ON STRONGLY ALMOST $\Delta^N_\nu$-SUMMABLE SEQUENCE SPACES ON INFINITE MATRIX DEFINED BY A SEQUENCE OF MODULI

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Abstract

The idea of difference sequences $X(\Delta) = \{ x = (x_k) : \Delta x \in X \}$ with $X = l_\infty$, $c$ and $c_0$ was introduced by Kizmaz. In this paper, we define strongly almost $\Delta^N_\nu$-summable sequences defined by a sequence of moduli, which generalizes the $\Delta^\alpha$-summable sequence spaces. Some equivalent statements with their inclusion relation are proved.

1. Introduction

Let $\omega$ be the set of all real or complex sequences and $l_\infty$, $c$ and $c_0$ be Banach spaces of bounded, convergent and null sequences $x = (x_k)$, respectively, normed by

$$\| x \|_\infty = \sup_k | x_k |, \text{ where } k \in \mathbb{N}.$$
The notion of difference sequence space was introduced by Kizmaz [1] in 1981 as follows:

\[ X(\Delta) = \{x = (x_k) : \Delta x \in X\} \text{ with } X = l_\infty, c \text{ and } c_0, \]

where \( \Delta x = (\Delta x_k) = x_k - x_{k+1} \) for all \( k \in \mathbb{N} \).

These are Banach spaces with the norm \( \|x\|_\Delta = |x_1| + \|\Delta x\|_\infty \).

Later, the difference sequence spaces were generalized by Et and Çolak [2] as follows:

\[ X(\Delta^n) = \{x = (x_k) \in \omega : (\Delta x) \in X\}, \text{ where } X = l_\infty, c \text{ and } c_0, \]

where \( \Delta^n x = (\Delta^n x_k) = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1} \) for all \( k \in \mathbb{N} \), and so

\[ \Delta^n x_k = \sum_{i=1}^{n} (-1)^{i} \binom{n}{i} x_{k+i}, \text{ normed by} \]

\[ \Delta^n x_k = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty. \]

Worthy it is to mention that recently, the sequence spaces \( X(\Delta^n) \) were generalized by Et and Esi [3] to the following sequence space:

\[ X(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in X\}, \]

for \( X = l_\infty, c \text{ and } c_0 \), where \( \Delta^n x = (\Delta^n x_k) \), \( \Delta^n x_k = (\Delta^n x_k - \Delta^n x_{k+1}) \), and

\[ \Delta^n x_k = \Delta^n x_k = (\Delta^n x_k - \Delta^n x_{k+1}), \]

such that

\[ \Delta^n x_k = \sum_{i=0}^{m} (-1)^{i} \binom{m}{i} v_{k+1} x_{k+i}, \]

and \( v = (v_k) \neq 0 \) is any fixed sequence of nonzero numbers for all \( k \in \mathbb{N} \) [3].
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It was proved that the generalized sequence space

$$X(\Lambda^m_v),$$

where $X = l_\infty$, $c$ or $c_0$,

is a Banach space with norm defined by

$$\|x\|_{\Lambda^m_v} = \sum_{i=1}^{m} |x_i| + \sup |\Lambda^m_v x_k|.$$

There is an important notion (modulus function) used in this paper, which was introduced by Nakano [4]. Now, we give the definition of the modulus function.

**Definition 2.1.** A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function, if

1. $f(t) = 0$ if and only if $t = 0$;
2. $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$;
3. $f$ is increasing; and
4. $f$ is continuous from the right of 0.

Because of (2) $|f(x) - f(y)| \leq f(x - y)$.

So that in view of (4) $f$ is continuous on $[0, \infty)$.

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, $(0 < p \leq 1)$ is unbounded and $f(x) = \frac{x}{1 + x}$ is bounded.

