EXISTENCE RESULTS FOR VECTOR VARIATIONAL-LIKE INEQUALITIES

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Abstract

In this paper, we consider and study a class of vector variational-like inequalities in Banach space without any generalized monotonicity by exploiting vector version of minimax inequality and obtain the existence results of solutions to the class of vector variational-like inequalities. The results presented here are different from [1, 5, 11], and extend and generalize the corresponding results in [7].

1. Introduction

A vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [6] in 1980. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Later on, vector variational inequalities have been investigated in abstract spaces, see [2, 3, 9]. It is

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worth noting that vector variational-like inequalities are important
generalization of vector variational inequalities related to the class of
connected sets which is much more general than the class of convex
sets (see [8, 10, 11]). Moreover, Under the monotonicity conditions, the
authors in [1, 5, 11] studied the vector variational (variational-like)
inequalities by using K-Fan lemma. On the other, without any
generalized monotonicity, the vector variational inequalities are studied
by using the Brouwer and Browder fixed pointed theorems in [1, 5],
respectively and in [7], Lai and Yao studied the existence of solutions of
the vector variational inequalities by minimax inequality due to Fan [4].

Inspired and motivated by the above research work, we study the
existence of the solutions of vector variational-like inequalities without
any monotonicity by using vector version of minimax inequality. The
results obtained in this paper are different from the corresponding
results in [1, 5, 11] and extend and generalize the corresponding results
in [7].

2. Preliminaries

Let $X$ be a Banach space. A nonempty subset $P$ of $X$ is called a
pointed, convex cone if $P + P \subset P$, $tP \subset P$ for all $t \geq 0$ and
$P \cap (-P) = \{0\}$. The partial order "\leq" on $X$ induced by a pointed cone is
defined by declaring $x \leq y$ if and only if $y - x \in P$ for all $x, y \in X$, and
in this case $P$ is called a positive cone in $X$. Furthermore, if such a partial
order is induced by a convex cone, it is called a linear order. A ordered
Banach space is a pair $(X, P)$, where $X$ is a real Banach space and $P$ is a
pointed convex cone. With linear order induced by $P$, the weak order "\preceq"
on ordered Banach space $(X, P)$ with $int P \neq \emptyset$ is defined as $x \preceq y$ if
and only if $y - x \not\in int P$ for all $x, y \in X$ where "int" denotes the interior.

Let $X$ and $Y$ be real Banach spaces. $L(X, Y)$ is the space of all
bounded linear mappings from $X$ into $Y$. We denote by $(l, x)$ the value of
$l \in L(X, Y)$ at $x \in X$. Let $K$ be a nonempty closed and convex subset of
$X$, $T : K \to L(X, Y)$ be a single-valued mapping, and a set-valued
mapping $C : K \to 2^Y$ be such that $C(x)$ is a closed, pointed and convex
cone of $Y$ with $intC(x) \neq \emptyset$ for all $x \in K$ and $\eta : K \times K \rightarrow X$ be two vector-valued mapping. In this paper, we consider the vector variational-like inequality problem, (denoted by VVIP) that is to find $x \in K$ such that

$$(Tx, \eta(y, x)) \notin -intC(x), \forall y \in K. \quad (2.1)$$

When $C(x) = P$ for all $x \in K$ and $(Y, P)$ is an ordered Banach space with weak order, (VVLI) becomes (VVLI$'$), that is to find $x \in K$ such that

$$(Tx, \eta(y, x)) < 0, \forall y \in K. \quad (2.2)$$

Furthermore, when $\eta(y, x) = y - x$, (VVLI) reduces to (VVI), that is to find $x \in K$ such that

$$(Tx, y - x) \notin -intC(x), \forall y \in K. \quad (2.3)$$

Lai and Yao [7] studied the existence of solution of vector variational-like inequalities (2.3) by minimax inequality in the case of nonmonotonicity conditions. In our paper, we study the existence results of (VVLI), which extend and generalize the results of [7] and different from the results of [1, 5, 11].

3. Main Results

In this section, we state and prove the existence results for vector variational-like inequalities without any generalized monotonicity assumption. To this end, the following result will be used.

