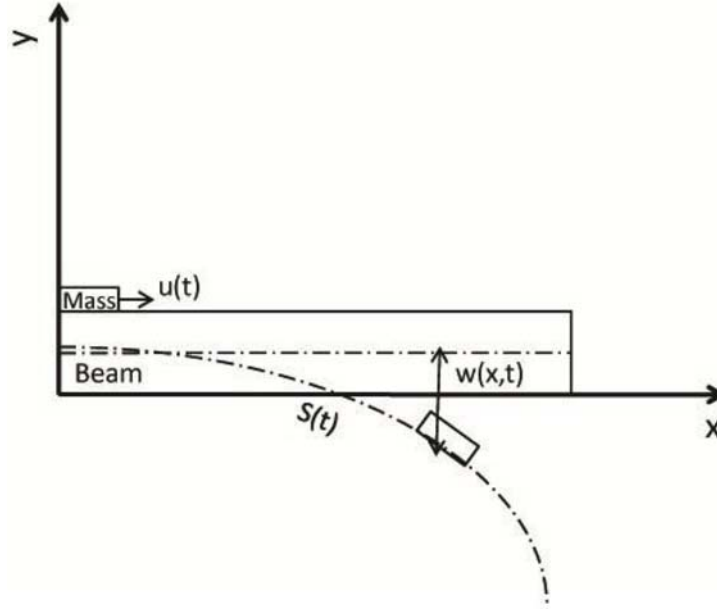


Figure 1. Geometric modelling description.

In the above figure, $w(x, t)$ indicates the curvature in one point of the beam that is in x distance of cantilever side of the beam at t time, and $S(t)$ is the longitudinal direction marked on the beam's length that the moving mass travels it.

3. Obtaining Equations of Motion

The system's equations of motion have been obtained by using Hamilton's law. The system's energy resulted from two components of beam and mass motion that can be obtained separately, so that kinetic energy of the beam can be expressed as follows:

$$T_b = \frac{A\rho}{2} \int_0^L \dot{w}^2 dx + \frac{1}{2} J_b \int_0^L \dot{\omega}^2 dx. \quad (3.1)$$

Similarly, kinetic energy of the moving mass is obtained as follows:

$$T_m = \left[\frac{1}{2} M(\dot{s}^2 + \dot{w}^2 + 2\dot{s}\dot{w}) + \frac{1}{2} J_m \dot{\omega}^2 \right]. \quad (3.2)$$

Potential energy of the system is only related to the energy stored in the beam due to the amount of its bending

$$V_b = \frac{EI}{2} \int_0^L \dot{w}^2 dx. \quad (3.3)$$

In the above equations, $(\dot{\quad})$ and (\quad) , respectively, indicate the differentiation with considering time and place parameters.

Based on the kinetic and potential energies obtained for the system, Lagrangian equation can be defined as follows using Hamilton's law:

$$L = T_b + T_m - V_b. \quad (3.4)$$

To calculate its minimum value, the following equation can be used:

$$\int_{t_1}^{t_2} \delta L dt = u(t)\delta s + q(x, t)\delta w. \quad (3.5)$$

To make the equations of motion, in the above equation, $u(t)$ is the force applied to the decentralized moving mass, and $q(x, t)$ is the force applied to the beam length. Therefore, Lagrangian equation can be expressed as a function of the following variables:

$$L = L(w', \dot{w}', \dot{w}, \dot{s}). \quad (3.6)$$

Given the above equation in relation to parameter, it can be written that

$$\int_{t_1}^{t_2} \int_0^L \left[\left(\frac{dL}{dw'} \right) \delta w' + \left(\frac{dL}{d\dot{w}'} \right) \delta \dot{w}' + \left(\frac{dL}{d\dot{w}} \right) \delta \dot{w} + \left(\frac{dL}{d\dot{s}} \right) \delta \dot{s} \right] dx dt = q(x, t)\delta w. \quad (3.7)$$

