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IDENTITY AND RANGE OF DERIVATION

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Abstract

Let A be a bounded linear operator acting on infinite dimensional separable Hilbert space $\mathcal{H}$. This paper has two purposes, the first is to give a large class of operators verifying $I \notin \overline{\operatorname{ran}(\delta_A)}$ and generalize Stampfli's results. The second purpose is to give a large class of operators verifying $A^* \notin \overline{\operatorname{ran}(\delta_A)}$ and generalize Ho's results.

1. Introduction

A derivation on a Banach algebra ξ is an endomorphism δ on ξ verifying $\delta(XY) = X\delta(Y) + \delta(X)Y$. In the case $\xi = B(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space and $B(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, we know that any derivation is an inner derivation of the form $\delta_A(\delta_A(X) = AX - XA)$ with $A \in B(\mathcal{H})$. The study of derivation led many work for the past years, and several problems of the range of derivation remain open [16].

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2. Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ denote the Banach space of all bounded linear operators on $\mathcal{H}$. A bounded operator A is called normal if $A^*A = AA^*$. Also, A is called p-hyponormal for $p > 0$ if $(A^*A)^p - (AA^*)^p \geq 0$, log-hyponormal if A is an invertible operator, which satisfies $\log(A^*A) \geq \log(AA^*)$ [15], semi-hyponormal if $p = \frac{1}{2}$ [18]. Throughout this paper, we consider the case where $p \in (0, 1]$. A is called hyponormal iff it is 1-hyponormal. We say that A is dominant if $\operatorname{ran}(A - \lambda) \subset \operatorname{ran}(A - \lambda)^*$ for all $\lambda \in \mathbb{C}$. We also say that A is p-quasihyponormal if $A^*((A^*A)^p - (AA^*)^p)A \geq 0$ and (p, k)-quasihyponormal if $A^{*k}((A^*A)^p - (AA^*)^p)A^k \geq 0 (k \in \mathbb{N})$. If $p = 1$, $k = 1$, and $(p = k = 1)$, then A is said k-quasihyponormal, p-quasihyponormal, and quasihyponormal, respectively. Let $(p - H)$, (HN), $(Q(p))$, and $(Q(p, k))$ denote the class of p-hyponormal, hyponormal, p-quasihyponormal, and (p, k)-quasihyponormal operators, respectively. A is called normaloid if $\|A\| = r(A)$, this class is denoted by (NL). These classes verify the strict inclusions [10, 11]:

$$(HN) \subset (p - H) \subset (Q(p)) \subset (Q(p, k)) \subset (NL).$$

The derivation induced by the operator $A \in B(\mathcal{H})$ is the operator δ_A defined by $\delta_A(X) = AX - XA$, $X \in B(\mathcal{H})$. The kernel of δ_A is called the commutant de A and denoted by $\{A\}'$.

Definition 2.1 ([16]). We say that $A \in B(\mathcal{H})$ is finite if the distance $\operatorname{dist}(I, \operatorname{ran}(\delta_A)) \geq 1$ from the identity to the range of δ_A.

In the following, for $A \in B(\mathcal{H})$, we will denote the spectrum, approximate spectrum, and the spectral radius of A by $\sigma(A)$, $\sigma_a(A)$, and

$r(A)$, respectively. The essential norm (resp., essential spectral radius) of A is denoted by $\|A\|_e$ (resp., $r_e(A)$). Also, $\text{ran}(A)$ (resp., $\overline{\text{ran}(A)}$) denote the range of A (resp., the closure of ran (A)).

3. The Classes of Operators Verifying $I \notin \overline{\text{ran}(\delta_A)}$

In this section, we shall give a large classes of operators satisfying $I \notin \overline{\text{ran}(\delta_A)}$.

Definition 3.1. An operator A is said to be weak finite (w-finite, in short) if $I \notin \overline{\text{ran}(\delta_A)}^w$, where $\overline{E}^w$ is the weak closure relatively to the weak operator topology of a subset in $B(\mathcal{H})$.

Definition 3.2. Let $A \in B(\mathcal{H})$. An isolated point λ in $\sigma(A)$ is called pole of order ν, if it is pole of order ν of $(z - A)^{-1}$ in the sense of analytic functions.

