THE CHARACTERIZATION OF NULL GENERALIZED HELICES IN 5-DIMENSIONAL LORENTZIAN SPACE

ESEN İYİGÜN

Department of Mathematics
Art and Science Faculty
Uludağ University
16059, Bursa
Turkey
e-mail: esen@uludag.edu.tr

Abstract

In this paper, we study null generalized helices by describing in view of harmonic curvatures to a null Frenet curve of osculating order 5 in 5-dimensional Lorentzian space by using the Frenet frame consisting of two null and three space-like vectors from [3].

1. Introduction

Let \( x = (x_1, x_2, x_3, x_4, x_5) \) and \( y = (y_1, y_2, y_3, y_4, y_5) \) be two nonzero vectors in Minkowski 5-space \( \mathbb{R}^5_1 \). We denote \( \mathbb{R}^5_1 \) shortly by \( \mathbb{L}^5 \).

For \( x, y \in \mathbb{L}^5 \),

\[
\langle x, y \rangle = -x_1y_1 + \sum_{i=2}^{5} x_i y_i,
\]

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is called Lorentzian inner product. The couple \( \{ \mathbb{R}^5_1, \langle \rangle \} \) is called Lorentzian space and briefly denoted by \( \mathbb{L}^5 \). Then a vector \( v \) of \( \mathbb{L}^5 \) is called

(i) time-like if \( \langle v, v \rangle < 0 \),

(ii) space-like if \( \langle v, v \rangle > 0 \) or \( v = 0 \),

(iii) null (or light-like) vector if \( \langle v, v \rangle = 0, v \neq 0 \).

An arbitrary curve \( \alpha = \alpha(t) \) in \( \mathbb{L}^5 \) can be locally be space-like, time-like or null (light-like), if all of its velocity vectors \( \alpha'(t) \) are, respectively, space-like, time-like or null [6].

2. Basic Definitions

**Definition 1** [6]. On a semi-Riemannian manifold \( M \subset \mathbb{L}^5 \), there is a unique connection \( \nabla \) such that

\[
[V, W] = \nabla V W - \nabla W V,
\]

and

\[
X \langle V, W \rangle = \langle \nabla_X V, W \rangle + \langle V, \nabla_X W \rangle,
\]

for all \( X, V, W \in \chi (\mathbb{L}^5) \). \( \nabla \) is called the Levi-Civita connection of \( \mathbb{L}^5 \).

**Definition 2.** Let \( \alpha : I \rightarrow \mathbb{L}^5 \) be a null curve in \( \mathbb{L}^5 \). The curve \( \alpha \) is called Frenet curve of osculating order 5, if its 5-th order derivatives \( \alpha'(t), \alpha''(t), \alpha'''(t), \alpha''''(t), \alpha''''''(t) \) are linearly independent and \( \alpha'(t), \alpha''(t), \alpha'''(t), \alpha''''(t), \alpha''''''(t) \) are no longer linearly independent for all \( t \in I \).

For each null Frenet curve of osculating order 5, one can associate an orthonormal 5-frame \( \{ T, N, W_1, W_2, W_3 \} \) along \( \alpha \) (such that \( \alpha'(t) = T \)) called the Frenet frame and functions \( \{ k_1, k_2, k_3, k_4, k_5 \} \) called the Frenet curvatures. Thus from [3], the Frenet equations of a null curve in a 5-dimensional Lorentz manifold are written down as follows:
where $\mathcal{V}$ is the Levi-Civita connection of $\mathbb{L}^5$; $h$ and $\{k_1, k_2, k_3, k_4, k_5\}$ are differential functions; $T$ and $N$ are null vectors; $W_1$, $W_2$, and $W_3$ are space-like vectors. In these equations by changing a suitable parameter $t$, we may take $h = 0$ and other equations stay unchanged. This parameter is called distinguished parameter of the curve [3]. That is,

$$\begin{align*}
\nabla_T T &= hT + k_1 W_1, \\
\nabla_T N &= -hN + k_2 W_1 + k_3 W_2, \\
\nabla_T W_1 &= -k_2 T - k_1 N + k_4 W_2 + k_5 W_3, \\
\nabla_T W_2 &= -k_3 T - k_4 W_1, \\
\nabla_T W_3 &= -k_5 W_1,
\end{align*}$$