After inserting Lagrangian equation in the (3.7) relationship and overall relationship (3.5), the equations obtained as follows: Considering the properties of the mentioned time integration in (3.8), and integrating the beam equations of motion in terms of two separate components of δw and δs in certain location of the path travelled by the mass on the beam ($x = s$):

$$\int_{t_1}^{t_2} f_k(q, \dot{q}) \delta \dot{q}_k dt = \int_{t_1}^{t_2} f_k(q, \dot{q}) \frac{d}{dx} \delta q_k dt = - \int_{t_1}^{t_2} \frac{df_k(q, \dot{q})}{dx} \delta q_k dt. \quad (3.8)$$

In this model, boundary conditions for the integration of partial differential equations of the beam motion are considered as

$$\int_0^L (A \rho \ddot{w} + EI \dot{\dot{w}} - J_b \dot{\dot{w}}) dx + M(\ddot{w} + \dot{s} \dot{w} + 2\dot{s} \dot{w} - J_m \dot{\dot{w}}) = q(x, t), \quad (3.9)$$

$$M \dot{s} + M(\ddot{w} + \dot{w} \dot{w}) = u(t). \quad (3.10)$$

On this integration, we have these boundary conditions

$$\left\{ \begin{array}{ll} (1) w(x, t) = 0 & \text{at } x = 0, \\ (2) \dot{w}(x, t) = 0 & \text{at } x = 0, \\ (3) EI \dot{\dot{w}}(x, t) = 0 & \text{at } x = L, \\ (4) EI \dot{\dot{w}}(x, t) = 0 & \text{at } x = L. \end{array} \right. \quad (3.11)$$

3.1. Non-dimensionalized equations of motion

To simplify the solution, the mentioned equations of motion in (3.9) and (3.10) relationships are rewritten as non-dimensionalized equations of motion using the dimensionless parameters defined in the following:

$$v = \frac{w}{L}, \quad x_0 = \frac{x}{L}, \quad s_0 = \frac{s}{L}, \quad (3.12)$$

$$\epsilon = \frac{M}{\rho AL}, \quad J_{nd} = \frac{2J_b}{\rho AL^3}, \quad J_{md} = \frac{2J_m}{\rho_m A_m L_{n_t}^3}. \quad (3.13)$$

Non-dimensionalized equations of motion are obtained as follows:

$$\int_0^L [(\ddot{v} + \dot{\dot{v}} - J_{nd} \dot{\dot{v}}) + \epsilon (\ddot{v} + \dot{s}_0 \dot{v} + 2\dot{s}_0 \dot{v} - J_{md} \dot{\dot{v}}) \delta_{\Delta}(x_0 - s_0)] dx_0 = \frac{q(x, t) L^3}{EI}, \quad (3.14)$$

$$\ddot{s}_0 + [\ddot{v} \dot{v} + \dot{v} \dot{v}] = \frac{\rho AL^3}{MEI} u(t), \quad (3.15)$$

where (δ_Δ) is Dirac delta function.

Based on the mentioned boundary conditions for the beam and the above dimensionless parameters, boundary conditions for the dimensionless equations of motion can be rewritten as follows:

$$\left\{ \begin{array}{ll} (1) v(x, t) = 0 & \text{at } x_0 = 0, \\ (2) \dot{v}(x, t) = 0 & \text{at } x_0 = 0, \\ (3) \dot{v}(x, t) = 0 & \text{at } x_0 = L, \\ (4) \dot{v}(x, t) = 0 & \text{at } x_0 = L. \end{array} \right. \quad (3.16)$$

3.2. Solving equations of motion

In the previous section, equations of motion governing the system obtained as partial differential equations that were non-linear, interdependent, and had two interdependent parameters of time and place. It is too difficult to solve these equations numerically and analytically therefore to simplify the problem, Rayleigh-Ritz method has been used that consists parameters of time and place as two independent functions, so partial differential equations can be approximated to ordinary differential equations. In this method, the basic assumption to solve the beam equations has been considered as follows:

$$v(x_0, \tau) = \sum_j \{\varphi_j(x_0) T_j(\tau)\}, \quad (3.17)$$

where $v(x_0, \tau)$ is the vertical displacement of a point in the beam in a distance of x_0 from the end of the beam's cantilever side at τ time. In this approximation, it is assumed that $\varphi_j(x_0)$ is a function of the specific shape so when it is inserted in (3.15) relationship, partial differential equation is changed into an ordinary differential equation that is in terms of time and can be solved easily using the usual solutions.