Let $X$ be a sequence space. For a modulus function $f$, the sequence space $X(f)$ is defined as

$$X(f) = \{x = (x_k) \in \omega : (f(|x_k|)) \in X\}.$$

An extension of $X(f)$ is obtained by considering a sequence of moduli $F = (f_k)$.

$$X(F) = \{x = (x_k)^{f_k}_{k=1} \in \omega : (f_k(|x_k|)) \in X\},$$

where $X = l_\infty$, $c$ and $c_0$ [5].
**Definition.** A sequence \( x \in l_\omega \) is said to be *almost convergent*, if all Banach limits of \( x \) coincide. Lorentz [6] proved that

\[
\hat{c} := \left\{ x = (x_k) \in \omega : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} x_{k+s}, \text{ uniformly in } S \right\}.
\]

Maddox (see [7] and [8]) has defined \( x \) to be strongly almost convergent to \( l \), if

\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l| = 0, \text{ uniformly in } S \text{ for some } l > 0.
\]

By \( \hat{c} \), we mean the set of all almost convergent sequences. Let \( \hat{c} \) and \( \hat{c}_0 \) denote the sets of all sequences which are almost convergent and almost convergent to zero. Just as convergence gives rise to strong convergence, it was quite natural to expect that almost convergent will give rise to a new type of convergence, namely, strong almost convergence. This concept was introduced and discussed by Maddox. If \( [\hat{c}] \) denote the set of strongly almost convergent sequences, then Maddox defined

\[
[\hat{c}] = \left\{ x = (x_k) \in \omega : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l| = 0, \text{ uniformly in } S \text{ for some } l > 0 \right\}.
\]

Spaces of strongly almost summable sequences were generalized by Nanda [9], as strongly almost summable, strongly almost summable to zero, and strongly almost bounded, respectively, as follows: Suppose \( p = (p_k) \) be a sequence of strictly positive real number. Then

\[
[\hat{c}, p] := \left\{ x = (x_k) \in \omega : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l|^{p_k} = 0, \text{ uniformly in } S \text{ for some } l > 0 \right\},
\]

\[
[\hat{c}, p]_0 := \left\{ x = (x_k) \in \omega : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \text{ uniformly in } S \right\},
\]
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\[ [c, p]_\infty := \left\{ x = (x_k) \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+\delta} - \eta|^p_k < \infty, \text{ uniformly in } S \right\} \]

(see [9] and [10]).

Let $A = (a_{nk})$ be an infinite matrix of non-negative real numbers and $p = (p_k)$ be a sequence such that $0 < p_k \leq \sup_k p_k = H < \infty$. We write $Ax = (A_nx)$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk} |x_k|^p_k$ converges for all $n$, then

\[ [A, p]_0 = \{ x : A_n(x) \to 0(n \to \infty) \}, \]

\[ [A, p] = \{ x : A_n(x - l) \to 0(n \to \infty), \text{ for some } l > 0 \}, \]

\[ [A, p]_{\infty} = \left\{ x : \sup_n A_n(x) < \infty \right\}, \]

are the spaces of strongly $A$-summable to zero, strongly $A$-summable, and strongly bounded sequences, respectively, defined by Maddox (see [7]).

In this paper, we introduce the space of strongly almost $\Lambda^N_\nu$-summable sequences, which generalizes the following spaces defined by Esi and Polat [11]:

\[ \left[ B_{\Lambda^N_\nu}, p \right]_0 = \{ x : T_m(x) \to 0(m \to \infty) \}, \]

\[ \left[ B_{\Lambda^N_\nu}, p \right] = \{ x : T_m(x - l) \to 0(m \to \infty) \}, \]

and

\[ \left[ B_{\Lambda^N_\nu}, p \right]_{\infty} = \left\{ x : \sup_m T_m(x) < \infty \right\}, \]

where $T_m(x) = \sum_k a(k, m) |\Lambda^N_\nu x|^p_k$ and $a(k, m) = \sum_j b_{mj} a_{jk}$ with $b_{mj} a_{jk}$ being of the same sign for each $m, j, k$. They are spaces of strongly
\( \Delta^N \)-summable to zero, \( \Delta^N \)-summable, and bounded sequences, respectively, which depends on the fixed chosen matrix \( B \).

