ON SUBCLASSES OF $p$-VALENT FUNCTIONS INVOLVING CERTAIN FRACTIONAL DERIVATIVE OPERATOR

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Abstract

In the present paper, we derive some subordination and superordination results for $p$-valent functions in the open unit disk, involving certain fractional derivative operator. Relevant connections of the results, which are presented in the paper, with various known results are also considered.

1. Introduction and Preliminaries

Let $H(U)$ denote the class of analytic functions in the open unit disk $U = \{z : |z| < 1\}$ and let $H(a, p)$ denote the subclass of the functions $f \in H(U)$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots \quad (a \in \mathbb{C}, \ p \in \mathbb{N}).$$

Also, let $A(p)$ be the class of functions $f \in H(U)$ of the form

Keywords and phrases: $p$-valent function, differential subordination, differential superordination, Hadamard product, fractional derivative operators.

Received June 23, 2011

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\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N}, \quad (1.1) \]

and set \( \mathcal{A} = \mathcal{A}(1) \). For functions \( f(z) \in \mathcal{A}(p) \), given by (1.1), and \( g(z) \) given by

\[ g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad p \in \mathbb{N}. \quad (1.2) \]

The Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is defined by

\[ (f \ast g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \quad z \in \mathcal{U}; \quad p \in \mathbb{N}. \quad (1.3) \]

Let \( f, g \in \mathcal{H}(\mathcal{U}) \), we say that the function \( f \) is subordinate to \( g \), if there exist a Schwartz function \( w \), analytic in \( \mathcal{U} \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in \mathcal{U}) \), such that \( f(z) = g(w(z)) \) for all \( z \in \mathcal{U} \).

This subordination is denoted by \( f \prec g \) or \( f(z) \prec g(z) \). It is well known that, if the function \( g \) is univalent in \( \mathcal{U} \), then \( f(z) \prec g(z) \), if and only if \( f(0) = g(0) \) and \( f(\mathcal{U}) \subset g(\mathcal{U}) \).

Let \( p(z), h(z) \in \mathcal{H}(\mathcal{U}) \), and let \( \Phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C} \). If \( p(z) \) and \( \Phi(p(z), zp'(z), z^2 p''(z); z) \) are univalent functions, and if \( p(z) \) satisfies the second-order superordination

\[ h(z) \prec \Phi(p(z), zp'(z), z^2 p''(z); z), \quad (1.4) \]

then \( p(z) \) is called to be a solution of the differential superordination (1.4). (If \( f(z) \) is subordinate to \( g(z) \), then \( g(z) \) is called to be superordinate to \( f(z) \).) An analytic function \( q(z) \) is called a subordinant, if \( q(z) \prec p(z) \) for all \( p(z) \) satisfies (1.4). An univalent subordinant \( \tilde{q}(z) \) that satisfies \( q(z) \prec \tilde{q}(z) \) for all subordinants \( q(z) \) of (1.4) is said to be the best subordinant.
Recently, Miller and Mocanu [6] obtained conditions on \( h(z), q(z), \) and \( \Phi \) for which the following implication holds true:

\[
h(z) < \Phi(p(z), zp'(z), z^2p'(z); z) \Rightarrow q(z) < p(z),
\]

with the results of Miller and Mocanu [6], Bulboaca [3] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [2]. Ali et al. [1] used the results obtained by Bulboaca [2] and gave the sufficient conditions for certain normalized analytic functions \( f(z) \) to satisfy

\[
q_1(z) \prec zf'(z) \prec q_2(z),
\]

where \( q_1(z) \) and \( q_2(z) \) are given univalent functions in \( \mathcal{U} \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \). Shanmugam et al. [10] obtained sufficient conditions for a normalized analytic functions to satisfy

\[
q_1(z) \prec f(z) \prec q_2(z),
\]

and

\[
q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z),
\]

where \( q_1(z) \) and \( q_2(z) \) are given univalent functions in \( \mathcal{U} \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \).