3.2.1. Obtaining function shape of $\varphi_j(x_0)$

In the method used, it is necessary that a mode shape function be clear for solving derived equations yet despite the existence of the mass, mode shapes of a beam cannot be used normally, while the involvement of decentralized mass parameter of location increases the accuracy of the mathematical model. Mode shape function can be obtained considering the initial and boundary conditions imposed on the beam in the mass location. In this regard, the defined method of Stanisic [5] has been used

$$v(x_0, \tau) = Z(x_0)e^{i\Omega\tau}. \quad (3.18)$$

In the defined mixed shape for vertical displacement of the beam, shows the non-dimensionalized frequency.

Homogeneous part of the equations of motion describes the beam's motion mode shape function that is expressed as follows:

$$\int_0^L [(\ddot{v} + \dot{v} - J_{nd}\dot{\dot{v}}) + \epsilon(\ddot{v} + \ddot{s}_0\dot{v} + 2\dot{s}_0\dot{v} - J_{md}\dot{\dot{v}})\delta_\Delta(x_0 - s_0)]dx_0 = 0. \quad (3.19)$$

By inserting (3.18) relationship in (3.19), and considering the two separate areas of the beam (left and right side of the beam) related to the mass location, the equations are obtained as follows:

$$\frac{d^4 Z_L}{dx_0^4} + (J_{nd} + J_{md})\Omega^2 \frac{d^2 Z_L}{dx_0^2} - \Omega^2 Z_L(x_0) = 0, \quad 0 \leq x_0 \leq s_0, \quad (3.20)$$

$$\frac{d^4 Z_R}{dx_0^4} + (J_{nd} + J_{md})\Omega^2 \frac{d^2 Z_R}{dx_0^2} - \Omega^2 Z_R(x_0) = 0, \quad s_0 \leq x_0 \leq L. \quad (3.21)$$

Common and general shape for the solution of (3.20) and (3.21) differential equations are considered as

$$Z(x_0) = A \sin(\beta x_0) + B \cos(\beta x_0) + C \sinh(\beta x_0) + D \cosh(\beta x_0), \quad (3.22)$$

where $\beta^2 = \pm\Omega$.

Considering boundary conditions $x_0 = 0$ and $x_L = 0$ in the functions $Z_L(x_0)$ and $Z_R(x_0)$, relationship (3.22) can be expressed as follows for the left and right side of the beam related to the decentralized mass location

$$Z_L(x_0) = A_L(\sin(\beta x_0) - \sinh(\beta x_0)) + B_L(\cos(\beta x_0) - \cosh(\beta x_0)), \quad 0 \leq x_0 < s_0, \quad (3.23)$$

$$Z_R(x_0) = A_R(\sin(\beta(x_0 - 1)) + \sinh(\beta(x_0 - 1))) \\ + B_R(\cos(\beta(x_0 - 1)) - \cosh(\beta(x_0 - 1))), \quad s_0 \leq x_0 \leq L. \quad (3.24)$$

Applying the imposed conditions on the beam (in the mass location) caused this situation, unknown coefficients A_L, B_L, A_R, B_R in the above equations can be calculated as $x_0 = s_0$.

