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TOPOLOGICAL INVARIANTS IN DIGITAL IMAGES

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Abstract

Topological invariants are very useful in various areas related to digital images and geometric modelling. In this paper, we study the simplicial homology groups of certain minimal simple closed surfaces, extend an earlier definition of the Euler characteristics of digital image, and show how to compute the Euler characteristic of several digital surfaces. In Example 4.4, we correct an error that appears in [16].

1. Introduction

Digital topology, introduced by Rosenfeld [21], plays an important role in computer vision, image processing, and computer graphics. As a result, many researchers (Rosenfeld, Kong, Kopperman, Kovalavsky, Malgouyres, Boxer, Chen, Rong, Kacynski, Mischaikow, Mrozek, Han, Karaca, and others) wish to characterize the properties of digital images with tools from topology (including algebraic topology).

In algebraic topology, the definition of homology groups is more sophisticated and less intuitive than the definition of homotopy groups. The digital simplicial homology group is an important tool for image analysis because a general algorithm to decide whether two distinct objects have isomorphic homology groups could be a very powerful tool for image analysis. Therefore, it is desirable to study the simplicial homology groups of digital images.

In [12], Chen and Rong have designed linear time algorithms to recognize and determine topological invariants such as the genus and homology groups in 3D. These properties can be used to identify patterns in 3D image recognition. They use Alexander duality to obtain the homology groups of a 3D object in 3D space. Several researchers have studied simplicial homology groups of digital images; see, e.g., [1], [18]. The current paper builds on [1] to expand our knowledge of the simplicial homology groups of digital images.

This paper is organized as follows. Some basic notions are provided in Section 2. In the next section, we compute the simplicial homology groups of certain minimal simple closed surfaces and present results of Arslan et

al. [1]. In the last section, we define Euler characteristics of digital images as a more general notion and compute Euler characteristics of certain digital surfaces. Among our results is a correction of an assertion of [16].

2. Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space, where $\mathbb{Z}$ is the set of integers. A (binary) digital image is a subset of $\mathbb{Z}^n$ with an adjacency relation. We use a variety of adjacency relations in the study of digital images.

Definition 2.1 [19]**.** (1) Two points p and q in $\mathbb{Z}$ are 2-adjacent, if $|p - q| = 1$ (see Figure 1).

(2) Two points p and q in $\mathbb{Z}^2$ are 8-adjacent, if they are distinct and differ by at most 1 in each coordinate (see Figure 2).

(3) Two points p and q in $\mathbb{Z}^2$ are 4-adjacent, if they are 8-adjacent and differ in exactly one coordinate (see Figure 2).

(4) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent, if they are distinct and differ by at most 1 in each coordinate (see Figure 3).

(5) Two points p and q in $\mathbb{Z}^3$ are 18-adjacent, if they are 26-adjacent and differ in at most two coordinates (see Figure 3).

(6) Two points p and q in $\mathbb{Z}^3$ are 6-adjacent, if they are 18-adjacent and differ in exactly one coordinate (see Figure 3).

The numbers $\{2, 8, 4, 26, 18, 6\}$ reflect the number of adjacent points, e.g., in $\mathbb{Z}^2$ a point has 8 8-adjacent points. More general, adjacency relations are studied in [17].

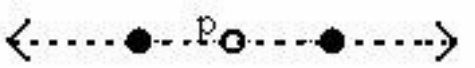

Figure 1. 2-adjacency.

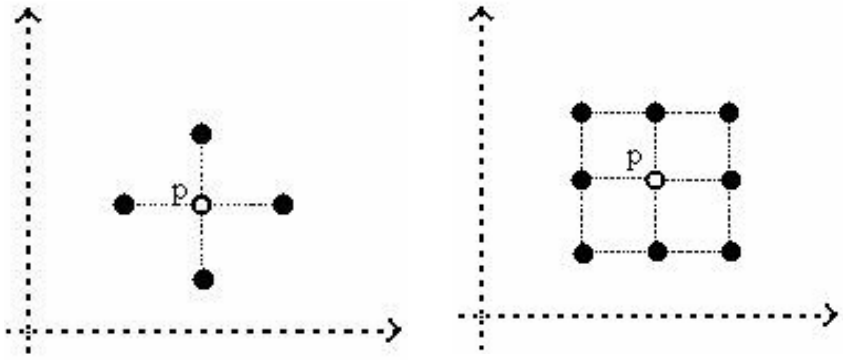

Figure 2. 4-adjacency, 8-adjacency.

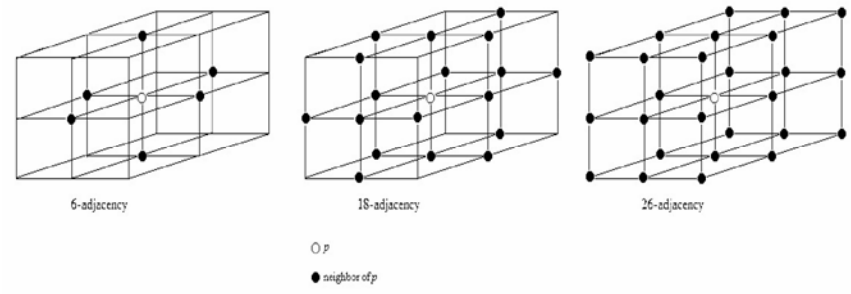

Figure 3. 6-adjacency, 18-adjacency, 26-adjacency.

Let $a, b \in \mathbb{Z}$ with $a < b$. A set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

is called a *digital interval* [4].

Let κ be an adjacency relation defined on $\mathbb{Z}^n$. A κ-neighbor of a lattice point p is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ-*connected* [17], if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$, and x_i and x_{i+1} are κ-neighbors, where $i = 0, 1, \ldots, r - 1$. A κ-*component* of a digital image X is a maximal κ-connected subset of X.

Definition 2.2 [5, 22]**.** Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. Then, the function $f : X \to Y$ is said to be (κ_0, κ_1)-continuous, if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous.

Proposition 2.3 [5, 22]**.** *Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. Then, the function $f : X \to Y$ is (κ_0, κ_1)-continuous, if and only if for every pair of κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y.*

Note that the proposition's characterization of continuity is what Chen calls an immersion, a gradually varied operator, or a gradually varied mapping in [10] and [11].

By a digital κ-path of length m from x to y in a digital image X, we mean a one-to-one $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$. If $f(0) = f(m)$, then the κ-path is said to be closed, and the function f is called a κ-*loop*. Let $f : [0, m-1]_{\mathbb{Z}} \to X$ be a $(2, \kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent, if and only if $j = i \pm 1 \mod m$. Then the set $f([0, m-1]_{\mathbb{Z}})$ is a simple closed κ-curve.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. A function $f : X \to Y$ is (κ_0, κ_1)-*isomorphism* [8], if f is (κ_0, κ_1)-continuous and bijective and further $f^{-1} : Y \to X$ is (κ_1, κ_0)-continuous, in which case we write $X \simeq_{(\kappa_0, \kappa_1)} Y$. Let us define the notion of interior, which plays an important role in establishing a connected sum.

Definition 2.4 [15]**.** Let $c^* := (x_0, x_1, \ldots, x_r)$ be a closed κ-curve in $\mathbb{Z}^2$, where $\{\kappa, \overline{\kappa}\} = \{4, 8\}$. A point x of the complement $\overline{c^*}$ of c^* in $\mathbb{Z}^2$ is said to be interior to c^*, if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points to c^* is denoted by $Int(c^*)$.

The digital images MSC_8^*, MSC_4^*, and $MSC_8'^*$, which are obtained from the minimal simple closed curves MSC_8, MSC_4, and MSC_8' in $\mathbb{Z}^2$, respectively, are essentially used in establishing the notion of a connected sum [15].

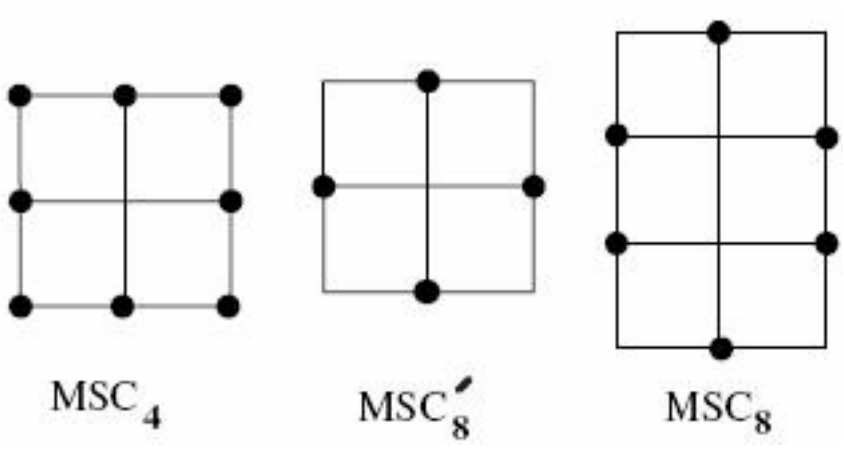

Figure 4 [16]**.** Minimal simple closed curves.