Let \( _2F_1(a, b; c; z) \) be the Gauss hypergeometric function defined for \( z \in \mathcal{U} \) by (see Srivastava and Karlsson [12])

\[
_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_nn!} z^n,
\]

where \( (\lambda)_n \) is the Pochhammer symbol defined, in terms of the Gamma function, by

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{when } n = 0, \\ \lambda(\lambda + 1)(\lambda + 2)\ldots(\lambda + n - 1), & \text{when } n \in \mathbb{N}, \end{cases}
\]

for \( \lambda \neq 0, -1, -2, \ldots \).
We recall the following definitions of fractional derivative operators, which were used by Owa [7] (see also [8]) as follows:

**Definition 1.1.** The fractional derivative operator of order \( \lambda \) is defined by

\[
D_\lambda^z f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi,
\]

(1.7)

where \( 0 \leq \lambda < 1 \), \( f(z) \) is analytic function in a simply-connected region of the \( z \)-plane containing the origin, and the multiplicity of \((z-\xi)^{-\lambda}\) is removed by requiring \( \log(z-\xi) \) to be real when \( z-\xi > 0 \).

**Definition 1.2.** Let \( 0 \leq \lambda < 1 \) and \( \mu, \eta \in \mathbb{R} \). Then, in terms of the familiar Gauss’s hypergeometric function \( \, _2F_1 \), the generalized fractional derivative operator \( J_{0, z}^{\lambda, \mu, \eta} \) is

\[
J_{0, z}^{\lambda, \mu, \eta} f(z) = \frac{d}{dz} \left( \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi) \, _2F_1 \left[ \mu - \lambda, 1 - \eta; 1 - \lambda; \frac{1-\xi}{z} \right] d\xi \right),
\]

(1.8)

where \( f(z) \) is analytic function in a simply-connected region of the \( z \)-plane containing the origin, with the order \( f(z) = O(|z|^\varepsilon) \), \( z \to 0 \), where \( \varepsilon > \max\{0, \mu - \eta\} - 1 \) and the multiplicity of \((z-\xi)^{-\lambda}\) is removed by requiring \( \log(z-\xi) \) to be real when \( z-\xi > 0 \).

**Definition 1.3.** Under the hypotheses of Definition 1.2, the fractional derivative operator \( J_{0, z}^{\lambda+m, \mu+m, \eta+m} \) of a function \( f(z) \) is defined by

\[
J_{0, z}^{\lambda+m, \mu+m, \eta+m} f(z) = \frac{d^m}{dz^m} J_{0, z}^{\lambda, \mu, \eta} f(z).
\]

(1.9)

Notice that

\[
J_{0, z}^{\lambda, \lambda, \eta} f(z) = D_\lambda^z f(z), \quad 0 \leq \lambda < 1.
\]

(1.10)
With the aid of the above definitions, we define a modification of the fractional derivative operator $M^\lambda_{0,z}^{\mu,\eta}$ by

$$M^\lambda_{0,z}^{\mu,\eta} f(z) = \frac{\Gamma(p + 1 - \mu)\Gamma(p + 1 - \lambda + \mu)}{\Gamma(p + 1)\Gamma(p + 1 - \mu + \eta)} z^\mu J^\lambda_{0,z}^{\mu,\eta} f(z),$$  
(1.11)

for $f(z) \in \mathcal{A}(p)$ and $\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}$.

Then, it is observed that $M^\lambda_{0,z}^{\mu,\eta} f(z)$ maps $\mathcal{A}(p)$ onto itself as follows:

$$M^\lambda_{0,z}^{\mu,\eta} f(z) = z^p + \sum_{n=1}^\infty \frac{(p + 1)_n(p + 1 - \mu + \eta)_n}{(p + 1 - \mu)_n(p + 1 - \lambda + \eta)_n} a_{p+n} z^{p+n}. \quad (1.12)$$