**Lemma 3.1** [4]. Let $E$ be a nonempty compact convex set of a Hausdorff topological vector space. Let $A$ be a subset of $E \times E$ having the following properties:

(i) $(x, x) \in A$ for all $x \in K$;

(ii) for each $x \in E$, the set $A_x = \{y \in E | (x, y) \in A\}$ is closed in $E$;

(iii) for each $y \in E$, the set $A_y = \{x \in E | (x, y) \notin A\}$ is convex.
Then there exists $y_0 \in E$ such that $E \times \{y_0\} \subset A$.

Now we can state and prove the main results of this paper.

**Theorem 3.2.** Let $X$ and $Y$ be real Banach spaces. Let $K$ be nonempty weakly compact convex subset of $X$. Let $C : K \to 2^Y$ be a set-valued mapping such that for all $x \in K$, $C(x)$ is a closed, pointed and convex cone in $Y$ with $\text{int} C(x) \neq \emptyset$, and a set-valued mapping $W : K \to 2^Y$ be defined by $W(x) = Y \setminus (\text{int} C(x))$ such that the graph of $W$ denoted by $\text{gph} W$ is weakly closed in $X \times Y$. Let $T : K \to L(X, Y)$ be a single-valued mapping such that for all $x \in K$, the mapping $y \mapsto (Ty, \eta(x, y))$ is continuous from the weak topology of $K$ to the weak topology of $Y$. Let $\eta : K \times K \to X$ be a vector-valued mapping such that

(a) $\eta(x, x) = 0$, $\forall x \in K$;

(b) $\eta(x, y)$ is affine with respect to $x$ if, for any given $y \in K$,

$$
\eta(tx_1 + (1-t)x_2, y) = t\eta(x_1, y) + (1-t)\eta(x_2, y), \quad \forall x_1, x_2 \in K, \ t \in R,
$$

with $x = tx_1 + (1-t)x_2 \in K$. Then there exists $x_0 \in K$ such that

$$(Tx_0, \eta(x, x_0)) \notin \text{int} C(x_0), \quad \forall x \in K.$$ 

**Proof.** Let $A = \{(x, y) \in K \times K | (Ty, \eta(x, y)) \notin \text{int} C(y)\}$. Then, it is clear that $(x, x) \in A$ for each $x \in K$. Next we show that for each $x \in K$, the set $A_x = \{y \in K | (x, y) \in A\}$ is weakly closed. To this end, let $\{y_\alpha\}$ be a net in $A_x$ converging weakly to some $y \in K$. For each $\alpha$, since $(x, y_\alpha) \in A$, we have

$$(Ty_\alpha, \eta(x, y_\alpha)) \notin \text{int} C(y_\alpha) \text{ or } (Ty_\alpha, \eta(x, y_\alpha)) \in y \setminus (\text{int} C(y_\alpha)).$$

By assumption, $(Ty_\alpha, \eta(x, y_\alpha))$ converges weakly to $(Ty, \eta(x, y))$. Since $\text{gph} W$ is weakly closed in $X \times Y$ we have

$$(Ty, \eta(x, y)) \in Y \setminus (\text{int} C(y)) \text{ or } (Ty, \eta(x, y)) \notin \text{int} C(y).$$

Thus, $y \in A_x$ and consequently $A_x$ is weakly closed.
Finally, we show that for each \( y \in K \), the set \( A_y = \{ x \in K \mid (x, y) \notin A \} \) is convex. To this end, let \( x_1, x_2 \in A_y \) and \( t_1 \geq 0, t_2 \geq 0 \) with \( t_1 + t_2 = 1 \). Then, \( A_y \) is convex. Since

\[
(T_y, t_1 \eta(x_1, y)) \in -\text{int}C(y),
\]

\[
(T_y, t_2 \eta(x_2, y)) \in -\text{int}C(y).
\]

As \( C(y) \) is convex cone and the condition of (b), we have

\[
(T_y, \eta(t_1 x_1 + t_2 x_2, y)) \in -\text{int}C(y)
\]

hence, \( t_1 x_1 + t_2 x_2 \in A_y \), and therefore \( A_y \) is convex.

Now by invoking Lemma 3.1, there exists \( x_0 \in K \) such that \( K \times \{ x_0 \} \subset A \). This implies that \( x_0 \in K \) and

\[
(Tx_0, \eta(x, x_0)) \notin \text{int}C(x_0) \quad \forall x \in K,
\]

which implies that the (VVLI) has a solution. This completes the proof.