- $MSC_8^* := MSC_8 \cup Int(MSC_8)$, where MSC_8 is a digital image, which is (8, 8)-isomorphic to the set

$$\{(0, 0), (1, 1), (1, 2), (0, 3), (-1, 2), (-1, 1)\}.$$

- $MSC_4^* := MSC_4 \cup Int(MSC_4)$, where MSC_4 is a digital image, which is (4, 4)-isomorphic to the set

$$\{(0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (1, 2), (0, 2), (0, 1)\}.$$

- $MSC_8'^* := MSC_8' \cup Int(MSC_8')$, where MSC_8' is a digital image, which is (8, 8)-isomorphic to the set

$$\{(0, 0), (1, 1), (-1, 1), (0, 2)\}.$$

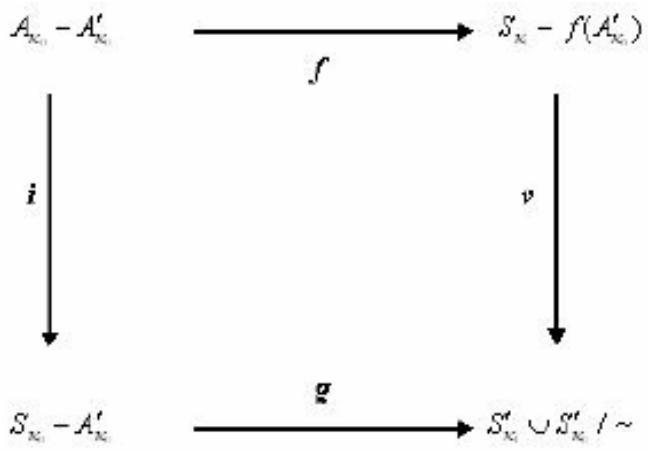

Figure 5 [15]**.** Diagram for a connected sum.

We recall a connected sum of two digital surfaces.

Definition 2.5 [15]**.** Let S_{κ_0} be a closed κ_0-surface in $\mathbb{Z}^{n_0}$ and let S_{κ_1} be a closed κ_1-surface in $\mathbb{Z}^{n_1}$ for $n_0, n_1 \geq 3$. Consider $A'_{\kappa_0} \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that $A'_{\kappa_0} \simeq_{(\kappa_0, 8)} Int(MSC_8^*)$, $A'_{\kappa_0} \simeq_{(\kappa_0, 4)} Int(MSC_4^*)$, or $A'_{\kappa_0} \simeq_{(\kappa_0, 8)} Int(MSC_8'^*)$. Let $f : A_{\kappa_0} \to f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0, κ_1)-isomorphism. Let $S'_{\kappa_i} = S_{\kappa_i} \setminus A'_{\kappa_i}$, $i \in \{0, 1\}$. Then the connected sum, denoted $S_{\kappa_0} \sharp S_{\kappa_1}$, is the quotient space $S_{\kappa_0} \cup S_{\kappa_1} / \sim$, where $i : A_{\kappa_0} \setminus A'_{\kappa_0} \to S'_{\kappa_0}$ is the inclusion map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} \setminus A'_{\kappa_0}$.

3. Homology Groups of n-Dimensional Digital Images

In algebraic topology, computing homology groups is easier than computing higher degree homotopy groups. Therefore, we prefer computing a homology group of a digital image to computing a homotopy group of a digital image. The simplicial homology groups of n-dimensional digital images from algebraic topology have been introduced in Arslan et al. [1]. In this section, we expand our knowledge of the simplicial homology group of digital images.

Definition 3.1 (see [24]). Let S be a set of nonempty subset of a digital image (X, κ). Then the members of S are called simplexes of (X, κ), if the followings hold:

(a) If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.

(b) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$ (note this implies every point p that belongs to a simplex determines a simplex $\{p\}$).

An m-simplex is a simplex S such that $|S| = m + 1$.

Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a *face* of P.

Definition 3.2 [1]. Let (X, κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. Then (X, κ) is called a *finite digital simplicial complex*.

(1) If P belongs to X, then every face of P also belongs to X.

(2) If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

The *dimension* of a digital simplicial complex X is the largest integer m such that X has an m-simplex.

Definition 3.3 [1]. $C_q^{\kappa}(X)$ is a free abelian group with basis all digital (κ, q)-simplices in X.

Corollary 3.4. *Let* $(X, \kappa) \subset \mathbb{Z}^n$ *be a digital simplicial complex of dimension* m. *Then for all* $q > m$, $C_q^{\kappa}(X)$ *is a trivial group.*

Definition 3.5 [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^{\kappa}(X) \to C_{q-1}^{\kappa}(X)$ defined by

$$\partial_q(< p_0, p_1, \ldots, p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, \ldots, \hat{p}_i, \ldots, p_q >, & q \le m; \\ 0, & q > m. \end{cases}$$

is called a *boundary homomorphism* (where $\hat{p}_i$ means delete the point p_i).

Proposition 3.6 [1]. *For all* $1 \le q \le m$, *we have*

$$\partial_{q-1} \circ \partial_q = 0.$$

Theorem 3.7 [1]. *Let* $(X, \kappa) \subset \mathbb{Z}^n$ *be a digital simplicial complex of dimension* m. *Then*,

$$C_*^{\kappa}(X): 0 \xrightarrow{\partial_{m+1}} C_m^{\kappa}(X) \xrightarrow{\partial_m} C_{m-1}^{\kappa}(X) \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0^{\kappa}(X) \xrightarrow{\partial_0} 0$$

is a chain complex.

Definition 3.8 [1]. Let (X, κ) be a digital simplicial complex.

(1) $Z_q^{\kappa}(X) = \operatorname{Ker} \partial_q$ is called the group of digital simplicial q-cycles.

(2) $B_q^{\kappa}(X) = \operatorname{Im} \partial_{q+1}$ is called the group of digital simplicial q-boundaries.

(3) $H_q^{\kappa}(X) = Z_q^{\kappa}(X) / B_q^{\kappa}(X)$ is called the q-th digital simplicial homology group.

Definition 3.9. Let $\varphi : (X, \kappa_0) \to (Y, \kappa_1)$ be a function between digital images. If for every digital (κ_0, m)-simplex P determined by the adjacency relation κ_0 in X, $\varphi(P)$ is a (κ_1, n)-simplex in Y for some $n \leq m$, then φ is called a *digital simplicial map*.

Definition 3.10. Let $\varphi : (X, \kappa_0) \to (Y, \kappa_1)$ be a digital simplicial map. For $q \geq 0$, we define a homomorphism $\varphi_\sharp : C_q^{\kappa 0}(X) \to C_q^{\kappa 1}(Y)$ by

$$\varphi_\sharp(\langle p_0, \ldots, p_q \rangle) = \langle \varphi_\sharp(p_0), \ldots, \varphi_\sharp(p_q) \rangle.$$

The following lemma immediately comes from Definition 3.10.

Lemma 3.11. *If* $\varphi : (X, \kappa_0) \to (Y, \kappa_1)$ *is a digital simplicial map, then*

$$\varphi_\sharp : C_q^{\kappa 0}(X) \to C_q^{\kappa 1}(Y)$$

is chain map; that is, $\varphi_\sharp \partial = \partial \varphi_\sharp$.

Theorem 3.12 [1]**.** *If* $f : X \to Y$ *is a digital* (κ_0, κ_1)*-isomorphism, then for all* $q \leq m$

$$H_q^{\kappa 0}(X) \cong H_q^{\kappa 1}(Y).$$

Theorem 3.13 [1]**.** *If* (X, κ) *is a single vertex, then*

$$H_q^{\kappa}(X) = \begin{cases} \mathbb{Z}, & q = 0, \\ 0, & q > 0. \end{cases}$$

Theorem 3.14. *Let* (X, κ) *be a directed digital simplicial complex of dimension* m.