(1)

From [3] again, since $T$ and $N$ are null vectors, $W_i, 1 \leq i \leq 3$, are space-like vectors, then we have

$$\begin{align*}
\langle T, T \rangle &= 0, \quad \langle N, N \rangle = 0, \quad \langle T, N \rangle = 1, \quad \langle T, W_i \rangle = 0, \quad \langle N, W_i \rangle = 0, \\
\langle W_i, W_j \rangle &= \delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j, 
\end{cases} \quad \text{for } i, j = 1, 2, 3.
\end{align*}$$

(2)

**Definition 3.** If a null curve $\alpha : I \rightarrow \mathbb{L}^5$ is a null Frenet curve of osculating order 5 and Frenet curvatures $k_i, 1 \leq i \leq 5$ are nonzero constant, then $\alpha$ is called a null $W$-curve of rank 5.
3. Null Generalized Helices in $\mathbb{L}^5$

**Definition 4** [8]. Assume that $\alpha \subset \mathbb{L}^5$ is a null generalized helix given by curvature functions $k_1, k_2, k_3, k_4, k_5$. Then the harmonic curvatures of $\alpha$ in $\mathbb{L}^5$ write down as follows:

$$H_i = \begin{cases} -\frac{k_2}{k_1}, & i = 1, \\ \frac{H'_1}{k_3}, & i = 2, \\ -\frac{k_4}{k_5} H_2, & i = 3. \end{cases}$$

(3)

**Definition 5** [4]. Let $\alpha$ be a time-like curve in $\mathbb{L}^5$ with $\alpha'(s) = V_1$. $X \in \chi(\mathbb{L}^5)$ being a constant unit vector field, if

$$\langle V_1, X \rangle = \cosh \varphi \ (\text{constant}),$$

then $\alpha$ is called a *general helix* (inclined curves) in $\mathbb{L}^5$, $\varphi$ is called *slope angle*, and the space $Sp\{X\}$ is called *slope axis*.

**Definition 6** [8]. A null curve $\alpha : I \rightarrow \mathbb{L}^5$ is said to be a generalized helix, if there exist a nonzero unit constant vector $X$ such that $\langle \alpha'(t), X \rangle \neq 0$, is constant. Then $Sp\{X\}$ is called *slope axis* and for the Frenet frame $\{T, N, W_1, W_2, W_3\}$, we have

$$\begin{align*}
\langle W_1, X \rangle &= 0, \\
\langle N, X \rangle &= H_1 \langle T, X \rangle, \\
\langle W_i, X \rangle &= H_i \langle T, X \rangle, \quad 2 \leq i \leq 5.
\end{align*}$$

(4)

Now, the Equation (1) can be given in terms of harmonic curvatures as follows.
Theorem 1. Let \( \alpha \) be a null Frenet curve of osculating order 5 in \( \mathbb{L}^5 \). Then

\[
\begin{align*}
\nabla_T T &= k_1 W_1, \\
\nabla_T N &= -k_1 H_1 W_1 + \frac{H'_1}{H_2} W_2, \\
\nabla_T W_1 &= k_1 H_1 T - k_1 N - \frac{H_3 k_5}{H_2} W_2 - \frac{H_2 k_4}{H_3} W_3, \\
\nabla_T W_2 &= -\frac{H'_1}{H_2} T + \frac{H_3 k_5}{H_2} W_1, \\
\nabla_T W_3 &= -\frac{H_2 k_4}{H_3} W_1,
\end{align*}
\]

where \( k_1, k_4, k_5 \) are Frenet curvatures of \( \alpha \); \( H_1, H_2, H_3 \) are harmonic curvatures of \( \alpha \); and \( \nabla \) is the Levi-Civita connection of \( \mathbb{L}^5 \).

