SOLVING PARTIAL DIFFERENTIAL EQUATIONS BY
MESHLESS PETROV-GALERKIN METHOD
COMBINED WITH RADIAL BASIS FUNCTIONS
AND LAGRANGE POLYNOMIALS

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Abstract

In this paper, we introduce a new meshless method for the numerical solution of partial differential equations. It is based on the Petrov-Galerkin method, in which the trial space is generated by the global radial basis functions, and the test space is generated by the Lagrange polynomials. This method of solution is also based on the unsymmetric weak formulation.

1. Introduction

Radial basis functions (RBFs) are well-known as traditional and powerful tools for multivariate interpolation from scattered data. In the past decade or so RBFs have also received increased attention as a meshless method for numerically solving partial differential equations (PDEs). The idea of using RBFs for solving PDEs was first proposed by
Kansa [3] who directly collocated the RBFs for approximate solution of the equations.

A meshfree method does not require a mesh to discretise the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. Up to now, many useful meshless methods have been provided, such as meshless Galerkin method [7], collocation method [3], subdomains method [6] and the meshless Petrov-Galerkin method [9].

In the present paper, the meshless Petrov-Galerkin method is based on the unsymmetric weak form, in which the trial space is generated by the global radial basis functions and the test space is generated by the Lagrange polynomials.

The organization of this paper is as follows: In Section 2, we review basic feature of multivariate interpolation by radial basis functions, then the meshless method is presented in Section 3. In Section 4, we present our numerical experiment, and the last Section contains conclusions.

2. Radial Basis Function Interpolation

In order to explain multivariate scattered data interpolation by radial basis functions, suppose a data vector \( f|_X = (f(x_1), \ldots, f(x_N))^T \in \mathbb{R}^n \) of function values, sampled from an unknown function \( f : \mathbb{R}^n \to \mathbb{R} \) at a scattered finite point set \( X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n, n \geq 1 \), is given. Scattered data interpolation requires computing a suitable interpolant \( Sf : \mathbb{R}^n \to \mathbb{R} \) satisfying \( Sf|_X = f|_X \). To this end, the interpolant \( Sf \) is assumed to have the form

\[
Sf(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|) + \sum_{k=1}^{M} c_k p_k(x), \quad x \in \mathbb{R}^n, \tag{1}
\]

where \( \|\| \) is the Euclidean norm on \( \mathbb{R}^n \) and \( \phi : [0, \infty) \to \mathbb{R} \) is a radial function. Moreover, \( p_1, \ldots, p_M \) form a basis for the \( M = \binom{m-1+n}{m-1} \).
dimensional linear space $\prod_{m-1}^{n}$ of all real-valued polynomials in $n$ variable of degree at most $m - 1$, where $m = m(\phi)$ is said to be the order of the basis function $\phi$.

Since enforcing the interpolation conditions $Sf(x_i) = f(x_i), i = 1, \ldots, N$ leads to a system of $N$ linear equations in the $N + M$ unknowns $\lambda_i$ and $c_k$ one usually adds the $M$ additional conditions

$$\sum_{i=1}^{N} \lambda_i p_j(x_i) = 0, \quad j = 1, 2, \ldots, M,$$

to ensure a unique solution.

In general, solving the interpolation problem based on the extended expansion (1) now amounts to solving a system of linear equations of the form

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix} = \begin{pmatrix} f_X \\ 0 \end{pmatrix},$$

where the pieces are given by $A_{ij} = \phi(\|x_j - x_i\|), i, j = 1, \ldots, N; P_{ik} = p_k(x_i), i = 1, \ldots, N, k = 1, \ldots, M; \Lambda = [\lambda_1, \ldots, \lambda_N]^T; C = [c_1, \ldots, c_M]^T$ and 0 is a zero vector of length $M$.