(1) $H_q^{\kappa}(X)$ *is a finitely generated abelian group for every* $q \geq 0$.

(2) $H_q^{\kappa}(X)$ *is a trivial group for all* $q > m$.

(3) $H_q^{\kappa}(X)$ *is a free abelian group, possibly zero.*

Proof. (1) We know that $C_q^{\kappa}(X)$ is finitely generated and abelian. Hence, its subgroup $Z_q^{\kappa}(X)$ is finitely generated and abelian. It is clear that its quotient $H_q^{\kappa}(X)$ is a finitely generated abelian group.

(2) It is clear from Corollary 3.4.

(3) By Corollary 3.4, $C_{m+1}^{\kappa}(X)$ is a trivial group. Then $B_m^{\kappa}(X)$ is also trivial. Thus, we have $H_m^{\kappa}(X) \cong Z_m^{\kappa}(X)$. Since a subgroup of a free abelian group is free abelian, the result holds. □

Theorem 3.15. *For each* $q \geq 0$, H_q^{κ} *is a covariant functor from the category of digital simplicial complexes and simplicial maps to the category of abelian groups.*

Proof. We know that $H_q^{\kappa}(X)$ is defined on objects X that are digital simplicial complexes. If $\varphi : (X, \kappa_0) \to (Y, \kappa_1)$ is a digital simplicial map, define

$$\varphi_* : H_q^{\kappa}(X) \to H_q^{\kappa}(Y)$$

by $\varphi_*(z + B_q^{\kappa}(X)) = \varphi_\sharp(z) + B_q^{\kappa}(Y)$, where $z \in Z_n^{\kappa}(X)$. It is easy to see that $[1_{(X, \kappa_0)}]_* = 1_{H_q^{\kappa 0}(X)}$ and that $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$. The assertion follows. □

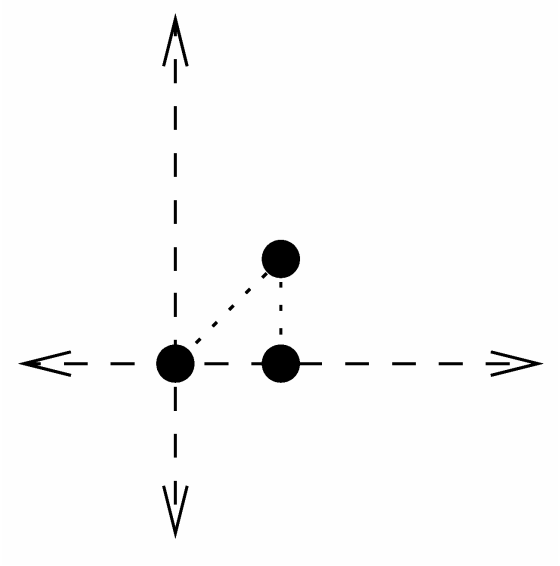

Figure 6. $(X, 8)$.

Theorem 3.16. *Let* $X = \{p_0 = (0, 0), p_1 = (1, 0), p_2 = (1, 1)\} \subset \mathbb{Z}^2$ *with an adjacency relation* $\kappa = 8$ (*see Figure* 6). *Then its digital simplicial homology groups are*

$$H_q^8(X) = \begin{cases} \mathbb{Z}, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

Proof. Assume that the points of X are ordered by $p_0 < p_1 < p_2$. From Theorem 3.14, we have

$$H_q^8(X) = \{0\} \quad \text{for } q > 2.$$

Moreover, $C_0^8(X)$, $C_1^8(X)$, and $C_2^8(X)$ are free abelian groups with bases

$$\{ < p_0 >, < p_1 >, < p_2 > \},$$

$$\{ < p_0p_1 >, < p_1p_2 >, < p_0p_2 > \}, \text{ and}$$

$$\{ < p_0p_1p_2 > \},$$

respectively. Thus, we have a short sequence

$$0 \xrightarrow{\partial_3} C_2^8(X) \xrightarrow{\partial_2} C_1^8(X) \xrightarrow{\partial_1} C_0^8(X) \xrightarrow{\partial_0} 0.$$

Clearly, $B_2^8(X) = \{0\}$. We have

$$\partial_2(a < p_0p_1p_2 >) = a(< p_1p_2 > - < p_0p_2 > + < p_0p_1 >),$$

so $Z_2^8(X) = \{0\}$. Therefore, $H_2^8(X) = \{0\}$.

From the description of ∂_2 above, we obtain

$$B_1^8(X) = \{a(< p_0p_1 > + < p_1p_2 > - < p_0p_2 >) \mid a \in \mathbb{Z}\}.$$

Further,

$$\partial_1(a < p_0p_1 > + b < p_0p_2 > + c < p_1p_2 >) = (-a - b) < p_0 > + (a - c) < p_1 > + (b + c) < p_2 > = 0$$

implies $a = -b = c$, so

$$Z_1^8(X) = \{a(< p_0p_1 > + < p_1p_2 > - < p_0p_2 >) \mid a \in \mathbb{Z}\} = B_1^8(X).$$

Thus we have $H_1^8(X) = \{0\}$.

Let

$$B = \{a < p_0 > + b < p_1 > + c < p_2 > \mid \{a, b, c\} \subset \mathbb{Z},\ a + b + c = 0\} \cong \mathbb{Z}^2.$$

We have, from the description of ∂_1 above, $B_0^8(X) \subset B$. To show the reverse containment, notice that an arbitrary member of B takes the form

$$a < p_0 > + b < p_1 > - (a + b) < p_2 > = \partial_1(- a < p_0p_1 > - b < p_1p_2 >) \in B_0^8(X).$$

Therefore, $B_0^8(X) = B \cong \mathbb{Z}^2$. Again using the short sequence, we get

$$Z_0^8(X) = \{a_0 < p_0 > + a_1 < p_1 > + a_2 < p_2 > \mid a_i \in \mathbb{Z}\ i = 0, 1, 2\} \cong \mathbb{Z}^3.$$

We claim that the quotient group $Z_0^8(X)/B_0^8(X)$ is isomorphic to one copy of $\mathbb{Z}$. Any 0-cycle $c_0 = a_0 < p_0 > + a_1 < p_1 > + a_2 < p_2 >$ can be written as

$$c_0 = \partial_1(a_1 < p_0p_1 > + a_2 < p_0p_2 >) + (a_0 + a_1 + a_2) < p_0 > .$$

This means that c_0 is homologous to the 0-chain $(a_0 + a_1 + a_2) < p_0 >$. Hence, the 0-chain is homologous to an integral multiple of $< p_0 >$. Therefore, $H_0^8(X)$ is isomorphic to the additive group $\mathbb{Z}$ of integers. We summarize

$$H_q^8(X) = \begin{cases} \mathbb{Z}, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

Theorem 3.17. *If* $MSC_8' \cong \{c_0 = (-1, 0), c_1 = (0, -1), c_2 = (0, 1), c_3 = (1, 0)\}$ (*see Figure* 4), *then its simplicial homology groups are*

$$H_q^8(MSC_8') = \begin{cases} \mathbb{Z}, & q = 0, 1, \\ 0, & q \neq 0, 1. \end{cases}$$

Proof. Assume that there is a dictionary order relation on the points of MSC_8' (see Figure 4). From Theorem 3.14, we have $H_q^8(MSC_8') = 0$ for every $q > 1$. Moreover, $C_1^8(MSC_8')$ and $C_0^8(MSC_8')$ are free abelian groups with bases

$$\{\langle c_0c_1 \rangle, \langle c_1c_3 \rangle, \langle c_2c_3 \rangle, \langle c_0c_2 \rangle\},$$

and

$$\{\langle c_0 \rangle, \langle c_1 \rangle, \langle c_2 \rangle, \langle c_3 \rangle\},$$

respectively.

Thus, we get the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^8(MSC_8') \xrightarrow{\partial_1} C_0^8(MSC_8') \xrightarrow{\partial_0} 0.$$

By the short sequence, it is easy to see that

$$Z_1^8(MSC_8') = \{a(\ <c_0c_1> + <c_1c_3> - <c_2c_3> - <c_0c_2>\ ) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Since $B_1^8(MSC_8') \cong \{0\}$, it follows that $H_1^8(MSC_8')$ is isomorphic to the additive group $\mathbb{Z}$ of integers.