2. Main Results

Let \( F = (f_k) \) be a sequence of moduli, \( \nu = (\nu_k) \) be any fixed sequence of nonzero complex numbers for all \( k \in \mathbb{N} \), \( p = (p_k) \) be any sequence space of strictly positive real numbers, \( A = (a_{nk}) \) be any infinite matrix of non-negative real numbers, and \( B = (b_{mk}) \) be a fixed chosen matrix, then we define the following sequence spaces:

\[
\left[ \hat{B}_{\Delta^N}, F, p \right]_{c_0} = \left\{ x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( |\Delta^N x_{k+s}| \right) \right]^{p_k} \to 0 (n \to \infty), \text{uniformly in } S \right\},
\]

\[
\left[ \hat{B}_{\Delta^N}, F, p \right]_c = \left\{ x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( |\Delta^N x_{k+s} - \delta| \right) \right]^{p_k} \to 0 (n \to \infty), \text{uniformly in } S \right\},
\]

and

\[
\left[ \hat{B}_{\Delta^N}, F, p \right]_{l_c} = \left\{ x = (x_k) \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( |\Delta^N x_{k+s}| \right) \right]^{p_k} < \infty, \text{uniformly in } S \right\}.
\]

Now if \( A = B = I \) (unity matrix), then

\[
\left[ \hat{B}_{\Delta^N}, F, p \right]_{c_0} = \left[ \hat{c}_{\Delta^N}, F, p \right]_{c_0},
\]
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$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_c = \left[ \hat{c}_{\Lambda^N_{\nu}}, F, p \right]_c,$$

and

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_{l_\infty} = \left[ \hat{c}_{\Lambda^N_{\nu}}, F, p \right]_{l_\infty}$$

(see [12]).

In the same vein, if $B = I$ (unity matrix), $f_k(x) = x$ and we let $\Lambda^N_{\nu} x_k = x_k$ in the above definition, we get well-known sequence spaces (strongly $A$-summable sequences) as follows:

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_{c_0} = \left[ A, p \right]_{c_0},$$

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_c = \left[ A, p \right]_c,$$

and

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_{l_\infty} = \left[ A, p \right]_{l_\infty}$$

(see [7]).

Also if $f_k(x) = x$, $\nu = (v_k) = (1, 1, 1, \ldots)$, then

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_{c_0} = \left[ B_{\Lambda^N_{\nu}}, p \right]_{c_0},$$

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_c = \left[ B_{\Lambda^N_{\nu}}, p \right]_c,$$

and

$$\left[ \hat{B}_{\Lambda^N_{\nu}}, F, p \right]_{l_\infty} = \left[ B_{\Lambda^N_{\nu}}, p \right]_{l_\infty}$$

(see [11]).
By specializing \( A, B, f_k(x), N, \) and \( v \), we obtain the spaces defined in Nanda [10].

**Theorem 2.1.** Let \( p = (p_k) \) be a bounded sequence, \( F = (f_k(x)) \) is a sequence of moduli. Then, \( \hat{B}_{N, F}, p \) \( c_0 \), \( \hat{B}_{N, F}, p \) \( c \), and \( \hat{B}_{N, F}, p \) \( l_\infty \) are linear spaces.

**Proof.** We shall only prove for \( \hat{B}_{N, F}, p \) \( c_0 \), others can be treated similarly.

Now, let \( x, y \in \hat{B}_{N, F}, p \) \( c_0 \) and \( \lambda, \mu \in \mathbb{C} \), there exist positive integers \( M_\lambda \) and \( N_\mu \), such that \( \|x\| \leq M_\lambda \) and \( \|y\| \leq N_\mu \). Since \( \Delta_v^N \) is linear,

\[
\frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \Delta_v^N (\lambda x_{k+s} + \mu y_{k+s}) \right) \right]^{p_k} \leq D \left( \max \{1, |M_\lambda|^{H} \} \right) \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \Delta_v^N x_{k+s} \right) \right]^{p_k} + D \left( \max \{1, |N_\mu|^{H} \} \right) \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \Delta_v^N y_{k+s} \right) \right]^{p_k} \rightarrow 0 \quad \text{as} \quad n \to \infty,
\]

where \( H = \sup_k p_k < \infty \).

Hence, \( \hat{B}_{N, F}, p \) \( c_0 \) is a linear space.

**Theorem 2.2.** The space \( \hat{B}_{N, F}, p \) \( c_0 \) is a paranormed space, paranormed by
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$$g(x) = \sum_{i=1}^{N} a(i, m) (f_k(|v_i x_i|)) + \sup \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \left| \Lambda^N_\nu x_{k+s} \right| \right) \right]^{\frac{p_k}{M}} , \quad \text{where}$$

$$M = \max \left( 1, \sup_k p_k \right) ; \quad \text{and} \quad \left[ \hat{B}_{\Lambda^N_\nu} , F , p \right] \quad \text{and} \quad \left[ \hat{B}_{\Lambda^N_\nu} , F , p \right]_{l_c} \quad \text{are paranormed}$$

by $g$ if $\inf_k p_k > 0$.