That is, equivalent [7] to $(A - \lambda)^{\nu} P_{\lambda} = 0$ and $(A - \lambda)^{\nu - 1} P_{\lambda} \neq 0$, where P_{λ} is the Riesz projection associated to λ.

We will give a more elegant proof of the Theorem 5 [12] with a slight generalization.

Lemma 3.3. *Every operator A with a pole of order ν is w-finite.*

Proof. The operator A can be written as $A = B \oplus C$ on $\mathcal{H} = \text{ran}(P_{\lambda}) \oplus \text{ran}(I - P)$, where $(B - \lambda)^{\nu} = 0$ on $\text{ran}(P_{\lambda})$. Since B is an algebraic operator, $I \notin \overline{\text{ran}(\delta_B)}^w$ by [12]. Therefore $I \notin \overline{\text{ran}(\delta_A)}^w$. □

Lemma 3.4. *Let $A \in B(\mathcal{H})$ and f be a function, which is analytic on the neighbourhood of $\sigma(A)$. If $AX_n - X_n A \to T \in \{A\}'$, then $f(A)X_n - X_n f(A) \to f'(A)T$.*

Proof. For a suitable Jordan curve γ, we have [9]

$$f(A) = \frac{1}{2i\pi}\int_\gamma f(\lambda)(\lambda - A)^{-1}d\lambda \text{ and } f'(A) = \frac{1}{2i\pi}\int_\gamma f(\lambda)(\lambda - A)^{-2}d\lambda.$$

Consequently,

$$f(A)X_n - X_n f(A) - f'(A)T = \frac{1}{2i\pi}\int_\gamma [f(\lambda)(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}T]d\lambda.$$

Since

$$(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}T = (\lambda - A)^{-1}[AX_n - X_nA - T](\lambda - A)^{-1}.$$

Hence,

$$\|f(\lambda)[(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}T]\| \leq \sup_{\lambda\in\gamma}\|f(\lambda)(\lambda - A)^{-2}\| \cdot \|AX_n - X_nA - T\|,$$

and the last term converge to 0 uniformly on γ. □

Theorem 3.5. *Let $A \in B(\mathcal{H})$. If $f(A)$ is normal or isometric for some analytic function f on the neighbourhood of $\sigma(A)$ such that f' is not identically zero on the neighbourhood of $\sigma(A)$, then $I \notin \overline{\operatorname{ran}(\delta_A)}$.*

Proof. From Lemma 3.3, we may suppose that A is without pole. Let $AX_n - X_nA \to I$, where $\{X_n\} \subset B(\mathcal{H})$. Then by Lemma 3.4,

$$f(A)X_n - X_nf(A) \to f'(A); \text{ hence } f'(A) \in \overline{\operatorname{ran}(\delta_{f(A)})} \cap \{f(A)\}'.$$

Since $f(A)$ is normal (resp., isometric), we get $f'(A) = 0$ [1] and, from the minimal equation theorem [7], f' is identically zero on the neighbourhood of $\sigma(A)$. □

Remark 3.6. (1) In Lemma 3.3, the hypotheses f' is not identically zero on the neighbourhood of $\sigma(A)$ is indispensable. Indeed, it known that there is an operator $A \in B(\mathcal{H})$ such that $I \in \overline{\operatorname{ran}(\delta_A)}$ [2], however, $f(A)$ is normal for every function f identically zero in the neighbourhood of $\sigma(A)$. So, the Theorem 5 of Stampfli as stated in [12] is false.

(2) From the proof of Lemma 3.4, we can affirm that $I \notin \overline{\operatorname{ran}(\delta_A)}$ if $\overline{\operatorname{ran}(\delta_{f(A)})} \cap \{f(A)\}' = \{0\}$ for some analytic function f on the neighbourhood of $\sigma(A)$ such that f' is not identically zero on the neighbourhood of $\sigma(A)$.

Theorem 3.7. *If* $A \in B(\mathcal{H})$ *is polynomially compact, then* $I \notin \overline{\operatorname{ran}(\delta_A)}$.