Let

$$B = \{a <c_0> + b <c_1> + c <c_2> + d <c_3> \mid \{a, b, c, d\} \subset \mathbb{Z},$$

$$a + b + c + d = 0\} \cong \mathbb{Z}^3.$$

We show that $B_0^8(MSC_8') = B$, as follows. We have

$$\partial_1(r <c_0c_1> + s <c_1c_3> + t <c_2c_3> + u <c_0c_2>\ ) = (-r-u) <c_0>$$

$$+ (r-s) <c_1> + (-t+u) <c_2> + (s+t) <c_3>\ .$$

It follows easily that $B_0^8(MSC_8') \subset B$. In order to show the reverse containment, we observe that an arbitrary member of B takes the form

$$a <c_0> + b <c_1> + c <c_2> - (a+b+c) <c_3> =$$

$$\partial_1(\ -a <c_0c_1> - (a+b) <c_1c_3> - c <c_2c_3>\ ).$$

The assertion follows.

Again using the short sequence, we have

$$Z_0^8(MSC_8') = \{a_0 <c_0> + a_1 <c_1> + a_2 <c_2> + a_3 <c_3> \mid a_i \in \mathbb{Z}\}$$

$$\cong \mathbb{Z}^4.$$

Any 0-cycle $w_0 = a_0 <c_0> + a_1 <c_1> + a_2 <c_2> + a_3 <c_3>$ can be written as

$$w_0 = \partial_1((\ -a_0 - a_2) <c_0c_1> + a_2 <c_0c_2> + a_3 <c_1c_3>\ )$$

$$+ (a_0 + a_1 + a_2 + a_3) <c_1>\ .$$

This means that w_0 is homologous to $(a_0 + a_1 + a_2 + a_3) < c_1 >$. Hence, the 0-chain is homologous to an integral multiple of $< c_1 >$. Therefore, $H_0^8(MSC_8')$ is isomorphic to the additive group $\mathbb{Z}$ of integers. Therefore, we obtain

$$H_q^8(MSC_8') = \begin{cases} \mathbb{Z}, & q = 0, 1, \\ 0, & q \neq 0, 1. \end{cases}$$

□

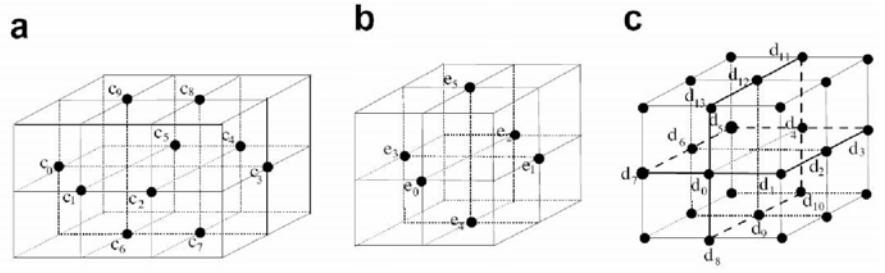

Figure 7. (a) MSS_{18}; (b) MSS_{18}'; (c) MSS_6.

Now we are ready to compute homology groups of minimal simple κ-surfaces (MSS_κ).

Theorem 3.18. *If*

$$MSS_{18} = \{c_0 = (0, 0, 1),\ c_1 = (1, 1, 1),\ c_2 = (1, 2, 1),\ c_3 = (0, 3, 1),$$
$$c_4 = (-1, 2, 1),\ c_5 = (-1, 1, 1),\ c_6 = (0, 1, 0),\ c_7 = (0, 2, 0),$$
$$c_8 = (0, 2, 2),\ c_9 = (0, 1, 2)\},$$

then its digital simplicial homology groups are

$$H_q^{18}(MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}^3, & q = 1, \\ 0, & q \neq 0, 1. \end{cases}$$

Proof. Let us direct MSS_{18} by the ordering $c_4 < c_5 < c_0 < c_6 < c_9 < c_7 < c_8 < c_3 < c_1 < c_2$. By Theorem 3.14, $H_q^{18}(MSS_{18})$ is a trivial group for all $q \geq 3$. Moreover, $C_0^{18}(MSS_{18})$, $C_1^{18}(MSS_{18})$, and $C_2^{18}(MSS_{18})$ are free abelian groups with bases

$\{\langle c_0\rangle, \langle c_1\rangle, \ldots, \langle c_9\rangle\}$

$\{\langle c_0c_1\rangle, \langle c_0c_6\rangle, \langle c_5c_0\rangle, \langle c_0c_9\rangle, \langle c_1c_2\rangle, \langle c_6c_1\rangle, \langle c_9c_1\rangle, \langle c_5c_6\rangle, \langle c_4c_5\rangle, \langle c_4c_8\rangle, \langle c_8c_2\rangle,$

$\langle c_9c_8\rangle, \langle c_8c_3\rangle, \langle c_4c_3\rangle, \langle c_4c_7\rangle, \langle c_7c_3\rangle, \langle c_7c_2\rangle, \langle c_3c_2\rangle, \langle c_5c_9\rangle, \langle c_6c_7\rangle\}$, and

$\{\langle c_0c_6c_1\rangle, \langle c_5c_0c_6\rangle, \langle c_0c_9c_1\rangle, \langle c_8c_3c_2\rangle, \langle c_5c_0c_9\rangle, \langle c_4c_8c_3\rangle, \langle c_4c_7c_3\rangle, \langle c_7c_3c_2\rangle\}$,

respectively. Thus, we obtain following short sequence:

$$0 \xrightarrow{\partial_3} C_2^{18}(MSS_{18}) \xrightarrow{\partial_2} C_1^{18}(MSS_{18}) \xrightarrow{\partial_1} C_0^{18}(MSS_{18}) \xrightarrow{\partial_0} 0.$$

Let

$$\partial_2(a_1\langle c_0c_6c_1\rangle + a_2\langle c_0c_9c_1\rangle + a_3\langle c_5c_0c_6\rangle + a_4\langle c_5c_0c_9\rangle + a_5\langle c_7c_3c_2\rangle + a_6\langle c_8c_3c_2\rangle$$
$$+ a_7\langle c_4c_7c_3\rangle + a_8\langle c_4c_8c_3\rangle) = (-a_1 - a_2)\langle c_0c_1\rangle + (a_3 + a_4)\langle c_5c_0\rangle + (a_1 + a_3)\langle c_0c_6\rangle$$
$$+ (a_2 + a_4)\langle c_0c_9\rangle + a_1\langle c_6c_1\rangle + a_2\langle c_9c_1\rangle + (a_5 + a_6)\langle c_3c_2\rangle - a_5\langle c_7c_2\rangle - a_6\langle c_8c_2\rangle$$
$$+ (-a_7 - a_8)\langle c_4c_3\rangle + (a_5 + a_7)\langle c_7c_3\rangle + (a_6 + a_8)\langle c_8c_3\rangle + a_7\langle c_4c_7\rangle$$
$$+ a_8\langle c_4c_8\rangle - a_3\langle c_5c_6\rangle - a_4\langle c_5c_9\rangle = 0.$$

From this equation, we must have $a_1 = a_2 = a_3 = \cdots = a_7 = a_8 = 0$. Therefore, $Z_2^{18}(MSS_{18}) = \{0\}$ and it follows that $H_2^{18}(MSS_{18}) = \{0\}$.