**Definition 2.1.** A function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is said to be conditionally positive definite of order $m$ iff for all sets $X = \{x_1, \ldots, x_N\} \subseteq \mathbb{R}^n$ consisting of pairwise distinct centers $x_j$ and all $\alpha \in \mathbb{R}^N \setminus \{0\}$ satisfying

$$\sum_{j=1}^{N} \alpha_j x_j^p = 0, \quad |p| < m, \quad p \in \mathbb{N}_0^n,$$

the inequality

$$\sum_{j, i=1}^{N} \alpha_j \alpha_i \Phi(x_j - x_i) > 0,$$

is valid. A conditionally positive definite function of order 0 is called a positive definite function.
From [4] and [5] we know that we have an unique interpolant \( Sf(x) \) of \( f \) if \( \phi \) is a conditional positive definite radial basis function of order \( m \).

The most prominent examples of conditional positive definite radial basis function of order \( m \) on \( \mathbb{R}^n \) are:

i. \((-1)^{\left\lceil \frac{\beta}{2} \right\rceil} \frac{r^\beta}{\beta!} \), \( \beta > 0 \), \( \beta \not\in 2\mathbb{N} \), \( m \geq \left\lceil \frac{\beta}{2} \right\rceil \) Polyharmonic splines,

ii. \((-1)^{k+1} r^{2k} \log(r) \), \( k \in \mathbb{N} \), \( m \geq k + 1 \), Polyharmonic splines,

iii. \((-1)^{\beta} (c^2 + r^2)^{\beta} \), \( \beta > 0 \), \( \beta \not\in \mathbb{N} \), \( m \geq \lceil \beta \rceil \) Multiquadrics,

iv. \((c^2 + r^2)^{\beta} \), \( \beta < 0 \), \( m \geq 0 \) Inverse Multiquadrics,

v. \( e^{-cr^2} \), \( c > 0 \), \( m \geq 0 \) Gaussian,

vi. \((1-p)^4(1+4r) \), \( n \leq 3 \), \( m \geq 0 \) Compactly support,

where for any \( x \in \mathbb{R} \), the symbol \( \lceil x \rceil \) denotes as usual the smallest integer greater than or equal to \( x \).

The convergence proofs in applying the RBFs for scattered data interpolation was given by Wu and Schaback [8]. For more details about radial basis functions see references [2] and [5].

### 3. The Meshless Petrov-Galerkin Method

Consider the linear Poisson’s equation, which can be written as

\[
\nabla^2 u(x) = f(x) \quad x \in \Omega, \tag{2}
\]

where \( f \) is a given source function, and the domain \( \Omega \) is enclosed by \( \Gamma = \Gamma_u \cup \Gamma_q \), with boundary conditions

\[
u = \bar{u} \quad \text{on} \; \Gamma_u, \tag{3a}
\]

and

\[
\frac{\partial u}{\partial n} = q = \bar{q} \quad \text{on} \; \Gamma_q, \tag{3b}
\]
where \( \mathbf{u} \) and \( \mathbf{q} \) are the prescribed potential normal flux respectively, on the boundaries \( \Gamma_u \) and \( \Gamma_q \); and \( n \) is the outward normal direction to the boundary \( \Gamma \).

An unsymmetric weak formulation (USWF) of the problem may be written as
\[
\int_{\Omega} (\nabla^2 \mathbf{u} - f) \mathbf{v} d\Omega = 0, \tag{4}
\]
where \( \mathbf{u} \) is the trial function, and \( \mathbf{v} \) is the test function. This USWF requires that \( \mathbf{u} \) can be at least \( C^1 \) continuous, while \( \mathbf{v} \) may be discontinuous, and hence is the label, “unsymmetric weak formulation”, in as much as the continuity requirements on \( \mathbf{u} \) and \( \mathbf{v} \) are not the same.