Proof. Let P be a polynomial of degree n for which $P(A)$ is compact. Suppose that $AX_n - X_nA \to I$, $\{X_n\} \subset B(\mathcal{H})$, and let $P^{(k)}$ denotes the derivative of order k of P. Then, by (3.4), we get $P(A)X_n - X_nP(A) \to P^{(1)}(A)$, this gives $P^{(1)}(A)$ is compact. Also $P^{(1)}(A)X_n - X_nP^{(1)}(A) \to P^{(2)}(A)$, implies $P^{(2)}(A)$ is compact. Continuing in this way, this yields $P^{(n)}(A)$ is compact, i.e., I is compact, contradiction. □

Remark 3.8. (1) The above theorem is an extension of the proposition of Stampfli [13] insuring that $I \notin \overline{\operatorname{ran}(\delta_A)}$ when A^n is compact.

(2) The above theorem gives a large class of quasi-nilpotent operators Q for which $I \notin \overline{\operatorname{ran}(\delta_Q)}$, including the universal quasi-nilpotent operators (i.e., Q^k compact). It is well known [8] that the set of universal quasi-nilpotent operators is dense in the set of quasi-nilpotent operators.

Definition 3.9. The reducing approximate spectrum of an operator $A \in B(\mathcal{H})$ denoted by $\sigma_{ra}(A)$ is the set of the scalars λ for which, there exists a normalized sequence $\{x_n\}$ in $\mathcal{H}$ verifying $(A - \lambda)x_n \to 0$ and $(A^* - \overline{\lambda})x_n \to 0.$

Note that for a dominant operator A (resp., log-hyponormal, semi-hyponormal), the reducing approximate spectrum $\sigma_{ra}(A)$ coincides with the approximate spectrum $\sigma_a(A)$ of A [4] (resp., [14], [18]).

Theorem 3.10. *Let* $A \in B(\mathcal{H})$. *If* $\sigma_{ra}(A) \neq \emptyset$, *then A is finite.*

Proof. Let $\lambda \in \sigma_{ra}(A)$ and x_n be a normalized sequence such that $(A - \lambda)x_n \to 0$ and $(A - \lambda)^* x_n \to 0$. If $X \in B(\mathcal{H})$, then we have

$$\|AX - XA - I\| = \|(A - \lambda)X - X(A - \lambda) - I\|$$

$$\geq |((A - \lambda)Xx_n, x_n) - (X(A - \lambda)x_n, x_n) - 1|.$$

Letting $n \to \infty$, we obtain $\|AX - XA - I\| \geq 1.$ □

In the two following propositions, we shall give some cases where the reducing approximate spectrum is not empty.

Proposition 3.11 ([3])**.** *Let* $A \in B(\mathcal{H})$. *If* $\|A\|_e = r_e(A)$ (*resp.*, $\|A\| = r(A)$), *there exist a scalar* λ *and an orthonormal* (*resp.*, *normalized*) *sequence* $\{x_n\}$ *such that* $|\lambda| = \|A\|_e$ (*resp.*, $|\lambda| = \|A\|$), $(A - \lambda)x_n \to 0$ *and* $(A^* - \overline{\lambda})x_n \to 0.$

Proposition 3.12. *Let* $A \in B(\mathcal{H})$.

(i) *If* $\mathrm{Re}A \geq 0$, *then* $\{\lambda \in \sigma(A): \mathrm{Re}\lambda = 0\} \subset \sigma_{ar}(A)$.

(ii) $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$, *where* $W(A)$ *denotes the numerical range of A.*

Proof. (i) Since $\lambda \in \sigma(A)$, there exists a normalized sequence $\{x_n\}$ such that $(A - \lambda)x_n \to 0$. Then $B = \mathrm{Re}A = \frac{1}{2}[(A - \lambda) + (A^* - \bar{\lambda})]$ verify $(Bx_n | x_n) \to 0$. Since $B \geq 0$, this gives $Bx_n \to 0$, implying $(A^* - \bar{\lambda}) \to 0$. Hence $\lambda \in \sigma_{ra}(A)$.