**Proof.** Obviously, $g(x) = g(-x), \forall x \in \left[ \hat{B}_{\Lambda^N_\nu} , F , p \right]_{c_0}$.

It is trivial that $\Lambda^N_\nu x_k = \overline{0}$ for $x = \overline{0}$, where $\overline{0} = (\theta, \theta, \theta, \ldots)$ and $\theta$ is zero element of $X$. Since $q(\overline{0}) = 0$ and $f_k(0) = 0$, we obtain $g(\overline{0}) = 0$, but if $a_k$ and $b_k$ are complex numbers, then

$$|a_k + b_k| \leq |a_k| + |b_k| \leq \frac{p_k}{M}.$$

Since $M \geq 1$, the above inequality implies that

$$\sum_{i=1}^{N} a(i, m) (f_k(|v_i x_i|)) + \sup \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \left| \Lambda^N_\nu (x_{k+s} + y_{k+s}) \right| \right) \right]^{\frac{p_k}{M}}$$

$$\leq \sum_{i=1}^{N} a(i, m) (f_k(|v_i x_i|)) + \sup \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \left| \Lambda^N_\nu (x_{k+s}) \right| \right) \right]^{\frac{p_k}{M}}$$

$$+ \sum_{i=1}^{N} a(i, m) (f_k(|v_i y_i|)) + \sup \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \left| \Lambda^N_\nu (y_{k+s}) \right| \right) \right]^{\frac{p_k}{M}}.$$

It follows that $g$ is subadditive. Next, let $\lambda$ be a nonzero scalar. The continuity of scalar multiplication follows from the inequality:

$$g(\lambda x) \leq k_1 \sum_{i=1}^{N} a(i, m) (f_k(|v_i x_i|)) + \sup \frac{1}{n} \left( \frac{p_k}{k_1^M} \right) \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \left| \Lambda^N_\nu x_{k+s} \right| \right) \right]^{\frac{p_k}{M}}$$

$$\leq \max \left( 1, \frac{p_k}{k_1^M} \right) g(x),$$

where $k_\lambda$ is an integer such that $|k| < k_\lambda$. 
Hence, the proof is complete and the others follow suit by applying the same techniques.

**Theorem 2.3.** For a sequence \( F = (f_k) \) of moduli, the following statements are equivalent:

(i) \[ \left\lfloor \hat{B}_{\Lambda_N}, p \right\rfloor_{l_{\infty}} \subseteq \left[ B_{\Lambda_N}, F, p \right]_{l_{\infty}}. \]

(ii) \[ \left\lfloor \hat{B}_{\Lambda_N}, p \right\rfloor_{c_0} \subseteq \left[ B_{\Lambda_N}, F, p \right]_{c_0}. \]

(iii) \[ \sup_{n} \left( \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} \right) < \infty (t > 0). \]

**Proof.** It is clear that (i) \( \Rightarrow \) (ii).

Now we prove that (ii) \( \Rightarrow \) (iii): Let \( \left\lfloor \hat{B}_{\Lambda_N}, p \right\rfloor_{c_0} \subseteq \left[ B_{\Lambda_N}, F, p \right]_{c_0} \).

Suppose that (iii) is not true. Then for some \( t \)

\[ \sup_{n} \left( \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} \right) = \infty, \]

and there exists a sequence \( (n_i) \) of positive integers such that

\[ \frac{1}{n_i} \sum_{k=1}^{n_i} a(k, m)f_k\left( \frac{1}{i} \right) > i \text{ for } i = 1, 2, 3, \ldots. \quad (1) \]

Now, we define \( x = (x_k) \) by

\[
x_k = \begin{cases} 
\frac{1}{i}, & \text{if } 1 \leq k \leq n_i, \quad i = 1, 2, 3, \ldots, \\
0, & \text{if } k > n_i.
\end{cases}
\]

Then \( x \in \left\lfloor \hat{B}_{\Lambda_N}, p \right\rfloor_{c_0} \) but by Equation (1), \( x \notin \left[ B_{\Lambda_N}, F, p \right]_{l_{\infty}} \), which contradict (ii).
Hence (iii) is true.