Let $\varphi_p(a, c; z)$ be the incomplete beta function defined for $z \in \mathcal{U}$ by

$$\varphi_p(a, c; z) = z^p + \sum_{n=1}^\infty \frac{(a)_n}{(c)_n} z^{p+n}, \quad p \in \mathbb{N},$$  
(1.13)

where $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ and $(\lambda)_n$ is given by (1.6), and using the Hadamard product, we define the following operator $\Omega^\lambda_{p}^{\mu,\eta} f(z) : \mathcal{U} \to \mathcal{U}$ by:

$$\Omega^\lambda_{p}^{\mu,\eta} f(z) = \varphi_p(a, c; z) * M^\lambda_{0,z}^{\mu,\eta} f(z).$$  
(1.14)

If $f(z) \in \mathcal{A}(p)$, then from (1.12) and (1.14), we can easily see that

$$\Omega^\lambda_{p}^{\mu,\eta} f(z) = z^p + \sum_{n=1}^\infty \frac{(a)_n}{(c)_n} z^{p+n}, \quad p \in \mathbb{N}.$$  
(1.15)

where $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, $\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1$, and $p \in \mathbb{N}$.

Notice that, if $a = c$, then we have $\Omega^\lambda_{p}^{\mu,\eta} f(z) = M^\lambda_{0,z}^{\mu,\eta} f(z)$, $\Omega^0_{p}^{0,\eta} f(z) = f(z)$, and $\Omega^1_{p}^{1,\eta} f(z) = \frac{zf(z)}{p}$. Also, we observe that the operator $\Omega^\lambda_{p}^{\mu,\eta} f(z)$ reduces to the following interesting operators considered earlier for different choices of $\lambda, \mu, a$, and $c$: 
(1) For \( \lambda = \mu = 0 \), we get the operator \( L_p(a, c)f(z) \), which is motivated from Carlson-Shaffer operator [4].

(2) For \( \lambda = \mu = 0, a = m + p, \) and \( c = 1 \), we get the operator \( D^{m+p-1} \), which is the Ruscheweyh derivative operator of order \( m + p - 1 \) (see [9, 13]).

It is easily verified from (1.15) that
\[
z(\Omega_p^{\lambda, \mu, \eta} f(z))' = (p - \mu)\Omega_p^{\lambda+1, \mu+1, \eta+1} f(z) + \mu \Omega_p^{\lambda, \mu, \eta} f(z). \tag{1.16}
\]

The object of this paper is to derive several subordination results defined by the Hadamard product involving certain fractional derivative operator. Furthermore, we obtain the previous results of Shanmugam et al. [11] as special cases of the results presented here.

In order to prove our results, we mention to the following known results, which shall be used in the sequel:

**Lemma 1.4** ([8]). Let \( \lambda, \mu, \eta \in \mathbb{R} \), such that \( \lambda \geq 0 \) and \( K > \max \{0, \mu - \eta\} - 1 \). Then
\[
J_{0,z}^{\lambda, \mu, \eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu}. \tag{1.17}
\]

**Definition 1.5** ([6]). Denoted by \( Q \) the set of all functions \( f \) that are analytic and injective in \( \mathbb{C} \), where
\[
E(f) = \{ \xi \in \partial \mathbb{U} : \lim_{z \to \infty} f(z) = \infty \},
\]
and are such that \( f'(\xi) \neq 0 \) for \( \xi \in \partial \mathbb{U} - E(f) \).

**Lemma 1.6** ([5]). Let the function \( q \) be univalent in the open unit disk \( \mathbb{U} \), and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(\mathbb{U}) \) with \( \varphi(w) \neq 0 \), when \( w \in q(\mathbb{U}) \). Set \( Q(z) = zq'(z)\varphi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \). Suppose that

1. \( Q \) is starlike univalent in \( \mathbb{U} \),
2. \( \text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > 0 \) for \( z \in \mathbb{U} \).
If
\[ \theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(q(z)) + zq'(z)\varphi(q(z)), \]
then \( p(z) < q(z) \) and \( q \) is the best dominant.

**Lemma 1.7 ([3])**. Let the function \( q \) be univalent in the open unit disk \( U \), and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \), when \( w \in q(U) \). Suppose that

1. \( \Re \left( \frac{\theta'(q(z))}{\varphi(q(z))} \right) > 0 \) for \( z \in U \),
2. \( zq'(z)\varphi(q(z)) \) is starlike univalent in \( U \).