Using the divergence theorem, we obtain the following symmetric weak formulation (SWF),
\[
\int_{\Gamma} \frac{\partial \mathbf{u}}{\partial n} \mathbf{v} d\Gamma - \int_{\Omega} (\nabla \mathbf{u} \nabla \mathbf{v} + p \mathbf{v}) d\Omega = 0. \tag{5}
\]
The SWF requires that both \( \mathbf{u} \) and \( \mathbf{v} \) be \( C^0 \) continuous, and the natural boundary “symmetric weak formulation”. Imposing the natural boundary condition (3b) in Eq. (5), we obtain
\[
\int_{\Gamma_u} q \mathbf{v} d\Gamma + \int_{\Gamma_q} \mathbf{q} v d\Gamma - \int_{\Omega} (\nabla \mathbf{u} \nabla \mathbf{v} + p \mathbf{v}) d\Omega = 0. \tag{6}
\]
For more details about weak forms, one can see [1] and [2].

The present petrov-Galerkin method is based on the unsymmetric weak form (4) so that it starts with finite dimensional spaces \( U_N \) and \( V_N \); \( U_N \) trial space is generated with the global RBFs, and \( V_N \) test space is generated with the Lagrange polynomials, and it is presented on the basis of SWF (6) by Zhang [9], in which the trial space is generated by the global RBFs and the test space is generated by the compactly supported RBFs.

In USWF (4), the collocation approach is used to impose both the essential as well as natural boundary conditions.
4. Numerical Experiment

Consider partial differential equation (2) with the exact solution

$$u = \sin(x + y),$$

(7)

where the potential boundary condition Eq. (3a) is prescribed on all boundaries according to Eq. (7); $\Omega = [1, 2] \times [3, 4]$ and $f = -2 \sin(x + y)$.

In this numerical experiment, we choose $n = n_x \times n_y$ collocation points all together ($n_x$ points on each horizontal lines and $n_y$ points on each vertical lines), and

$$U_N = \{u_N : u_N = \sum_{i=1}^{N} \alpha_i \phi(\|x - x_i\|)\},$$

$$V_N = \{v_N : v_N = \sum_{i=1}^{N} \beta_i L_i(x)\},$$

where $L_i(x)$ is Lagrange polynomials in $x_i$ points.

Numerical results obtained by this method are displayed in Tables 1, 2 and 3.

Remark

(1) The expression $E$ in Tables is

$$E = \frac{1}{wcs} \sum_{i=1}^{wcs} \left| \frac{u_{num}(x_i) - u_{exact}(x_i)}{u_{exact}(x_i)} \right|,$$

where $\{x_i\}_{i=1}^{wcs}$ are $wcs$ uniformly distributed points in $\Omega$; we always set $wcs = 2n$ in this numerical experiment.

(2) In the numerical computation, $\phi$ can be any other radial basis functions.

(3) The expression $E_{max}$ in Tables is maximum error of $wcs$ points in $\Omega$.

(4) We use Gauss quadratures method for the computation of integrals where $m$ is Gaussian points number.
### Table 1. Solving PDEs by Petrov-Galerkin method combined with $\phi(r) = (r^2 + 1)^{0.5}$ and Lagrange polynomials

<table>
<thead>
<tr>
<th>$n_x$</th>
<th>$n_y$</th>
<th>$n$</th>
<th>$m$</th>
<th>wcs</th>
<th>$E$</th>
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### Table 2. Solving PDEs by Petrov-Galerkin method combined with $\phi(r) = (r^2 + 1)^{1.5}$ and Lagrange polynomials

<table>
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<tr>
<th>$n_x$</th>
<th>$n_y$</th>
<th>$n$</th>
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### Table 3. Solving PDEs by Petrov-Galerkin method combined with $\phi(r) = (r^2 + 0.5)^{1.5}$ and Lagrange polynomials

<table>
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<tr>
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5. Conclusions

In this paper, we have provided a new meshless method for the numerical solution of the partial differential equations. The numerical experiment indicate that it is an accurate and efficient numerical method. As we have seen, in this method of solution, cost of computations is less than Zhang Petrov-Galerkin method [9].

References