(ii) By replacing A by $\alpha A + \beta$, where α, β are appropriate scalars, the assumption $\lambda \in \partial W(A) \cap \sigma(A)$ may be reduced to $0 \in \partial W(A) \cap \sigma(A)$, with $\mathrm{Re}A \geq 0$. Since $0 \in \partial W(A) \subset \sigma_a(A)$, (i) shows that $0 \in \sigma_{ar}(A)$. □

Remark 3.13. It follows from the previous proposition that every normaloid operator is finite.

Definition 3.14. $A \in B(\mathcal{H})$ is said to be class (G_1) if $\|(A - zI)^{-1}\| = [\mathrm{dist}(z, \sigma(A))]^{-1}$ for all $z \notin \sigma(A)$.

Proposition 3.15. *If $A \in B(\mathcal{H})$ is class (G_1), then A is finite.*

Proof. Since A is of class (G_1), by [5], we can suppose that $\sigma_a(A) = \sigma_p(A)$, where $\sigma_p(A)$ denotes the point spectrum of A, consequently, $\partial\sigma(A) \subset \sigma_a(A) = \sigma_p(A)$. Let $\lambda_0 \in \partial\sigma(A)$ and show that there exists a normalized sequence $\{x_n\}$ such that

$$(A - \lambda_0)x_n \to 0 \text{ and } (A^* - \overline{\lambda_0}) \to 0.$$

For $n = 1, 2, 3, \ldots,$ let $D_n = \{\lambda : |\lambda - \lambda_0| \leq \frac{1}{n}\}$, since $\lambda_0 \in \partial\sigma(A)$, D_n contains the points μ_n of the resolvent of A, such that $|\mu_n - \lambda_0| < \frac{1}{2n}$.

Let λ_n such that $\mathrm{dist}(\mu_n, \sigma(A)) = |\mu_n - \lambda_n|$, with this way $\lambda_n \in \sigma(A)$, with λ_n belonging to the boundary of a closed disc centered at μ_n, which not contains points of $\sigma(A)$. Since A is of class (G_1), it follows from [6] that $\ker(A - \lambda_n) = \ker(A - \lambda_n)^*$; hence λ_n is an eigenvalue, and therefore $(A - \lambda_n)x_n = (A - \lambda_n)^* x_n = 0$, where x_n is a normalized vector. Then

$$(A-\lambda_0)x_n = (A-\lambda_n)x_n + (\lambda_n-\lambda_0)x_n = (\lambda_n-\lambda_0)x_n.$$

Consequently,

$$\|(A-\lambda_0)x_n\| = \|\lambda_n-\lambda_0\| \le \frac{1}{n}, \text{ hence } (A-\lambda_0)x_n \to 0,$$

also $(A-\lambda_0)^* x_n \to 0.$ By Theorem 3.10, A is finite. □

Definition 3.16. Let a be an element of a C^*-algebra $\mathcal{A}$. a is said

(i) dominant, if there exists $M_\lambda \ge 1$ such that

$$(a^*-\overline{\lambda})(a-\lambda) - M_\lambda^{-2}(a-\lambda)(a^*-\overline{\lambda}) \ge 0 \text{ for all } \lambda \in \mathbb{C};$$

(ii) of class (G_1), if $\|(a-zI)^{-1}\| = [\operatorname{dist}(z, \sigma(a))]^{-1}$ for all $z \notin \sigma(a)$;

(iii) semi-hyponormal, if $(a^*a)^{1/2} - (aa^*)^{1/2} \ge 0$;

(iv) log-hyponormal, if a is invertible and such that $\log(a^*a) \ge \log(aa^*)$;

(v) p-quasihyponormal, if $a^*((a^*a)^p - (aa^*)^p)a \ge 0\,(0 < p \le 1)$;

(vi) (p, k)-quasihyponormal, if $a^{*k}((a^*a)^p - (aa^*)^p)a^k \ge 0$ $(0 < p \le 1, k \in \mathbb{N})$;

(vii) finite, if $\operatorname{dist}(e, \operatorname{ran}(\delta_a)) \ge 1$, where e is the unit of $\mathcal{A}$.