If \( p(z) \in H(q(0), 1) \cap Q \) with \( p(U) \subseteq D \), and \( \theta(p(z)) + zp'(z)\varphi(p(z)) \) is univalent in \( U \), and

\[ \theta(q(z)) + zq'(z)\varphi(q(z)) < \theta(p(z)) + zp'(z)\varphi(p(z)), \]
then \( q(z) < p(z) \) and \( q \) is the best subordinant.

**2. Subordination and Superordination for \( p \)-Valent Functions**

We begin with the following result involving differential subordination between analytic functions:

**Theorem 2.1.** Let
\[ \left( \frac{z^p}{\Omega_p^{\lambda, \mu, \eta}f(z)} \right)^\gamma \in H(U) \]
and let the function \( q(z) \) be analytic and univalent in \( U \) such that \( q(z) \neq 0 \), \( (z \in U) \). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). Let

\[ \Re \left( 1 + \frac{\zeta}{\beta} q(z) + \frac{2\delta}{\beta} (q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q(z)^2} \right) > 0, \quad (2.1) \]

\( (\alpha, \delta, \zeta, \beta \in \mathbb{C}; \beta \neq 0) \).
\[ \Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z) = \alpha + \xi \left( \frac{z^p}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right)^\gamma + \delta \left( \frac{z^p}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right)^{2\gamma} + \beta \gamma (p - \mu) \left[ 1 - \frac{\Omega_p^{\lambda+1,\mu+1,\eta+1} f(z)}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right]. \tag{2.2} \]

If \( q \) satisfies the following subordination:

\[ \Psi_{\lambda,\mu,\eta}(\gamma, \xi, \beta, \delta, f)(z) \prec \alpha + \xi q(z) + \delta q(z)^2 + \beta \frac{zq'(z)}{q(z)}, \]

\( (\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0), \)

then

\[ \left( \frac{z^p}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right)^\gamma \prec q(z), \quad (\gamma \in \mathbb{C} \setminus \{0\}), \tag{2.3} \]

and \( q \) is the best dominant.

**Proof.** Let the function \( p(z) \) be defined by

\[ p(z) = \left( \frac{z^p}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right)^\gamma, \quad (z \in \mathbb{U}; \gamma \in \mathbb{C} \setminus \{0\}). \]

So that, by a straightforward computation, we have

\[ \frac{zp'(z)}{p(z)} = \gamma \left[ p - \frac{z(\Omega_p^{\lambda,\mu,\eta} f(z))'}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right]. \]

By using the identity (1.16), we obtain

\[ \frac{zp'(z)}{p(z)} = \gamma \left[ (p - \mu) \frac{\Omega_p^{\lambda+1,\mu+1,\eta+1} f(z)}{\Omega_p^{\lambda,\mu,\eta} f(z)} \right]. \]
By setting $\theta(w) = \alpha + \xi w + \delta w^2$ and $\varphi(w) = \frac{\beta}{w}$, it can be easily verified that $\theta$ is analytic in $\mathbb{C}$, $\varphi$ is analytic in $\mathbb{C} \setminus \{0\}$, and that $\varphi(w) \neq 0$ ($w \in \mathbb{C} \setminus \{0\}$). Also, by letting

$$Q(z) = zq'(z)\varphi(q(z)) = \beta \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \delta(q(z))^2 + \beta \frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in $\mathcal{U}$ and that

$$\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{\xi}{\beta} q(z) + \frac{2\delta}{\beta} (q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq'(z)}{q'(z)} \right\} > 0.$$

The assertion (2.3) of Theorem 2.1 now follows by an application of Lemma 1.6.