We can derive the following corollary from Theorem 3.2.

**Corollary 3.3.** Let \( X \) and \( Y \) be real Banach spaces. Let \( K \) be a nonempty compact convex subset of \( X \). Let \( C : K \to 2^Y \) be a set-valued mapping such that for each \( x \in K \), \( C(x) \) is a closed pointed and convex cone and \( \text{int}C(x) \neq \emptyset \), and \( W : K \to 2^Y \) be defined by \( W(x) = Y \setminus (-\text{int}C(x)) \) such that \( gphW \) is weakly closed in \( X \times Y \). Let \( T : K \to L(X, Y) \) be continuous from the weak topology of \( K \) to the norm topology of \( Y \). Let \( \eta : K \times K \to X \) be such that

(a) \( \eta(x, x) = 0, \quad \forall x \in K \);

(b) \( \eta(x, y) \) is affine with respect to \( x \) if, for any given \( y \in K \),

\[
\eta(tx_1 + (1 - t)x_2, y) = t\eta(x_1, y) + (1 - t)\eta(x_2, y), \quad \forall x_1, x_2 \in K, \ t \in R,
\]

with \( x = tx_1 + (1 - t)x_2 \in K \). Then there exists \( x_0 \in K \) such that

\[
(Tx_0, \eta(x, x_0)) \notin \text{int}C(x_0) \quad \text{for all} \ x \in K;
\]
(c) \( \forall x \in K, \eta(x, y) \) is weakly continuous in the first argument.

Then there exists \( x_0 \in K \) such that

\[
(Tx_0, \eta(x, x_0)) \notin \text{int}C(x_0), \forall x \in K.
\]

**Proof.** It suffices to check that for each \( x \in K \), the mapping \( y \mapsto (Ty, \eta(x, y)) \) is continuous from weak topology of \( K \) to the weak topology of \( Y \). To this end, let \( x \in K \) be arbitrary but fixed, and let \( T_x : K \to Y \) be defined by \( T_x y = (Ty, \eta(x, y)) \), \( \forall y \in K \). Let \( \{y_\alpha\} \) be any net in \( K \) converging weakly to some \( y \in K \). By assumption, we have

\[
|Ty_\alpha - Ty|_{L(X,Y)} \to 0.
\]

Since the net \( \{y_\alpha\} \) is weakly convergent and the condition of (c), it is bounded. Therefore,

\[
|(Ty_\alpha - Ty, \eta(x, y_\alpha))| \leq \|Ty_\alpha - Ty\|_{L(X,Y)} \|\eta(x, y_\alpha)\| \xrightarrow{X} 0
\]

and hence \( (Ty_\alpha - Ty, \eta(x, y_\alpha)) \) converges weakly to 0 in \( Y \). On the other hand, as \( Ty \in L(X,Y) \), \( Ty \) is continuous from the weak topology of \( X \) to the weak topology of \( Y \). Consequently, we have

\[
T_x y_\alpha = (Ty_\alpha, \eta(x, y_\alpha)) = (Ty_\alpha - Ty, \eta(x, y_\alpha)) + (Ty, \eta(x, y_\alpha))
\]

converges weakly to \( (Ty, \eta(x, y)) = T_x y \). Hence the operator \( T_x \) is continuous from the weak topology of \( K \) to the weak topology of \( Y \). The result then follows from Theorem 3.2.

From Corollary 3.3, we have the following result.

**Corollary 3.4.** Let \( X \) be a real Banach space, \( (Y, C) \) be an ordered Banach space, where \( C \) is a pointed, closed and convex cone in \( Y \) with \( \text{int}C(x) \neq \emptyset \), such that \( Y \setminus (-\text{int}C) \) is weakly closed. Let \( K, T, \eta \) be as in Corollary 3.3. Then there exists \( x_0 \in K \) such that

\[
(Tx_0, \eta(x, x_0)) \notin \text{int}C(x_0), \forall x \in K.
\]

**Remark 3.5.** Theorem 3.2, Corollaries 3.3 and 3.4 extend and generalize the corresponding results in [7].
References