Observe

$$\partial_1(a_1\langle c_0c_1\rangle + a_2\langle c_0c_6\rangle + a_3\langle c_5c_0\rangle + a_4\langle c_0c_9\rangle + a_5\langle c_1c_2\rangle + a_6\langle c_6c_1\rangle + a_7\langle c_9c_1\rangle$$
$$+ a_8\langle c_5c_6\rangle + a_9\langle c_4c_5\rangle + a_{10}\langle c_4c_8\rangle + a_{11}\langle c_8c_2\rangle + a_{12}\langle c_9c_8\rangle + a_{13}\langle c_8c_3\rangle$$
$$+ a_{14}\langle c_4c_3\rangle + a_{15}\langle c_4c_7\rangle + a_{16}\langle c_7c_3\rangle + a_{17}\langle c_7c_2\rangle + a_{18}\langle c_3c_2\rangle + a_{19}\langle c_5c_9\rangle$$

$$+ a_{20}\langle c_6 c_7 \rangle) = (-a_1 - a_2 + a_3 - a_4)\langle c_0 \rangle + (a_1 - a_5 + a_6 + a_7)\langle c_1 \rangle$$

$$+ (a_5 + a_{11} + a_{17} + a_{18})\langle c_2 \rangle + (a_{13} + a_{14} + a_{16} - a_{18})\langle c_3 \rangle$$

$$+ (-a_9 - a_{10} - a_{14} - a_{15})\langle c_4 \rangle + (-a_3 - a_8 + a_9 - a_{19})\langle c_5 \rangle$$

$$+ (a_2 - a_6 + a_8 - a_{20})\langle c_6 \rangle + (a_{15} - a_{16} - a_{17} + a_{20})\langle c_7 \rangle$$

$$+ (a_{10} - a_{11} + a_{12} - a_{13})\langle c_8 \rangle + (a_4 - a_7 - a_{12} + a_{19})\langle c_9 \rangle.$$

So, we get

$$Z_1^{18}(MSS_{18}) =$$

$$\{a_1 < c_0 c_1 > + a_2 < c_0 c_6 > + a_3 < c_5 c_0 > + (-a_1 - a_2 + a_3) < c_0 c_9 >$$

$$+ a_5 < c_1 c_2 > + a_6 < c_6 c_1 > + (-a_1 + a_5 - a_6) < c_9 c_1 > + a_8 < c_5 c_6 >$$

$$+ a_9 < c_4 c_5 > + a_{10} < c_4 c_8 > + a_{11} < c_8 c_2 >$$

$$+ (-a_2 - a_5 + a_6 - a_8 + a_9) < c_9 c_8 >$$

$$+ (-a_2 - a_5 + a_6 - a_8 + a_9 + a_{10} - a_{11}) < c_8 c_3 > + a_{14} < c_4 c_3 >$$

$$+ (-a_9 - a_{10} - a_{14}) < c_4 c_7 > + a_{16} < c_7 c_3 >$$

$$+ (a_2 - a_6 + a_8 - a_9 - a_{10} - a_{14} - a_{16}) < c_7 c_2 >$$

$$+ (-a_2 - a_5 + a_6 - a_8 + a_9 + a_{10} - a_{11} + a_{14} + a_{16}) < c_3 c_2 >$$

$$+ (-a_3 - a_8 + a_9) < c_5 c_9 > + (a_2 - a_6 + a_8) < c_6 c_7 > \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^{11}.$$

We saw above that $\operatorname{Ker} \partial_2 = Z_2^{18}(MSS_{18}) \cong \{0\}$. Therefore, ∂_2 is one-to-one. It follows that $B_1^{18}(MSS_{18}) \cong C_2^{18}(MSS_{18}) \cong \mathbb{Z}^8$.

Any 1-cycle

$$w_1 = a_1 < c_0 c_1 > + a_2 < c_0 c_6 > + a_3 < c_5 c_0 > + (-a_1 - a_2 + a_3) < c_0 c_9 >$$

$$+ a_5 < c_1 c_2 > + a_6 < c_6 c_1 > + (-a_1 + a_5 - a_6) < c_9 c_1 > + a_8 < c_5 c_6 >$$

$$+a_9 < c_4c_5 > +a_{10} < c_4c_8 > +a_{11} < c_8c_2 >$$

$$+(-a_2 - a_5 + a_6 - a_8 + a_9) < c_9c_8 >$$

$$+(-a_2 - a_5 + a_6 - a_8 + a_9 + a_{10} - a_{11}) < c_8c_3 > +a_{14} < c_4c_3 >$$

$$+(-a_9 - a_{10} - a_{14}) < c_4c_7 > +a_{16} < c_7c_3 >$$

$$+(a_2 - a_6 + a_8 - a_9 - a_{10} - a_{14} - a_{16}) < c_7c_2 >$$

$$+(-a_2 - a_5 + a_6 - a_8 + a_9 + a_{10} - a_{11} + a_{14} + a_{16}) < c_3c_2 >$$

$$+(-a_3 - a_8 + a_9) < c_5c_9 > +(a_2 - a_6 + a_8) < c_6c_7 >$$

can be written as

$$w_1 = \partial_2[a_6 < c_0c_6c_1 > +(a_2 - a_6) < c_5c_0c_6 > +(-a_1 + a_5 - a_6) < c_0c_9c_1 >$$

$$+(-a_2 + a_3 + a_6) < c_5c_0c_9 > +(a_9 + a_{10} + a_{14} + a_{16}) < c_7c_3c_2 > +a_{10} < c_4c_8c_3 >$$

$$+(-a_{11}) < c_8c_3c_2 > +(-a_9 - a_{10} - a_{14}) < c_4c_7c_3 >]$$

$$+a_5(< c_0c_1 > - < c_0c_9 > + < c_1c_2 > - < c_9c_8 > - < c_8c_3 > - < c_3c_2 >)$$

$$+(a_2 - a_6 + a_8)(< c_5c_6 > - < c_9c_8 > - < c_8c_3 > + < c_7c_2 > - < c_3c_2 > - < c_5c_9 > + < c_6c_7 >)$$

$$+a_9(< c_4c_5 > + < c_9c_8 > + < c_8c_3 > - < c_4c_3 > + < c_5c_9 >).$$

Since an arbitrary 1-cycle is homologous to the sum of multiples of the three 2-chains

$$< c_0c_1 > - < c_0c_9 > + < c_1c_2 > - < c_9c_8 > - < c_8c_3 > - < c_3c_2 >,$$

$$< c_5c_6 > - < c_9c_8 > - < c_8c_3 > + < c_7c_2 > - < c_3c_2 > - < c_5c_9 > + < c_6c_7 >,$$

and

$$< c_4c_5 > + < c_9c_8 > + < c_8c_3 > - < c_4c_3 > + < c_5c_9 >,$$

each of which is easily seen to be a 1-cycle, it follows that $H_1^{18}(MSS_{18}) \cong \mathbb{Z}^3$.

Again using the short sequence, we have

$$B_0^{18}(MSS_{18}) = \{a_0 < c_0 > + a_1 < c_1 > + a_2 < c_2 > + a_3 < c_3 > + a_4 < c_4 > + a_5 < c_5 > + a_6 < c_6 > + a_7 < c_7 > + a_8 < c_8 > + a_9 < c_9 > | \sum_{i=0}^{9} a_i = 0,\ a_i \in \mathbb{Z}\} \cong \mathbb{Z}^9.$$

$$Z_0^{18}(MSS_{18}) = \{a_0 < c_0 > + a_1 < c_1 > + a_2 < c_2 > + a_3 < c_3 > + a_4 < c_4 > + a_5 < c_5 > + a_6 < c_6 > + a_7 < c_7 > + a_8 < c_8 > + a_9 < c_9 > | a_i \in \mathbb{Z},\ i = 0, \ldots, 9\} \cong \mathbb{Z}^{10}.$$

Any 0-cycle $w_0 = a_0 < c_0 > + a_1 < c_1 > + \cdots + a_9 < c_9 >$ can be written as

$$w_0 = \partial_1(-a_0 < c_0 c_1 > + (a_0 + a_1 + a_2 + a_3) < c_9 c_1 > + (-a_4 - a_5 - a_6 - a_7) < c_5 c_9 > + a_8 < c_9 c_8 > + (a_2 + a_3) < c_1 c_2 > - a_3 < c_3 c_2 > - (a_4 + a_7) < c_4 c_5 > + a_7 < c_4 c_7 > - a_6 < c_6 c_5 >) + (a_0 + a_1 + \cdots + a_9) < c_9 > .$$

This means that w_0 is homologous to $(a_0 + a_1 + \cdots + a_9) < c_9 >$. Hence, the 0-chain is homologous to an integral multiple of $< c_9 >$. Therefore, $H_0^{18}(MSS_{18})$ is isomorphic to the additive group $\mathbb{Z}$ of integers. Thus, we have the required result

$$H_q^{18}(MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}^3, & q = 1, \\ 0, & q \neq 0, 1. \end{cases}$$

□

Theorem 3.19. *The digital simplicial homology groups of* MSS'_{18} *are*

$$H_q^{18}(MSS'_{18}) = \begin{cases} \mathbb{Z}, & q = 0, 2, \\ 0, & q \neq 0, 2. \end{cases}$$

Proof. Let

$$MSS'_{18} = \{e_0 = (1, 1, 0),\ e_1 = (0, 2, 0),\ e_2 = (-1, 1, 0),\ e_3 = (0, 0, 0),$$
$$e_4 = (0, 1, -1),\ e_5 = (0, 1, 1)\}.$$