Proof. By using Equations (1) and (3), we obtain the proof of the theorem.

Corollary 2. If \( h = 0 \) and \( k_1 = 0 \) in \( \nabla_T T = h T + k_1 W_1 \), then \( \alpha \) is a null geodesics in \( \mathbb{L}^5 \).

Theorem 3. Let \( \alpha : I \to \mathbb{L}^5 \) be a null curve in \( \mathbb{L}^5 \). Then

\[
\begin{align*}
\langle \nabla_T T, W_1 \rangle &= -\frac{k_2}{H_1}, \\
\langle \nabla_T T, W_2 \rangle &= \langle \nabla_T T, W_3 \rangle = \langle \nabla_T N, W_3 \rangle = \langle \nabla_T W_1, W_1 \rangle = 0, \\
\langle \nabla_T W_2, W_2 \rangle &= \langle \nabla_T W_2, W_3 \rangle = \langle \nabla_T W_3, W_2 \rangle = \langle \nabla_T W_3, W_3 \rangle = 0, \\
\langle \nabla_T N, W_1 \rangle &= -k_1 H_1, \\
\langle \nabla_T N, W_2 \rangle &= \frac{H'_1}{H_2}, \\
\langle \nabla_T W_1, W_2 \rangle &= -\frac{H_3 k_5}{H_2}, \\
\langle \nabla_T W_1, W_3 \rangle &= -\frac{H_2 k_4}{H_3}, \\
\langle \nabla_T W_2, W_1 \rangle &= -\langle \nabla_T W_1, W_2 \rangle, \\
\langle \nabla_T W_3, W_1 \rangle &= -\langle \nabla_T W_1, W_3 \rangle,
\end{align*}
\]
where \( T \) and \( N \) are null vectors; \( W_1, W_2, \) and \( W_3 \) are space-like vectors; \( H_1, H_2, \) and \( H_3 \) are harmonic curvatures of \( \alpha; \) \( \nabla \) is the Levi-Civita connection of \( \mathbb{L}^5; \) and \( k_1, k_2, k_4, k_5 \) are Frenet curvatures of \( \alpha. \)

**Proof.** By using Equations (1), (2), and (3), we obtain the proof of the theorem. \( \square \)

**Theorem 4.** Let \( \alpha : I \rightarrow \mathbb{L}^5 \) be a null curve in \( \mathbb{L}^5 \) and \( X \) be a nonzero unit constant vector field (time-like or space-like) of \( \mathbb{L}^5. \) Then

\[
\begin{cases}
(i) \quad \langle \nabla_T T, X \rangle = \langle \nabla_T W_1, X \rangle = \langle \nabla_T W_3, X \rangle = 0, \\
(ii) \quad \langle \nabla_T N, X \rangle = H_1^\prime (T, X), \\
(iii) \quad \langle \nabla_T W_2, X \rangle = -\frac{H_1^\prime}{H_2} (T, X),
\end{cases}
\]

where \( H_1 \) and \( H_2 \) are harmonic curvatures of \( \alpha. \)

**Proof.**

(i) \( \langle \nabla_T T, X \rangle = \langle k_1 W_1, X \rangle = k_1 \langle W_1, X \rangle = 0, \) \( (\langle W_1, X \rangle = 0), \)

\[
\langle \nabla_T W_1, X \rangle = \langle (-k_2 T - k_1 N + k_4 W_2 + k_5 W_3), X \rangle \\
= -k_2 \langle T, X \rangle - k_1 \langle N, X \rangle + k_4 \langle W_2, X \rangle + k_5 \langle W_3, X \rangle \\
= -k_2 \langle T, X \rangle - k_1 H_1 (T, X) + k_4 \langle W_2, X \rangle + k_5 \langle W_3, X \rangle \\
= -k_2 \langle T, X \rangle + k_4 H_2 (T, X) + k_5 H_3 (T, X) \\
= (k_4 H_2 - k_4 H_2) \langle T, X \rangle \\
\Rightarrow \langle \nabla_T W_1, X \rangle = 0,
\]