So we have following conditions:

$$\begin{cases} Z_L = Z_R, \\ \dot{Z}_L = \dot{Z}_R, \\ \ddot{Z}_L = \ddot{Z}_R. \end{cases} \quad (3.25)$$

Based on the mentioned conditions, unknowns can be obtained in terms of each other, so that if they are expanded, there will be

$$A_L = A(\beta^4 - \Omega^2(J_{nd} + J_{md})\beta^2 - \Omega^2), \quad (3.26)$$

$$B_L = B(\beta^4 - \Omega^2(J_{nd} + J_{md})\beta^2 - \Omega^2), \quad (3.27)$$

$$A_R = -C(\beta^4 + \Omega^2(J_{nd} + J_{md})\beta^2 - \Omega^2), \quad (3.28)$$

$$B_R = -D(\beta^4 + \Omega^2(J_{nd} + J_{md})\beta^2 - \Omega^2). \quad (3.29)$$

After calculating the invariables, vertical displacement function can be expressed as follows:

$$v(x_0, \tau) = Z(x_0; s_0)T(\tau), \quad (3.30)$$

where

$$Z(x_0; s_0) = \begin{cases} Z_L(x_0; s_0), & 0 \leq x_0 < s_0, \\ Z_R(x_0; s_0), & s_0 \leq x_0 \leq L. \end{cases} \quad (3.31)$$

3.2.2. Equations of motion considering two shapes of the first mode

To show the solution, the first two mode shapes of the beam's vertical displacement function series have been used

$$v(x_0, \tau) = \sum_{i=1}^2 Z_i(x_0; s_0)T_i(\tau). \quad (3.32)$$

Using relationship (3.32), equations of kinetic and potential energy in the system components can be expressed as follows:

$$\begin{aligned} T_b &= \frac{A\rho}{2L^2} \int_0^L \sum_{i=1}^2 \sum_{j=1}^2 (\dot{T}_i(\tau)\dot{T}_j(\tau)Z_i(x_0; s_0)Z_j(x_0; s_0))^2 dx_0 \\ &+ \frac{J_b}{L^2} \int_0^L \sum_{i=1}^2 \sum_{j=1}^2 (\dot{T}_i(\tau)\dot{T}_j(\tau)\dot{Z}_i(x_0; s_0)\dot{Z}_j(x_0; s_0))^2 dx_0, \end{aligned} \quad (3.33)$$

$$\begin{aligned} T_m &= \frac{1}{2} M\dot{s}_0^2 + \frac{M}{2L^2} \left[\sum_{i=1}^2 \sum_{j=1}^2 (\dot{T}_i(\tau)\dot{T}_j(\tau)Z_i(x_0; s_0)Z_j(x_0; s_0))^2 \right] \\ &- \frac{M\dot{s}_0}{L^2} \left[\sum_{i=1}^2 \sum_{j=1}^2 \dot{T}_i(\tau)\dot{T}_j(\tau)\dot{Z}_i(x_0; s_0)\dot{Z}_j(x_0; s_0) \right] \\ &+ \frac{J_m}{L^2} \left[\sum_{i=1}^2 \sum_{j=1}^2 (\dot{T}_i(\tau)\dot{T}_j(\tau)\dot{Z}_i(x_0; s_0)\dot{Z}_j(x_0; s_0))^2 \right], \end{aligned} \quad (3.34)$$

$$V_b = \frac{EI}{2L} \int_0^L \sum_{i=1}^2 \sum_{j=1}^2 \dot{T}_i(\tau)\dot{T}_j(\tau)\dot{Z}_i(x_0; s_0)\dot{Z}_j(x_0; s_0) dx_0. \quad (3.35)$$

As it is clear in the above equations, there are spatial integrals that are defined as integral invariables C_i