The points of MSS'_{18} are directed as follows:

$$e_2 < e_3 < e_4 < e_5 < e_1 < e_0.$$

From Theorem 3.14, it is clear that

$$H_q^{18}(MSS'_{18}) = 0, \text{ for all } q > 2.$$

Moreover, $C_0^{18}(MSS'_{18})$, $C_1^{18}(MSS'_{18})$, and $C_2^{18}(MSS'_{18})$ are free abelian groups with bases, respectively,

$$\{\langle e_0\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_5\rangle\},$$

$$\{\langle e_4e_1\rangle, \langle e_1e_0\rangle, \langle e_4e_0\rangle, \langle e_2e_4\rangle, \langle e_2e_1\rangle, \langle e_3e_4\rangle, \langle e_2e_3\rangle, \langle e_3e_0\rangle, \langle e_5e_1\rangle,$$
$$\langle e_5e_0\rangle, \langle e_2e_5\rangle, \langle e_3e_5\rangle\},$$

and

$$\{\langle e_4e_1e_0\rangle, \langle e_2e_4e_1\rangle, \langle e_2e_3e_4\rangle, \langle e_3e_4e_0\rangle, \langle e_5e_1e_0\rangle, \langle e_2e_5e_1\rangle, \langle e_2e_3e_5\rangle, \langle e_3e_5e_0\rangle\}.$$

Thus, we get the following short sequence:

$$0 \xrightarrow{\partial_3} C_2^{18}(MSS'_{18}) \xrightarrow{\partial_2} C_1^{18}(MSS'_{18}) \xrightarrow{\partial_1} C_0^{18}(MSS'_{18}) \xrightarrow{\partial_0} 0.$$

We first find the kernel of ∂_2. We have

$$\partial_2(a_1\langle e_4e_1e_0\rangle + a_2\langle e_2e_4e_1\rangle + a_3\langle e_2e_3e_4\rangle + a_4\langle e_3e_4e_0\rangle + a_5\langle e_5e_1e_0\rangle + a_6\langle e_2e_5e_1\rangle$$
$$+ a_7\langle e_2e_3e_5\rangle + a_8\langle e_3e_5e_0\rangle) = (a_1 + a_2)\langle e_4e_1\rangle + (a_1 + a_5)\langle e_1e_0\rangle$$
$$+ (-a_1 + a_4)\langle e_4e_0\rangle + (a_2 - a_3)\langle e_2e_4\rangle + (-a_2 - a_6)\langle e_2e_1\rangle + (a_3 + a_4)\langle e_3e_4\rangle$$
$$+ (a_3 + a_7)\langle e_2e_3\rangle + (-a_4 - a_8)\langle e_3e_0\rangle + (a_5 + a_6)\langle e_5e_1\rangle + (-a_5 + a_8)\langle e_5e_0\rangle$$
$$+ (a_6 - a_7)\langle e_2e_5\rangle + (a_7 + a_8)\langle e_3e_5\rangle. \tag{3.1}$$

Solving the equation

$$(a_1 + a_2)\langle e_4e_1\rangle + (a_1 + a_5)\langle e_1e_0\rangle + (-a_1 + a_4)\langle e_4e_0\rangle + (a_2 - a_3)\langle e_2e_4\rangle$$
$$+(-a_2 - a_6)\langle e_2e_1\rangle + (a_3 + a_4)\langle e_3e_4\rangle + (a_3 + a_7)\langle e_2e_3\rangle + (-a_4 - a_8)\langle e_3e_0\rangle$$
$$+(a_5 + a_6)\langle e_5e_1\rangle + (-a_5 + a_8)\langle e_5e_0\rangle + (a_6 - a_7)\langle e_2e_5\rangle + (a_7 + a_8)\langle e_3e_5\rangle = 0,$$

we must have

$$a_1 = -a_2 = -a_3 = a_4 = -a_5 = a_6 = a_7 = -a_8.$$

Hence,

$$Z_2^{18}(MSS'_{18}) = \{a(\langle e_4e_1e_0\rangle - \langle e_2e_4e_1\rangle - \langle e_2e_3e_4\rangle + \langle e_3e_4e_0\rangle - \langle e_5e_1e_0\rangle$$
$$+\langle e_2e_5e_1\rangle + \langle e_2e_3e_5\rangle - \langle e_3e_5e_0\rangle) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Since $B_2^{18}(MSS'_{18}) \cong \{0\}$,

$$H_2^{18}(MSS'_{18}) \cong \mathbb{Z}.$$

Let

$$\partial_1(a_1\langle e_4e_1\rangle + a_2\langle e_1e_0\rangle + a_3\langle e_4e_0\rangle + a_4\langle e_2e_4\rangle + a_5\langle e_2e_1\rangle + a_6\langle e_3e_4\rangle + a_7\langle e_2e_3\rangle$$
$$+ a_8\langle e_3e_0\rangle + a_9\langle e_5e_1\rangle + a_{10}\langle e_5e_0\rangle + a_{11}\langle e_2e_5\rangle + a_{12}\langle e_3e_5\rangle) = 0.$$

Then, we get

$$(a_2 + a_3 + a_8 + a_{10})\langle e_0\rangle + (a_1 - a_2 + a_5 + a_9)\langle e_1\rangle + (-a_4 - a_5 - a_7 - a_{11})\langle e_2\rangle$$
$$+(-a_6 + a_7 - a_8 - a_{12})\langle e_3\rangle + (-a_1 - a_3 + a_4 + a_6)\langle e_4\rangle + (-a_9 - a_{10} + a_{11} + a_{12})\langle e_5\rangle = 0.$$

Solving the equation above, we must have

$$a_6 = a_1 + a_3 - a_4,$$
$$a_9 = -a_1 + a_2 - a_5,$$
$$a_{10} = -a_2 - a_3 - a_8,$$
$$a_{11} = -a_4 - a_5 - a_7,$$
$$a_{12} = -a_6 + a_7 - a_8 = -a_1 - a_3 + a_4 + a_7 - a_8.$$

Hence, we get

$$Z_1^{18}(MSS'_{18}) = \{a_1 < e_4e_1 > + a_2 < e_1e_0 > + a_3 < e_4e_0 > + a_4 < e_2e_4 >$$
$$+ a_5 < e_2e_1 > + (a_1 + a_3 - a_4) < e_3e_4 > + a_7 < e_2e_3 >$$
$$+ a_8 < e_3e_0 > + (-a_1 + a_2 - a_5) < e_5e_1 >$$
$$+ (-a_2 - a_3 - a_8) < e_5e_0 > + (-a_4 - a_5 - a_7) < e_2e_5 >$$
$$+ (-a_1 - a_3 + a_4 + a_7 - a_8) < e_3e_5 > | \; a_i \in \mathbb{Z} \quad i = 1, 2, 3, 4, 5, 7, 8\}$$
$$\cong \mathbb{Z}^7.$$

On the other hand, from the Equation (3.1), we have

$$B_1^{18}(MSS'_{18}) = \{h_1 < e_4e_1 > + h_2 < e_1e_0 > + h_3 < e_4e_0 > + h_4 < e_2e_4 >$$
$$+ h_5 < e_2e_1 > + (h_1 + h_3 - h_4) < e_3e_4 > + h_7 < e_2e_3 >$$
$$+ h_8 < e_3e_0 > + (-h_1 + h_2 - h_5) < e_5e_1 >$$
$$+ (-h_2 - h_3 - h_8) < e_5e_0 > + (-h_4 - h_5 - h_7) < e_2e_5 >$$
$$+ (-h_1 - h_3 + h_4 + h_7 - h_8) < e_3e_5 > | h_i \in \mathbb{Z} \quad i = 1, 2, 3, 4, 5, 7, 8\}$$
$$\cong \mathbb{Z}^7.$$

Since $B_1^{18}(MSS'_{18}) = Z_1^{18}(MSS'_{18})$, it follows that $H_1^{18}(MSS'_{18})$ is isomorphic to the trivial group.