Theorem 3.17. *If* $a \in \mathcal{A}$ *is dominant* (*resp*., *of class* (G_1), *semi-hyponormal*, *log-hyponormal*, (p, k)*-quasihyponormal*), *then* $e \notin \overline{\operatorname{ran}(\delta_a)}$.

Proof. We known [8] that there exist an $*$-isomorphism isometric Ψ and a Hilbert space $\mathcal{H}$, with Ψ preserving the order, so $\Psi(a)$ is a dominant operator (resp., of class (G_1), semi-hyponormal, log-hyponormal, (p, k)-quasihyponormal). Hence by Theorem 3.10, $\Psi(a)$ is finite, otherwise $I_{\mathcal{H}} \notin \overline{\operatorname{ran}(\delta_{\Psi(a)})}$, since Ψ is isometric, it follows that $e \notin \overline{\operatorname{ran}(\delta_a)}$. □

Corollary 3.18. *Let* $A \in B(\mathcal{H})$, *then* $I \notin \overline{\operatorname{ran}(\delta_A)}$ *in one of the following conditions*:

(i) $A =$ *dominant* + *compact*;

(ii) $A =$ *semi-hyponormal* + *compact*;

(iii) $A = T +$ *compact, where* T *is of class* (G_1);

(iv) $A =$ *log-hyponormal* + *compact*;

(v) $A = (p, k)$*-quasihyponormal* + *compact*.

Proof. Since the Calkin algebra is a C^*-algebra, $\pi(A)$ as an element of the algebra is dominant (resp., semi-hyponormal, of class (G_1), log-hyponormal, (p, k)- quasihyponormal) and so $\pi(I) \notin \overline{\operatorname{ran}(\delta_{\pi(A)})}$ by Theorem 3.17. If $X \in B(\mathcal{H})$, then

$$\|I - AX - XA\| = \|\pi(I) - \pi(A)\pi(X) - \pi(X)\pi(A)\| \geq \|\pi(I)\| = 1.$$

Hence $I \notin \overline{\operatorname{ran}(\delta_A)}$. □

These results generalize those of Stampfli [13].

4. The Classes of Operators Verifying $A^* \notin \overline{\operatorname{ran}(\delta_A)}$

Ho [9] showed that if $\|A\| = r(A)$ and $A \neq 0$, then $A^* \notin \overline{\operatorname{ran}(\delta_A)}$. We will extend this result and give other classes of operators satisfying $A^* \notin \overline{\operatorname{ran}(\delta_A)}$.

Lemma 4.1. *Let* $A \in B(\mathcal{H})$ *such that* $\sigma_{ra}(A) - \{0\}$ *is not empty, then* $A^* \notin \overline{\operatorname{ran}(\delta_A)}$.

Proof. Let $\lambda \in \sigma_{ra}(A) - \{0\}$, then there exists a normalized sequence $\{x_n\}$ such that

$$(A - \lambda)x_n \to 0 \text{ and } (A^* - \overline{\lambda})x_n \to 0.$$

Let $X \in B(\mathcal{H})$, then

$$\begin{aligned}\|AX - XA - A^*\| &= \|(A-\lambda)X - X(A-\lambda) - (A^* - \overline{\lambda}) - \overline{\lambda}\| \\ &\geq |((A-\lambda)Xx_n, x_n) - (X(A-\lambda)x_n, x_n) \\ &\quad - ((A^* - \overline{\lambda})x_n, x_n) - \overline{\lambda}|.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\|AX - XA - A^*\| \geq |\lambda|.$$

Hence $A^* \notin \overline{R(\delta_A)}$. □

Corollary 4.2. *Let* $A \in B(\mathcal{H})$, *then* $A^* \notin \overline{\operatorname{ran}(\delta_A)}$ *in one of the following conditions*:

(i) A = *non-quasinilpotent dominant* + *compact*;

(ii) A = *non-quasinilpotent semi-hyponormal* + *compact*;

(iii) A = T + *compact*, *where* T *is of class* (G_1) *non-quasinilpotent*;

(iv) A = *log-hyponormal* + *compact*;

(v) $A = (p, k)$*-non-quasinilpotent quasihyponormal* + *compact* (*non-quasinilpotent p-quasihyponormal* + *compact*).