$$\begin{aligned}
C_1 &= \int_0^L \dot{Z}_i(x_0; s_0) \dot{Z}_j(x_0; s_0) dx_0 \\
&= \int_0^{s(t)-\delta} \dot{Z}_{iL}(x_0; s_0) \dot{Z}_{jL}(x_0; s_0) dx_0 + \int_{s(t)+\delta}^L \dot{Z}_{iR}(x_0; s_0) \dot{Z}_{jR}(x_0; s_0) dx_0,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
C_2 &= \int_0^L Z_i(x_0; s_0) Z_j(x_0; s_0) dx_0 \\
&= \int_0^{s(t)-\delta} Z_{iL}(x_0; s_0) Z_{jL}(x_0; s_0) dx_0 + \int_{s(t)+\delta}^L Z_{iR}(x_0; s_0) Z_{jR}(x_0; s_0) dx_0,
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
C_3 &= \int_0^L \dot{Z}'_i(x_0; s_0) \dot{Z}'_j(x_0; s_0) dx_0 \\
&= \int_0^{s(t)-\delta} \dot{Z}'_{iL}(x_0; s_0) \dot{Z}'_{jL}(x_0; s_0) dx_0 + \int_{s(t)+\delta}^L \dot{Z}'_{iR}(x_0; s_0) \dot{Z}'_{jR}(x_0; s_0) dx_0.
\end{aligned} \tag{3.38}$$

In order to calculate the above integral invariables, Simpson's three-point numerical integration method has been used that in the two left and right sides of the beam, two points in Simpson method are the same first and end points of the interval, and the third point is considered as the midpoint of the interval, so that

$$\begin{aligned}
I &= \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) \right. \\
&\quad \left. + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx. \tag{3.39}
\end{aligned}$$

Then, mode shape function can be rewritten as follows:

$$\begin{aligned}
Z_{iL}(x_0; s_0) &= A_{iL}(s_0) (\sin(\beta_i x_0) - \sinh(\beta_i x_0)) \\
&\quad + B_{iL}(s_0) (\cos(\beta_i x_0) - \cosh(\beta_i x_0)), \tag{3.40}
\end{aligned}$$

$$\begin{aligned}
Z_{iR}(x_0, s_0) = & A_{iR}(s_0)(\sin(\beta_i(x_0 - 1)) - \sinh(\beta_i(x_0 - 1))) \\
& + B_{iR}(s_0)(\cos(\beta_i(x_0 - 1)) - \cosh(\beta_i(x_0 - 1))). \quad (3.41)
\end{aligned}$$

Therefore, unknown quantities in kinetic and potential energy equations include is $T_1(\tau)$, $T_2(\tau)$, and $S_0(\tau)$. By inserting new mode shape function in the kinetic and potential energy equations in the form of a finite series, and combining the response in the Lagrangian equation, a system of ordinary differential equations can be achieved that its general form is as follows:

$$M \begin{bmatrix} \ddot{T}_1 \\ \ddot{T}_2 \\ \ddot{S}_0 \end{bmatrix} + K \begin{bmatrix} T_1 \\ T_2 \\ S_0 \end{bmatrix} + N = U, \quad (3.42)$$

where K and M are stiffness and mass matrix, the mass is non-linear and N indicates other non-linear parameters in the system. In the following, the resulting equations of motion are solved using a forward difference method.

4. Simulating the Problem to Solve the Equations Numerically

In the present study, a beam with a square cross-section having defined specifications according to Table 1, and a decentralized mass in the form of a beam with a rectangular cross-sectional rigid according to Table 2 have been considered.

Table 1. Specifications of a flexible Euler-Bernoulli cantilevered beam with a square cross-section

Parameter of Beam	Unit	Sign	Value
Length	cm	L	100
Width	cm	H	5
Thickness	cm	W	5
Density	kg/m ³	ρ	2710
Mass	kg	M	6.775
Inertia Momentum	kg.m ²	J_m	3.3875
Elasticity module	MPa	E	70

Table 2. Specifications of a decentralized moving mass in the form of a beam with a rectangular cross-section

Parameter of Beam	Unit	Sign	Parameter
Length	cm	L	10
Width	cm	H	1
Thickness	cm	W	1
Density	kg/m ³	ρ	2710
Mass	kg	M	0.0271
Inertia Momentum	kg.m ²	J_m	0.0001355
Elasticity module	MPa	E	70

4.1. Solving the problem with the assumption of a centralized moving mass

In this section, assuming that the moving mass moment of inertia equals to *zero*, the problem is simulated and solved. So natural frequency of the system has been obtained for three specific positions of the moving mass on the longitudinal direction of its motion on the beam (at the beginning, middle and the end of beam), and its results presented in Table 3.