Again using the short sequence, we have

$$Z_0^{18}(MSS'_{18}) = \{a_0 < e_0 > + a_1 < e_1 > + a_2 < e_2 > + a_3 < e_3 > + a_4 < e_4 >$$
$$+ a_5 < e_5 > | \; a_i \in \mathbb{Z} \quad i = 0, 1, \ldots, 5\}$$
$$\cong \mathbb{Z}^6.$$

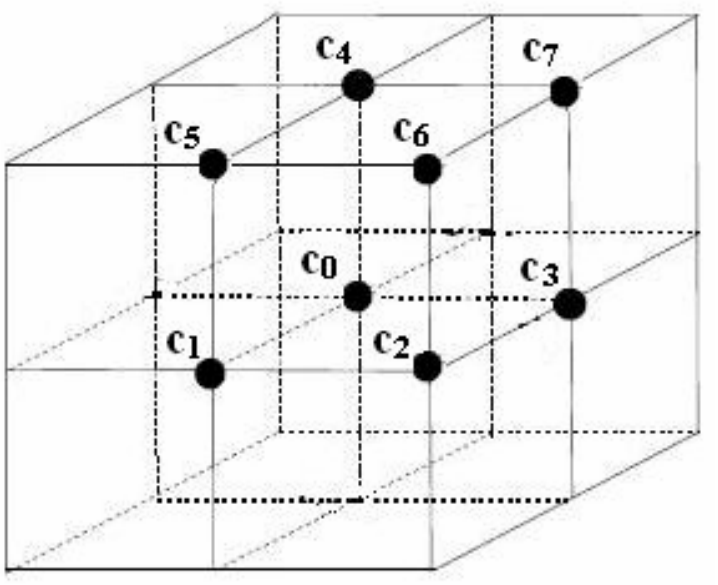

Figure 8 [16]. MSS'_6.

Any 0-cycle $w_0 = a_0 < e_0 > + a_1 < e_1 > + \cdots + a_5 < e_5 >$ can be written as

$$w_0 = \partial_1(-a_1 < e_1e_0 > -a_2 < e_2e_3 > -(a_2 + a_3) < e_3e_0 > -a_4 < e_4e_0 > -a_5 < e_5e_0 >)$$

$$+ (a_0 + a_1 + a_2 + a_3 + a_4 + a_5) < e_0 > .$$

This means that w_0 is homologous to $(a_0 + a_1 + \cdots + a_5) < e_0 >$. Hence, the 0-chain is homologous to an integral multiple of $< e_0 >$. Therefore, $H_0^{18}(MSS'_{18})$ is isomorphic to the additive group $\mathbb{Z}$ of integers. Therefore, we have the required result

$$H_q^{18}(MSS'_{18}) = \begin{cases} \mathbb{Z}, & q = 0, 2, \\ 0, & q \neq 0, 2. \end{cases}$$

□

Theorem 3.20. *Let* $MSS'_6 = \{c_0 = (0, 0, 0), c_1 = (1, 0, 0), c_2 = (1, 1, 0), c_3 = (0, 1, 0), c_4 = (0, 0, 1), c_5 = (1, 0, 1), c_6 = (1, 1, 1), c_7 = (0, 1, 1)\}$ (*see Figure* 8). *Then its digital simplicial homology groups are*

$$H_q^6(MSS'_6) = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}^5, & q = 1, \\ 0, & q \neq 0, 1. \end{cases}$$

Proof. Assume that there is a dictionary order relation on the points of MSS'_6. From Theorem 3.14, we have $H_q^6(MSS'_6) = \{0\}$ for every $q > 1$. Moreover, $C_0^8(MSS'_6)$ and $C_1^6(MSS'_6)$ are free abelian groups with bases

$\{\langle c_0 \rangle, \langle c_1 \rangle, \langle c_2 \rangle, \langle c_3 \rangle, \langle c_4 \rangle, \langle c_5 \rangle, \langle c_6 \rangle, \langle c_7 \rangle\}$, and

$\{\langle c_0c_1 \rangle, \langle c_0c_4 \rangle, \langle c_0c_3 \rangle, \langle c_1c_2 \rangle, \langle c_1c_5 \rangle, \langle c_2c_6 \rangle, \langle c_3c_2 \rangle, \langle c_3c_7 \rangle, \langle c_4c_5 \rangle,$

$\langle c_4c_7 \rangle, \langle c_5c_6 \rangle, \langle c_7c_6 \rangle\}$,

respectively. Thus, we get the following short sequence:

$$0 \xrightarrow{\partial_2} C_1^6(MSS'_6) \xrightarrow{\partial_1} C_0^6(MSS'_6) \xrightarrow{\partial_0} 0.$$

$$\partial_1(a_1\langle c_0c_1 \rangle + a_2\langle c_0c_4 \rangle + a_3\langle c_0c_3 \rangle + a_4\langle c_1c_2 \rangle + a_5\langle c_1c_5 \rangle + a_6\langle c_2c_6 \rangle + a_7\langle c_3c_2 \rangle$$

$$+ a_8\langle c_3c_7 \rangle + a_9\langle c_4c_5 \rangle + a_{10}\langle c_4c_7 \rangle + a_{11}\langle c_5c_6 \rangle + a_{12}\langle c_7c_6 \rangle) = (-a_1 - a_2 - a_3)\langle c_0 \rangle$$

$$+ (a_1 - a_4 - a_5)\langle c_1 \rangle + (a_4 - a_6 + a_7)\langle c_2 \rangle + (a_3 - a_7 - a_8)\langle c_3 \rangle$$

$$+ (a_2 - a_9 - a_{10})\langle c_4 \rangle + (a_5 - a_{10} + a_{12})\langle c_5 \rangle + (a_6 + a_{10} + a_{11})\langle c_6 \rangle$$

$$+ (a_8 + a_9 - a_{11})\langle c_7 \rangle.$$

Solving the equation

$$(-a_1 - a_2 - a_3)\langle c_0 \rangle + (a_1 - a_4 - a_5)\langle c_1 \rangle + (a_4 - a_6 + a_7)\langle c_2 \rangle$$

$$+ (a_3 - a_7 - a_8)\langle c_3 \rangle + (a_2 - a_9 - a_{10})\langle c_4 \rangle + (a_5 - a_{10} + a_{12})\langle c_5 \rangle$$

$$+(a_6 + a_{10} + a_{11})\langle c_6\rangle + (a_8 + a_9 - a_{11})\langle c_7\rangle = 0,$$

we have

$$a_3 = -a_1 - a_2,\quad a_5 = a_1 - a_4,\quad a_7 = -a_4 + a_6,\quad a_8 = -a_1 - a_2 + a_4 - a_6,$$

$$a_{10} = a_2 - a_9, a_{11} = -a_1 - a_2 + a_4 - a_6 + a_9, a_{12} = -a_6 - a_{11} = -a_1 + a_4 - a_6 - a_9.$$

Hence, we get

$$Z_1^6(MSS_6') = \{a_1\langle c_0c_1\rangle + a_2\langle c_0c_4\rangle + (-a_1 - a_2)\langle c_0c_3\rangle + a_4\langle c_1c_2\rangle$$

$$+(a_1 - a_4)\langle c_1c_5\rangle + a_6\langle c_2c_6\rangle + (-a_4 + a_6)\langle c_3c_2\rangle + (-a_1 - a_2 + a_4 - a_6)\langle c_3c_7\rangle$$

$$+ a_9\langle c_4c_5\rangle + (a_2 - a_9)\langle c_4c_7\rangle + (a_1 - a_4 + a_9)\langle c_5c_6\rangle$$

$$+(-a_1 + a_4 - a_6 - a_9)\langle c_7c_6\rangle \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^5.$$

Since $B_1^6(MSS_6') \cong \{0\}$, it follows that $H_1^6(MSS_6') \cong \mathbb{Z}^5$.

Again using the short sequence, we have

$$Z_0^6(MSS_6') = \{a_0 < c_0 > + a_1 < c_1 > + a_2 < c_2 > + a_3 < c_3 > + a_4 < c_4 >$$

$$+ a_5 < c_5 > + a_6 < c_6 > + a_7 < c_7 > \mid a_i \in \mathbb{Z}\}$$

$$\cong \mathbb{Z}^8.$$

Any 0-cycle $w_0 = a_0 < c_0 > + a_1 < c_1 > + a_2 < c_2 > + a_3 < c_3 > + a_4 < c_4 >$ $+ a_5 < c_5 > + a_6 < c_6 > + a_7 < c_7 >$ can be written as

$$w_0 = \partial_1((a_1 + a_2 + a_5 + a_6) < c_0c_1 > + a_2 < c_1c_2 > + (a_3 + a_7) < c_0c_3 >$$

$$+ a_4 < c_0c_4 > + (a_5 + a_6) < c_1c_5 > + a_6 < c_5c_6 > + a_7 < c_3c_7 >)$$

$$+ \sum_{i=0}^{7} a_i < c_0 > .$$

Thus, w_0 is homologous to the 0-chain

$$(a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7) < c_0 > .$$

Hence, the 0-chain is homologous to an integral multiple of $< c_0 >$. Therefore, $H_0^6(MSS_6')$ is isomorphic to the additive group $\mathbb{Z}$ of integers. Therefore, we obtain

$$H_q^6(MSS_6') = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}^5, & q = 1, \\ 0, & q \neq 0, 1. \end{cases}$$

□

4. Euler Characteristics of Digital Images

Han [16] has defined the Euler characteristic of a digital surface. In this section, we will define Euler characteristic for digital images in general. We will imitate the definition from algebraic topology [23].