**Remark 1.** For the choices $q(z) = \frac{1 + Az}{1 + Bz}$, $-1 \leq B < A \leq 1$ and

$q(z) = \left(\frac{1 + z}{1 - z}\right)\epsilon$, $0 < \epsilon \leq 1$, in Theorem 2.1, we get the following results (Corollaries 2.2 and 2.3) below:

**Corollary 2.2.** Assume that (2.1) holds. If $f \in \mathcal{A}(p)$ and

$$\Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) \prec \alpha + \xi \left(\frac{1 + Az}{1 + Bz}\right) + \delta \left(\frac{1 + Az}{1 + Bz}\right)^2 + \frac{\beta(A - B)z}{(1 + Az)(1 + Bz)},$$

($\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0$),

where $\Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z)$ is as defined in (2.2), then

$$\left(\frac{z^p}{\Omega_{\lambda, \mu, \eta}^f(z)}\right)^\gamma \prec \frac{1 + Az}{1 + Bz}, \quad (\gamma \in \mathbb{C} \setminus \{0\}),$$

and $\frac{1 + Az}{1 + Bz}$ is the best dominant.
Corollary 2.3. Assume that (2.1) holds. If \( f \in A(p) \) and

\[
\Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) < \alpha + \xi \left( \frac{1 + z}{1 - z} \right)^\gamma + \delta \left( \frac{1 + z}{1 - z} \right)^{2\gamma} + \frac{2\beta \xi z}{1 - z^2},
\]

(\( \lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; \ p \in \mathbb{N}; \ \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C}; \gamma \neq 0; \ \beta \neq 0 \)),

where \( \Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) \) is as defined in (2.2), then

\[
\left( \frac{z^p}{\Omega_{\lambda, \mu, \eta} f(z)} \right)^\gamma < \left( \frac{1 + z}{1 - z} \right)^\gamma, \quad (\gamma \in \mathbb{C} \setminus \{0\}),
\]

and \( \left( \frac{1 + z}{1 - z} \right)^\gamma \) is the best dominant.

Remark 2. Taking \( p = 1 \) and \( \lambda = \mu = 0 \) in Theorem 2.1, Corollaries 2.2 and 2.3, we obtain the subordination results for linear operator due to Shanmugam et al. ([11], Theorem 3.1, Corollaries 3.2 and 3.3, respectively).

Next, by appealing to Lemma 1.7 of the preceding section, we prove the following:

Theorem 2.4. Let \( q \) be analytic and univalent in \( U \) such that \( q(z) \neq 0 \) and \( \frac{zq'(z)}{q(z)} \) be starlike univalent in \( U \). Further, let us assume that

\[
\Re \left\{ \frac{2\delta}{\beta} (q(z))^2 + \frac{\xi}{\beta} q(z) \right\} > 0, \quad (\delta, \xi, \beta \in \mathbb{C}; \beta \neq 0). \tag{2.4}
\]

If \( f(z) \in A(p) \),

\[
0 \neq \left( \frac{z^p}{\Omega_{\lambda, \mu, \eta} f(z)} \right)^\gamma \in \mathcal{H}(q(0), 1] \cap Q,
\]

and \( \Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) \) is univalent in \( U \), then

\[
\alpha + \xi q(z) + \delta (q(z))^2 + \beta \frac{zq'(z)}{q(z)} < \Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z),
\]
implies
\[ q(z) = \left( \frac{z^p}{\Omega^\lambda_{\mu, \eta} f(z)} \right)^\gamma, \quad (\gamma \in \mathbb{C} \setminus \{0\}), \] (2.5)
and \( q \) is the best subordinant, where \( \Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) \) is as defined in (2.2).

**Proof.** By setting \( \theta(w) = \alpha + \xi w + \delta w^2 \) and \( \varphi(w) = \frac{\beta}{w} \), it can be easily observed that \( \theta \) is analytic in \( \mathbb{C} \), \( \varphi \) is analytic in \( \mathbb{C} \setminus \{0\} \), and that \( \varphi(w) \neq 0 \) (\( w \in \mathbb{C} \setminus \{0\} \)). Since \( q \) is convex (univalent) function, it follows that
\[
\text{Re} \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = \text{Re} \left\{ \frac{2\delta}{\beta} (q(z))^2 + \frac{\xi}{\beta} q(z) \right\} > 0,
\]
(\( \delta, \xi, \beta \in \mathbb{C}; \beta \neq 0 \)).