Table 3. First six natural frequencies for the system given the moving mass as centralized

Number of Natural Frequency	Beginning of beam	Middle of beam	End of beam
1	2.3729	5.3765	12.081
2	2.8013	6.7445	15.068
3	29.174	41.393	38.049
4	31.628	44.429	44.45
5	44.86	67.5725	50.777
6	91.673	94.460	100.14

In the following, to simulate the problem, the parameters defined in previous sections are obtained according to Table 4.

Table 4. Parameters obtained for the system with centralized moving mass

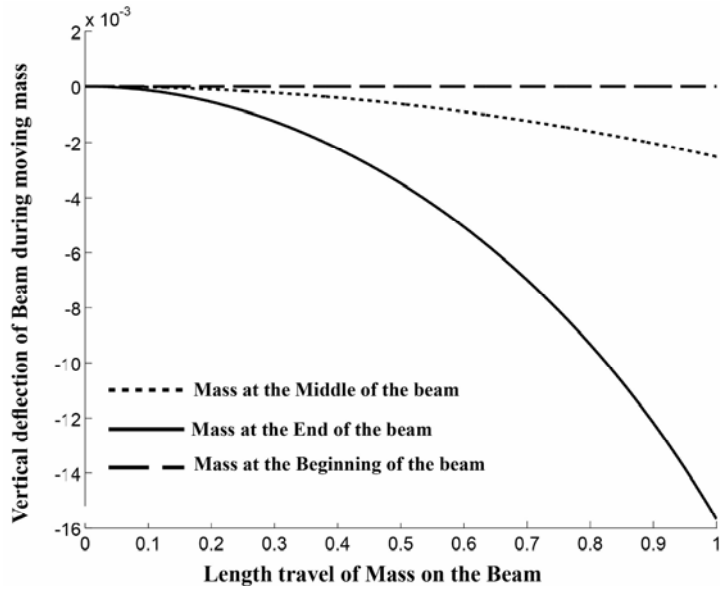
Parameters	Beginning of beam	Middle of beam	End of beam
Ω_1	1.0463	5.3717	27.1220
Ω_2	1.4583	8.4531	42.1916
β_1	1.0229	2.3177	5.2079
β_2	1.2076	2.9074	6.4955

In order to calculate the integral invariables in calculating the system's Lagrangian equation, Simpson's three-point numerical integration method has been used and its results for the three different positions of the mass on the beam can be seen in Table 5.

Table 5. Integral invariables obtained for the system with centralized moving mass

Invariables	Beginning of beam	Middle of beam	End of beam
C_1	- 0.8367	- 4.9355	- 73.7439
C_2	- 2.5166	- 15.9162	- 428.62
C_3	- 5.0762	- 39.3766	- 2509.6

Based on the data obtained, the beam's deflection in three different modes is shown in Figure 2.

**Figure 2.** The beam's deflection for three different modes of mass location in the system with centralized moving mass.

4.2. Solving the problem with the assumption of a decentralized moving mass

In this section, the problem is solved considering that the moving mass moment of inertia equals a constant non-zero value. So natural frequency of the system has been obtained for three specific points of the moving mass on the longitudinal direction of its motion on the beam (at the beginning, middle and the end of beam), and its results presented in Table 6.

Table 6. First six natural frequencies for the system given the moving mass as decentralized

Number of Natural Frequency	Beginning of beam	Middle of beam	End of beam
1	2.6277	8.4817	44.113
2	2.9156	9.9065	47.391
3	30.364	44.45	73.862
4	33.274	70.889	83.54
5	44.531	112.73	652.57
6	88.410	141.99	693.18

In the following, to simulate the problem, the parameters defined in previous sections are obtained according to Table 7.