(ii) \( \langle \nabla_T N, X \rangle = \langle (k_2 W_1 + k_3 W_2), X \rangle \\
= k_2 \langle W_1, X \rangle + k_3 \langle W_2, X \rangle \\
= k_3 \langle W_2, X \rangle \)
\[ k_3 H_2 \langle T, X \rangle \]
\[ \Rightarrow \langle \nabla_T N, X \rangle = H'_1 \langle T, X \rangle. \]

(iii) \[ \langle \nabla_T W_2, X \rangle = \langle (-k_3 T - k_4 W_1), X \rangle \]
\[ = -k_3 \langle T, X \rangle - k_4 \langle W_1, X \rangle \]
\[ = -k_3 \langle T, X \rangle \]
\[ \Rightarrow \langle \nabla_T W_2, X \rangle = -\frac{H'_1}{H_2} \langle T, X \rangle. \]

**Corollary 5** [8]. \( \alpha \) is a null helix in \( \mathbb{L}^5 \) \( \iff \) \( 2H_1 + (H_2)^2 + (H_3)^2 = \) constant.

**Definition 7.** A null curve \( \alpha : I \rightarrow \mathbb{L}^5 \) is said to be a generalized helix, if there exist harmonic curvatures \( H_1, H_2, \) and \( H_3 \) such that
\[ H'_1 + H'_2 + H'_3 = 0. \]

**Corollary 6.** \( H'_2 = -\frac{H'_1}{H_2} \) and \( H'_3 = 0. \)

**Proof.** From [8],
\[ \langle W_i, X \rangle = H_i \langle T, X \rangle, \quad 2 \leq i \leq 5. \]

Thus
\[ H'_2 = \frac{\langle \nabla_T W_2, X \rangle}{\langle T, X \rangle} = -\frac{H'_1}{H_2} \frac{\langle T, X \rangle}{\langle T, X \rangle} = -\frac{H'_1}{H_2}, \]
and
\[ H'_3 = \frac{\langle \nabla_T W_3, X \rangle}{\langle T, X \rangle} = 0. \]
4. Examples

**Example 1.** Let \( \alpha : I \to \mathbb{L}^5 \) be the null curve defined by

\[
\alpha(t) = (\sinh t, \cosh t, 1, 0, -t), \quad t \in \mathbb{R},
\]

and \( X = (0, 0, 0, 1) \) a unit constant vector field in \( \mathbb{L}^5 \). The tangent vector of \( \alpha \) is

\[
T = \alpha'(t) = (\cosh t, \sinh t, 0, 0, -1),
\]

and \( \langle T, T \rangle = 0 \), so \( \alpha \) is a null curve in \( \mathbb{L}^5 \). Also, \( \langle T, X \rangle = -1 = \text{constant} \). Therefore, the curve \( \alpha \) is a null helix.

**Example 2.** Let \( \alpha : I \to \mathbb{L}^5 \) be the null curve defined by

\[
\alpha(t) = (t, 0, \sin t, \cos t, 1), \quad t \in \mathbb{R},
\]

and \( X = (1, 0, 0, 0, 0) \) a unit constant vector field in \( \mathbb{L}^5 \). The tangent vector of \( \alpha \) is

\[
T = \alpha'(t) = (1, 0, \cos t, -\sin t, 0),
\]

and \( \langle T, T \rangle = 0 \), so \( \alpha \) is a null curve in \( \mathbb{L}^5 \). Also, \( \langle T, X \rangle = -1 = \text{constant} \). Therefore, the curve \( \alpha \) is a null helix. Moreover, the frame \( \{T, N, W_1, W_2, W_3\} \) is a distinguished Frenet frame along \( \alpha \), where from (2),

\[
N = \frac{1}{2} (-1, 0, \cos t, -\sin t, 0),
\]

\[
W_1 = (0, 0, \sin t, \cos t, 0),
\]

\[
W_2 = (0, 1, 0, 0, 0),
\]

\[
W_3 = (0, 0, 0, 0, 1).
\]
Thus, from (4), we can find the following results:

\[ H_1 = -\frac{1}{2}, \quad H_2 = H_3 = 0. \]