Next we prove that (iii) $\Rightarrow$ (i):

Suppose (iii) is true and $x \in \left[ \mathring{B}_{\Delta^N}, p \right]_{l_\infty}$. If we assume that $x \notin \left[ B_{\Delta^N}, F, p \right]_{l_\infty}$, then

$$\sup_{s, n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \Delta^N_{\nu} x_{k+s} \right) \right]^{p_k} = \infty. \quad (2)$$

If we take $t = \left| \Delta^N_{\nu} x_{k+s} \right|$ for each $k$ and fixed $s$, then by Equation (2),

$$\sup_{s, n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \Delta^N_{\nu} x_{k+s} \right) \right]^{p_k} = \infty,$$

which contradicts (iii). Hence $\left[ \mathring{B}_{\Delta^N}, p \right]_{l_\infty} \subseteq \left[ B_{\Delta^N}, F, p \right]_{l_\infty}$.

**Theorem 2.4.** Let $1 \leq p_k \leq \sup_k p_k < \infty$ for a sequence of moduli $F = (f_k)$, the following statements are equivalent:

(i) $\left[ \mathring{B}_{\Delta^N}, F, p \right]_{l_0} \subseteq \left[ B_{\Delta^N}, p \right]_{l_0}$,

(ii) $\left[ \mathring{B}_{\Delta^N}, F, p \right]_{c_0} \subseteq \left[ B_{\Delta^N}, p \right]_{c_0}$,

(iii) $\inf_n \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} > 0 (t > 0)$.

**Proof.** Clearly (i) $\Rightarrow$ (ii):

Now we prove (ii) $\Rightarrow$ (iii): Let $\left[ \mathring{B}_{\Delta^N}, F, p \right]_{c_0} \subseteq \left[ B_{\Delta^N}, p \right]_{l_\infty}$. Suppose that (iii) does not hold. Then
\[
\inf \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = 0 \quad (t > 0), \quad (3)
\]

and there exists a sequence \((n_i)\) of positive integers such that

\[
\frac{1}{n_i} \sum_{k=1}^{n_i} a(k, m) [f_k(i)]^{p_k} < \frac{1}{i}, \quad \text{for } i = 1, 2, 3, \ldots.
\]

Now, we define the sequence \(x = (x_k)\) by

\[
x_k = \begin{cases} 
  i, & \text{if } 1 \leq k \leq n_i, \text{ for } i = 1, 2, 3, \ldots, \\
  0, & k > n_i.
\end{cases}
\]

By Equation (3), \(x \in \tilde{B}_{\Delta_i^N, F, p} c_0\) but \(x \notin \tilde{B}_{\Delta_i^N, p} c_0\), which contradicts (ii).

Hence (iii) is true.

Next we prove that (iii) \(\Rightarrow\) (i): Let (iii) be true and \(x \in \tilde{B}_{\Delta_i^N, F, p} c_0\), i.e.,

\[
\lim \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(\Delta_i^N x_{k+s})]^{p_k} = 0, \text{ uniformly in } S. \quad (4)
\]

Suppose that \(x \notin \tilde{B}_{\Delta_i^N, p} c_0\). Then for some number \(\varepsilon_0 > 0\) and positive integer \(n_0\), we have \(|\Delta_i^N x_{k+s}| \geq \varepsilon_0\) for some \(s \geq s'\) and \(1 \leq k \leq n_0\). Therefore,

\[
[f_k(\varepsilon_0)]^{p_k} \leq [f_k(\Delta_i^N x_{k+s})]^{p_k},
\]

and hence \(\lim \frac{1}{n} \sum_{k=1}^{n} [f_k(\varepsilon_0)]^{p_k} = 0\), which contradicts (iii). Thus

\[
\tilde{B}_{\Delta_i^N, F, p} c_0 \subseteq B_{\Delta_i^N, p} c_0.
\]
Theorem 2.5. Let $1 \leq p_k \leq \sup n p_k < \infty$. The inclusion $\left[ \hat{B}_{\Lambda^N}, F, p \right]_{l_\infty} \subseteq \left[ B_{\Lambda^N}, p \right]_{c_0}$ holds, if and only if $\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = \infty$ for $t > 0$.