Table 7. Parameters obtained for the system with decentralized moving mass

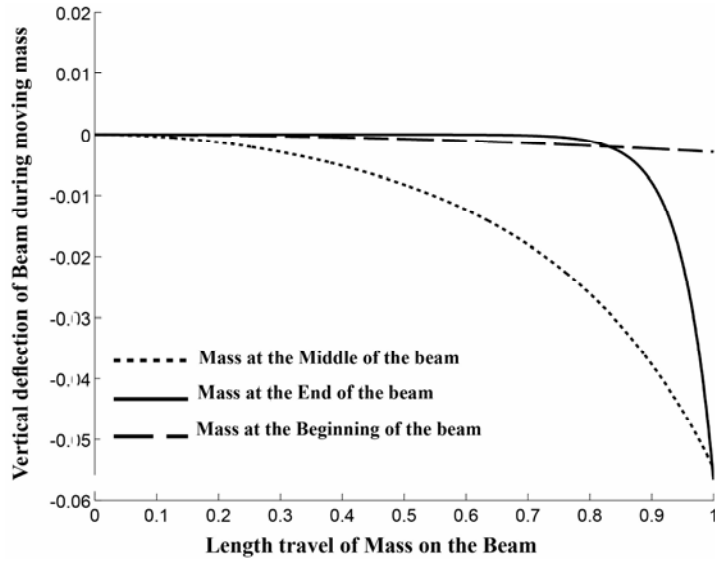
Parameters	Beginning of beam	Middle of beam	End of beam
Ω_1	1.2831	13.3685	361.6167
Ω_2	1.5797	18.2371	417.3563
β_1	1.1327	3.6563	19.0162
β_2	1.2569	4.2705	20.4293

In order to calculate the integral invariables in calculating the system's Lagrangian equation, Simpson's three-point numerical integration method has been used and its results for the three different positions of the mass on the beam can be seen in Table 8.

Table 8. Integral invariables obtained for the system with decentralized moving mass

Invariables	Beginning of beam	Middle of beam	End of beam
C_1	-0.9571	-14.5981	-6.28E ⁷
C_2	-2.8811	-58.1060	-1.2401E ⁹
C_3	-5.8274	-217.0484	-2.4488E ¹⁰

Based on the data obtained, the beam's deflection in three different modes is shown in Figure 3.

**Figure 3.** The beam's deflection for three different modes of mass location in the system with decentralized moving mass.

4.3. Comparing the results for two different modes of the system

In this section, two different modes of the system consist of centralized moving mass and decentralized moving mass have been compared.

Given the parameters obtained in the previous steps, it can be realized that the more the moving mass closes to the end of the beam, the moving mass moment of inertia will more increase. So that differences of the parameters will sometimes reach three times than their initial values.

The beam's deflection has been compared in two different modes of moment of inertia in two locations of the moving mass on the beam. The beam's deflection in the two modes when the moving mass is at the beginning and in the middle of the beam can be seen, respectively, in Figures 4 and 5.

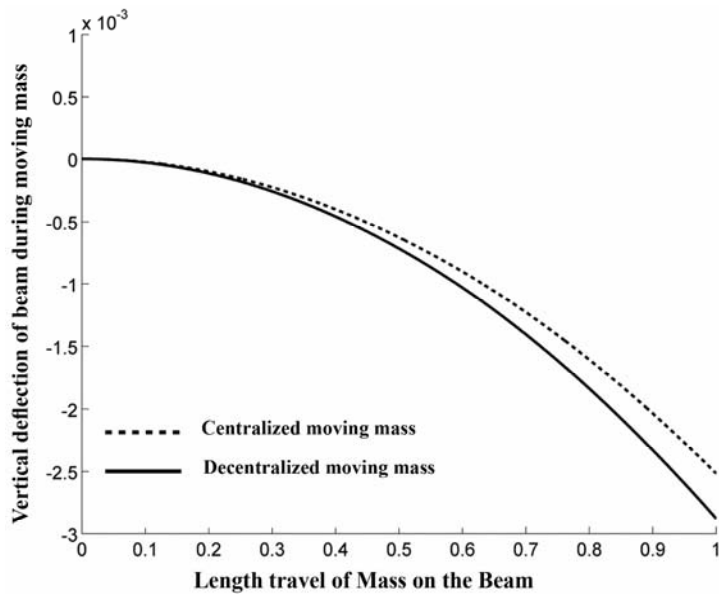


Figure 4. The beam's deflection for two different modes of moment of inertia in the beginning of the beam.