The following is a generalization of ([16], Definition 12).

Definition 4.1. Let (X, κ) be a digital image of dimension m, and for each $q \geq 0$, let α_q be the number of digital (κ, q)-simplexes in X. The Euler characteristic of X, denoted by $\chi(X, \kappa)$, is defined by

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \alpha_q.$$

The following result, a generalization of ([16], Theorem 5.1) is motivated by its Euclidean analogue.

Theorem 4.2. *If (X, κ) is a digital image of dimension m, then*

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \operatorname{rank} H_q^{\kappa}(X).$$

Proof. Consider the digital chain complex $C_*^{\kappa}(X)$

$$0 \xrightarrow{\partial_{m+1}} C_m^{\kappa}(X) \xrightarrow{\partial_m} C_{m-1}^{\kappa}(X) \longrightarrow \ldots \longrightarrow C_1^{\kappa}(X) \xrightarrow{\partial_1} C_0^{\kappa}(X) \xrightarrow{\partial_0} 0.$$

Each $C_q^\kappa(X)$ is a free abelian group of rank α_q. Since $H_q^\kappa(X) = Z_q^\kappa(X)/\ B_q^\kappa(X)$, we have

$$\text{rank } H_q^\kappa(X) = \text{rank } Z_q^\kappa(X) - \text{rank } B_q^\kappa(X).$$

For each $q \geq 0$, there is an exact sequence

$$0 \to Z_q^\kappa(X) \to C_q^\kappa(X) \to B_{q-1}^\kappa(X) \to 0,$$

and

$$\alpha_q = \text{rank } C_q^\kappa(X) = \text{rank } Z_q^\kappa(X) + \text{rank } B_{q-1}^\kappa(X).$$

Hence, we get

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \alpha_q = \sum_{q=0}^{m} (-1)^q (\text{rank } Z_q^\kappa(X) + \text{rank } B_{q-1}^\kappa(X))$$

$$= \sum_{q=0}^{m} (-1)^q \text{rank } Z_q^\kappa(X) + \sum_{q=0}^{m} (-1)^q \text{rank } B_{q-1}^\kappa(X).$$

Changing index of summation in the last sum and using the fact that $\text{rank } B_{-1}^\kappa(X) = 0 = \text{rank } B_m^\kappa(X)$, we have

$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \text{rank } Z_q^\kappa(X) + \sum_{q=0}^{m} (-1)^{q+1} \text{rank } B_q^\kappa(X)$$

$$= \sum_{q=0}^{m} (-1)^q (\text{rank } Z_q^\kappa(X) - \text{rank } B_q^\kappa(X))$$

$$= \sum_{q=0}^{m} (-1)^q \text{rank } H_q^\kappa(X).$$

□

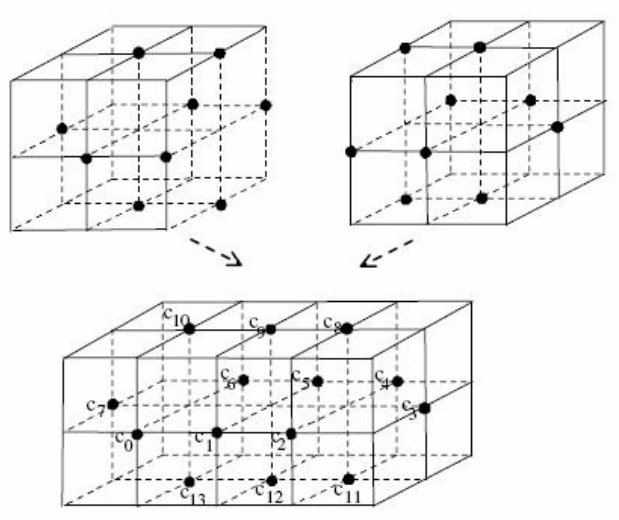

Figure 9 [16]. $MSS_{18} \sharp MSS_{18}$.

Theorem 4.3. *If* $(X, \kappa_0) \subset Z^{n_0}$ *and* $(Y, \kappa_1) \subset Z^{n_1}$ *are* (κ_0, κ_1)*-isomorphic, then*

$$\chi(X, \kappa_0) = \chi(Y, \kappa_1).$$

Proof. Let X and Y be (κ_0, κ_1)-isomorphic. Then, by Theorem 3.12, their homology groups are the same. From Theorem 4.2, the result holds.

□

Example 4.4. From the definition of Euler characteristics, we have

$\chi(MSS_6, 6) = \alpha_0 - \alpha_1 = 26 - 48 = -22$ (see Figure 7).

$\chi(MSS'_6, 6) = \alpha_0 - \alpha_1 = 8 - 12 = -4$ (see Figure 8).

$\chi(MSS_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 10 - 20 + 8 = -2$ (see Figure 7).

$\chi(MSS'_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 6 - 12 + 8 = 2$ (see Figure 7).

$\chi(MSS_6 \sharp MSS_6, 6) = \alpha_0 - \alpha_1 = 42 - 80 = -38$ (see Figure 10).

$\chi(MSS_{18} \sharp MSS_{18}, 18) = \alpha_0 - \alpha_1 + \alpha_2 = 14 - 28 + 8 = -6$ (see Figure 9).

Note that the last assertion of Example 4.4 corrects ([16], Example 5.3).

Example 4.5. We can alternately compute Euler numbers by using Theorem 4.2.

From Theorem 3.18,

$$\chi(MSS_{18}, 18) = \text{rank } H_0^{18}(MSS_{18}) - \text{rank } H_1^{18}(MSS_{18}) = 1 - 3 = -2.$$

From Theorem 3.19,

$$\chi(MSS'_{18}, 18) = \text{rank } H_0^{18}(MSS'_{18}) - \text{rank } H_1^{18}(MSS'_{18}) + \text{rank } H_2^{18}(MSS'_{18})$$
$$= 1 - 0 + 1 = 2.$$

From Theorem 3.20,

$$\chi(MSS'_6, 6) = \text{rank } H_0^6(MSS'_6) - \text{rank } H_1^6(MSS'_6) = 1 - 5 = -4.$$

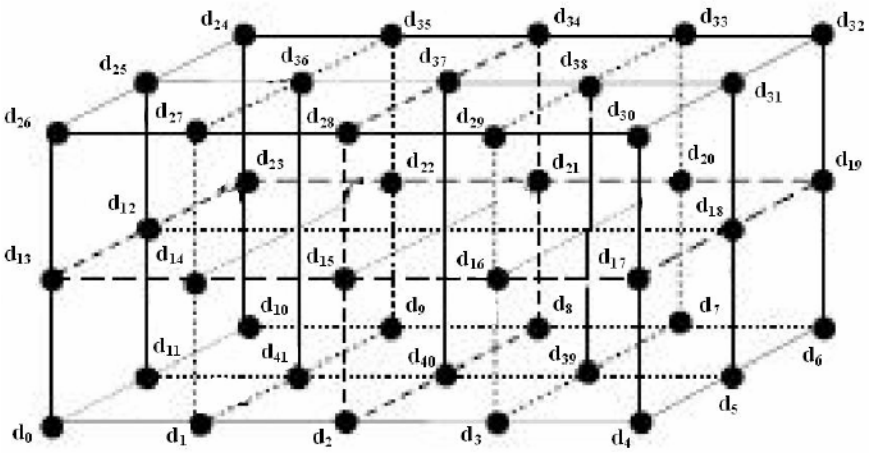

Figure 10 [16]. $MSS_6 \sharp MSS_6$.

5. Further Remarks

We have studied the simplicial homology groups of digital images. In Section 3, we focused our attention on computing homology groups of certain fundamental digital images by what might be termed direct

methods, based on their definitions. In Section 4, we obtained several results concerning the Euler characteristic of a digital surface, including the correction of an assertion that appeared in [16].

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