Proof. In all cases, we have $\sigma_{ra}(A\text{-compact})$ is not empty. □

Remark 4.3. It was given an example in [13] of a quasinilpotent dominant operator, which is a nonzero operator.

Theorem 4.4. *Let* $A \in B(\mathcal{H})$.

(i) *If* $\|A\| = r(A)$, *then* $\operatorname{dist}(A^*, \operatorname{ran}(\delta_A)) \geq \|A\|$.

(ii) *If* $\|A\|_e = r_e(A)$, *then* $\operatorname{dist}(A^*, \operatorname{ran}(\delta_A)) \geq \|A\|_e$.

(iii) *If* $\|A^2\| = r(A^2)$ *and* $A \neq 0$, *then* $A^* \notin \overline{\operatorname{ran}(\delta_A)}$.

(iv) *If* $\|A^2\|_e = r_e(A^2)$ *and A is not compact, then* $A^* \notin \overline{\operatorname{ran}(\delta_A)}$.

Proof. (i) By Proposition 3.11, there is a scalar λ and a normalized sequence $\{x_n\}$ such that $|\lambda| = \|A\|$, $(A - \lambda)x_n \to 0$ and $(A^* - \overline{\lambda})x_n \to 0$. If $X \in B(\mathcal{H})$, then

$$\|AX - XA - A^*\| = \|(A - \lambda)X - X(A - \lambda) - (A^* - \overline{\lambda}) - \overline{\lambda}\|$$

$$\geq |((A - \lambda)Xx_n, x_n) - (X(A - \lambda)x_n, x_n)$$

$$- ((A^* - \overline{\lambda})x_n, x_n) - \overline{\lambda}|.$$

(ii) The proof is the same as (i).

(iii) By Proposition 3.11, there exist a scalar λ and a normed system $\{x_n\}$ such that $|\lambda| = \|A^2\|$, $(A^2 - \lambda)x_n \to 0$ and $(A^{*2} - \overline{\lambda})x_n \to 0$.

If we suppose that $A^* \in \overline{\operatorname{ran}(\delta_A)}$, there exists a sequence $\{X_n\} \subset B(\mathcal{H})$ such that $AX_n - X_nA \to A^*$, so, $A^2X_n - X_nA^2 \to AA^* + A^*A$. For $\epsilon > 0$, we can find $X \in B(\mathcal{H})$ verifying

$$\|A^2X_n - X_nA^2 - (AA^* + A^*A)\| = \|(A^2 - \lambda)X - X(A^2 - \lambda) - (AA^* + A^*A)\| \leq \epsilon.$$

Then, for all positive integer n, we obtain

$$|((A^2 - \lambda)Xx_n, x_n) - (X(A^2 - \lambda)x_n, x_n) - (\|A^*x_n\| + \|Ax_n\|)| \leq \epsilon.$$

Letting $n \to \infty$, this yields

$$\lim(\|A^*x_n\|^2 + \|Ax_n\|^2) \leq \epsilon.$$

Since ϵ is arbitrary, we have $\lim(\|A^* x_n\|^2 + \|Ax_n\|^2) = 0$, otherwise, $Ax_n \to 0$ and $A^* x_n \to 0$. In particular, $A^2 x_n \to 0$; or $(A - \lambda)x_n \to 0$, where $\lambda x_n \to 0$. We deduce that $\lambda = 0$, this gives $A^2 = 0$. But $A^2 X_n - X_n A^2 \to AA^* + A^* A$, where $AA^* + A^* A = 0$ implying $A = 0$.

(iv) The equality $\|A^2\| = r(A^2)$ give a scalar λ and a normed system $\{x_n\}$ such that $|\lambda| = \|A^2\|_e$, $(A^2 - \lambda)x_n \to 0$ and $(A^{*2} - \overline{\lambda})x_n \to 0$. If we suppose that $A^* \in \overline{\operatorname{ran}(\delta_A)}$, we obtain, with the same approach as in (iii), that $[A^2] = 0$, otherwise A^2 is compact. Then, from $A^2 X_n - X_n A^2 \to AA^* + A^* A$, we deduce that $AA^* + A^* A$ is compact, which implies that A is compact.

□

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