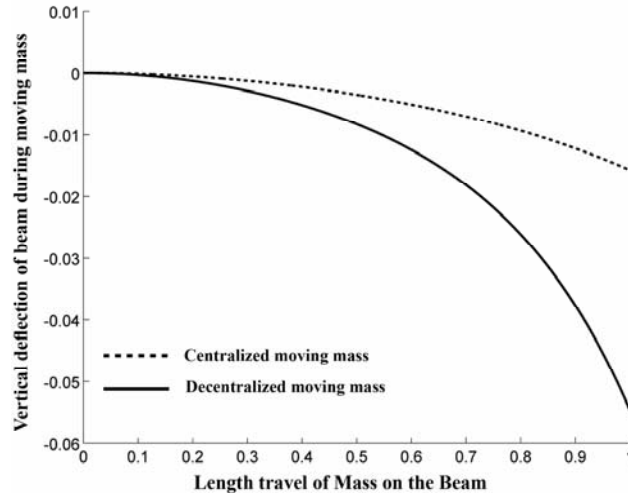


Figure 5. The beam's deflection for two different modes of moment of inertia in the middle of the beam.

Finally, to better understand the effect of moving mass moment of inertia on the system, normalized spatial vertical changes of the moving mass have been studied and compared in two different assumptions when the mass located in the middle of the beam. The results of this comparison can be seen in Figure 6.

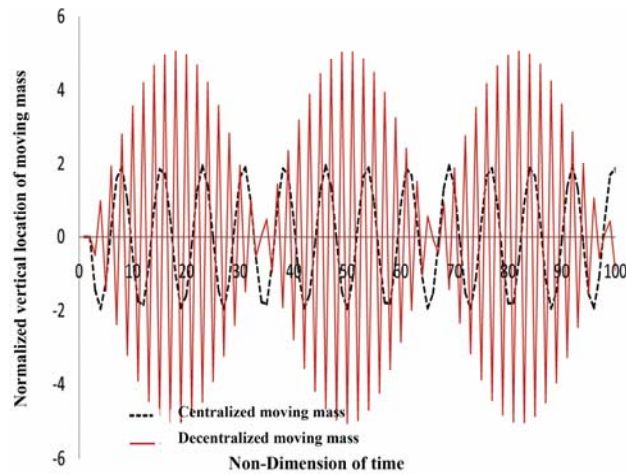


Figure 6. Comparison of normalized vertical position of the moving mass in two different studied modes.

5. Conclusion

In the present paper, vibration behaviour of the flexible Euler-Bernoulli cantilevered beam subjected to centralized and decentralized moving masses has been studied and the results presented in three specific positions of the moving mass. When there is no exact solution for solving the problem, the important point is that to what extent the proposed model can be validated by the future experimental results. So, responding the proposed example in this paper, solving it with the above-mentioned complex method, obtaining parameters and beam's deflection, and reporting the results, it's possible to compare them with the experimental data in the future.

Natural frequency in the system with centralized moving mass on the flexible Euler-Bernoulli beam is much lower than that of the system having decentralized moving mass so that there no significant difference in early frequencies, whereas according to the data obtained, the difference will be more six times at high frequencies (6-th frequency of the system). Thus, at high frequencies, the mass's moment of inertia effect is of great importance, and cannot be ignored in problems. It is so clear that the more the mass closes to the end of its path on the beam, the system's natural frequency more increases.

It should be considered that the proposed model can be used in designing engineering algorithms to reduce the effects of the moving mass on the beam's vibration behaviour or optimizing the mass motion to decrease the beam's fluctuations because of its parametric solution.

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