The assertion (2.5) of Theorem 2.4 now follows by an application of Lemma 1.7.

Combining Theorems 2.1 and 2.4, we get the following sandwich theorem:

**Theorem 2.5.** Let \( q_1 \) and \( q_2 \) be univalent in \( \mathcal{U} \) such that \( q_1(z) \neq 0 \) and \( q_2(z) \neq 0 \), \( z \in \mathcal{U} \) with \( \frac{z q_1'(z)}{q_1(z)} \) and \( \frac{z q_2'(z)}{q_2(z)} \) being starlike univalent. Suppose that \( q_1 \) satisfies (2.4) and \( q_2 \) satisfies (2.1). If \( f(z) \in \mathcal{A}(p) \),
\[
\left( \frac{z^p}{\Omega^\lambda_{\mu, \eta} f(z)} \right)^\gamma \in \mathcal{H}(q(0), 1] \cap Q,
\]
and \( \Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) \) is univalent in \( \mathcal{U} \), then
\[
\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta \frac{zq_1'(z)}{q_1(z)} < \Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z) < \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta \frac{zq_2'(z)}{q_2(z)},
\]

\((\lambda \geq 0; \mu < p + 1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}; \alpha, \delta, \xi, \gamma, \beta, \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)\)

implies

\[q_1(z) < \left(\frac{z^p}{\Omega_p^{\lambda, \mu, \eta} f(z)}\right)^\gamma < q_2(z), \quad (\gamma \in \mathbb{C} \setminus \{0\}), \quad (2.6)\]

and \(q_1\) and \(q_2\) are, respectively, the best subordinant and the best dominant, where \(\Psi_{\lambda, \mu, \eta}(\gamma, \xi, \beta, \delta, f)(z)\) is as defined in (2.2).

**Remark 3.** Taking \(p = 1\) and \(\lambda = \mu = 0\) in Theorems 2.4 and 2.5, we obtain the superordination and sandwich results for linear operator due to Shanmugam et al. ([11], Theorems 3.6 and 3.7, respectively).

**Remark 4.** For \(\lambda = \mu = 0\) and \(a = c\) in Theorem 2.5, we get the following result:

**Theorem 2.6.** Let \(q_1\) and \(q_2\) be univalent in \(U\) such that \(q_1(z) \neq 0\) and \(q_2(z) \neq 0\), \((z \in U)\) with \(\frac{zq_1'(z)}{q_1(z)}\) and \(\frac{zq_2'(z)}{q_2(z)}\) being starlike univalent. Suppose that \(q_1\) satisfies (2.4) and \(q_2\) satisfies (2.1). If \(f(z) \in A(p)\),

\[\left(\frac{z^p}{f(z)}\right)^\gamma \in H[q(0), 1] \cap Q,\]

and let

\[\Psi_1(\gamma, \xi, \beta, \delta, f)(z) = \alpha + \xi \left(\frac{z^p}{f(z)}\right)^\gamma + \delta \left(\frac{z^p}{f(z)}\right)^{2\gamma} + \beta \gamma \left(p - \frac{zf'(z)}{f(z)}\right)\]

is univalent in \(U\), then
ON SUBCLASSES OF p-VALENT FUNCTIONS ...

\[\alpha + \xi q_1(z) + \delta (q_1(z))^2 + \beta \frac{zq_1'(z)}{q_1(z)} < \Psi_1(\gamma, \xi, \beta, \delta, f)(z)\]

\[< \alpha + \xi q_2(z) + \delta (q_2(z))^2 + \beta \frac{zq_2'(z)}{q_2(z)},\]

\((p \in \mathbb{N}; \alpha, \delta, \xi, \beta \in \mathbb{C}; \gamma \neq 0; \beta \neq 0)\)

implies

\[q_1(z) \prec \left(\frac{z^p}{f'(z)}\right)^\gamma < q_2(z), \quad (\gamma \in \mathbb{C} \setminus \{0\}), \quad (2.7)\]

and \(q_1\) and \(q_2\) are, respectively, the best subordinant and the best dominant.

References