Proof. Let $\left[ \hat{B}_{\Lambda^N}, F, p \right]_{l_\infty} \subseteq \left[ B_{\Lambda^N}, p \right]_{c_0}$. Suppose $\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = \infty$, for $t > 0$ does not hold. Then, there exists a number $t_0 > 0$ and the sequence $(n_i)$ of positive integer such that

$$\frac{1}{n_i} \sum_{k=1}^{n_i} a(k, m) [f_k(t)]^{p_k} \leq M < \infty, \quad i = 1, 2, 3, \ldots$$

(5)

Now we define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, & \text{if } 1 \leq k \leq n_i, \text{ for } i = 1, 2, 3 \ldots, \\ 0, & k > n_i. \end{cases}$$

Thus by Equation (5), $x \in \left[ \hat{B}_{\Lambda^N}, F, p \right]_{l_\infty}$ but $x \notin \left[ B_{\Lambda^N}, p \right]_{c_0}$, so that

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = \infty$$

must hold.

Conversely, let $\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = \infty$ holds. If $x \in \left[ \hat{B}_{\Lambda^N}, F, p \right]_{l_\infty}$, then for each $s$ and $n$

$$\frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k\left(\left[ \Lambda^N x_{k+s} \right]\right)]^{p_k} \leq M < \infty.$$  

(6)
Suppose that $x \in \left[ \mathring{B}_{\Delta}^N, p \right]_{c_0}$. Then for some numbers $\varepsilon_0 > 0$ and positive integer $s_0$ and index $n_0$, we have $|\Delta^N x_{k+s}| \geq \varepsilon_0$ for $s \geq s_0$, therefore, $[f_k(\varepsilon_0)]^{p_k} \leq [f_k(\Delta^N x_{k+s})]^{p_k}$, and hence for each $k$ and $s$, we get

$$\frac{1}{n} \sum_{k=1}^{n} [f_k(\varepsilon_0)]^{p_k} \leq M < \infty \text{ for } M > 0,$$

by Equation (6), which contradicts

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(\Delta^N x_{k+s})]^{p_k} = \infty.$$

Hence

$$\left[ \mathring{B}_{\Delta}^N, F, p \right]_{c_0} \subseteq \left[ B_{\Delta}^N, p \right]_{c_0}.$$

**Theorem 2.6.** Let $1 \leq p_k \leq \sup_n p_k < \infty$. The inclusion $\left[ \mathring{B}_{\Delta}^N, p \right]_{c_0} \subseteq \left[ B_{\Delta}^N, F, p \right]_{c_0}$ holds, if and only if

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = 0 \text{ for } t > 0.$$

**Proof.** Suppose that $\left[ \mathring{B}_{\Delta}^N, F, p \right]_{c_0} \subseteq \left[ B_{\Delta}^N, F, p \right]_{c_0}$, but

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) [f_k(t)]^{p_k} = l \neq 0.$$

Define the sequence $x = (x_k)$ by

$$x_k = t_o \sum_{k=1}^{k-u} (-1)^{N+k-u-1} \binom{N+k-u-1}{k-u}, \text{ for } k = 1, 2, \ldots,$$

then $x \in \left[ B_{\Delta}^N, F, p \right]_{c_0}$, by Equation (5).
Hence the equation in Theorem 2.5 above must hold.

Conversely, let \( \left[ B_{\Lambda^N}, p \right] \), and suppose that the equation in Theorem 2.6 holds. Then for every \( k \) and \( s \), \( \left| \Lambda^N x_{k+s} \right| \leq M < \infty \). Therefore,

\[
\left[ f_k \left( \left| \Lambda^N x_{k+s} \right| \right) \right]^{p_k} \leq \left[ f_k (M) \right]^{p_k},
\]

and

\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k \left( \left| \Lambda^N x_{k+s} \right| \right) \right]^{p_k} \leq \lim_{n} \frac{1}{n} \sum_{k=1}^{n} a(k, m) \left[ f_k (M) \right]^{p_k} = 0.
\]

By the hypothesis of Theorem 2.6. Hence \( x \in \left[ B_{\Lambda^N}, F, p \right] \).

References