**Example 3.** Let

\[ \alpha(t) = \left( \sqrt{3} \sinh t, \sqrt{3} \cosh t, t, \cos t, \sin t \right), \quad t \in R, \]

\[ V_1 = \alpha'(t) = \left( \sqrt{3} \cosh t, \sqrt{3} \sinh t, 1, -\sin t, \cos t \right), \]

where \( \langle \alpha'(t), \alpha'(t) \rangle = -1 \), which shows \( \alpha(s) \) is time-like curve and \( X = (1, 0, 0, 0) \) a unit constant vector field in \( \mathbb{L}^5 \). Then,

\[ \langle V_1, X \rangle = -\sqrt{3} \cosh t = \text{constant}. \]

Thus \( \alpha(t) \) is a general helix in \( \mathbb{L}^5 \).

**Example 4.** Let \( \alpha : I \rightarrow \mathbb{L}^5 \) be the null curve defined by

\[ \alpha(t) = \frac{1}{\sqrt{2}} (\sinh t, \cosh t, 0, \cos t, \sin t), \quad t \in R. \]

The tangent vector of \( \alpha \) is

\[ T = \alpha'(t) = \frac{1}{\sqrt{2}} (\cosh t, \sinh t, 0, -\sin t, \cos t), \]

and \( \langle T, T \rangle = 0 \), so \( \alpha \) is a null curve in \( \mathbb{L}^5 \). Moreover,

\[ \nabla_T T = \frac{1}{\sqrt{2}} (\sinh t, \cosh t, 0, -\cos t, -\sin t), \]

and

\[ \langle \nabla_T T, \nabla_T T \rangle = 1 > 0, \]

\( \nabla_T T \) is a space-like vector field, so we can take \( \nabla_T T = W_1 \), which implies that \( h = 0 \) and \( k_1 = 1 \) in the first equation of (1). Thus, \( h = 0 \) implies that \( t \) is the distinguished parameter for \( \alpha \) and by Corollary 2, \( \alpha \) is a non-null geodesic in \( \mathbb{L}^5 \). By taking the derivative of \( W_1 \) with respect to \( T \), we have
\[ \nabla_T W_1 = \frac{1}{\sqrt{2}} (\cosh t, \sinh t, 0, \sin t, -\cos t). \]

Choosing
\[ W_2 = \frac{1}{\sqrt{2}} (\sinh t, \cosh t, 0, \cos t, \sin t), \]
and taking the derivative with respect to \( T \), we have
\[ \nabla_T W_2 = \frac{1}{\sqrt{2}} (\cosh t, \sinh t, 0, -\sin t, \cos t) = T. \]

This implies that \( k_3 = -1, k_4 = 0 \) from \( \nabla_T W_2 = -k_3 T - k_4 W_1 \) and we obtain
\[ N = \frac{1}{\sqrt{2}} (-\cosh t, -\sinh t, 0, -\sin t, \cos t). \]

By taking the derivative of \( N \) with respect to \( T \), we have
\[ \nabla_T N = \frac{1}{\sqrt{2}} (-\sinh t, -\cosh t, 0, -\cos t, -\sin t) = -W_2. \]

This implies that \( k_2 = 0 \) in the second equation of (1). Choosing
\[ W_3 = \frac{1}{\sqrt{2}} (-\cosh t, -\sinh t, 0, \sin t, -\cos t), \]
and taking the derivative with respect to \( T \), we have
\[ \nabla_T W_3 = \frac{1}{\sqrt{2}} (-\sinh t, -\cosh t, 0, \cos t, \sin t) = -W_1. \]

This implies that \( k_5 = 1 \) in the fourth equation of (1). Thus, the harmonic curvatures of \( \alpha \) are
\[ H_1 = H_2 = H_3 = 0. \]
References


