A CLASSIFICATION OF REDUCTIVE
PREHOMOGENEOUS VECTOR SPACES

\((G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda'_1)(n = 2, 3)\)

OF SEPARATED TYPE WITH
FULL SCALARS

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Abstract

In this paper, we classify all reductive prehomogeneous vector spaces
\((G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda'_1, V(m_1) \otimes V(n) + V(m_2) \otimes V(n'))(n = 2, 3)\) of separated type with full scalars. They are obtained from reduced ones by castling transformations and we give the list of reduced such prehomogeneous vector spaces. We consider everything over the complex number field \(C\).

2010 Mathematics Subject Classification: Primary 11S90; Secondary 20G20.

Keywords and phrases: prehomogeneous vector spaces, classification.

Received June 17, 2011

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1. Introduction

If a rational representation $\rho : G \rightarrow GL(V)$ of an algebraic group $G$ has a Zariski-dense $G$-orbit, we call $(G, \rho, V)$ a prehomogeneous vector space (abbrev. PV). If $G^\circ$ is the connected component of $G$, then $(G, \rho, V)$ is a PV, if and only if $(G^\circ, \rho|_{G^\circ}, V)$ is a PV. Hence, in view of classification, we do not care the connectedness of $G$. When there is no confusion, we sometimes write $(G, \rho)$ or $(G, V)$ instead of $(G, \rho, V)$. When $G$ is reductive, we call it a reductive PV. For the basic facts of a PV, see [7]. Let $\rho_i : G_i \rightarrow GL(m_i)$ be a rational representation of an algebraic group $G_i(i = 1, ..., k)$. Then, we denote the representation $\rho = (\rho_1 \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \rho_k)$ of $G = G_1 \times \cdots \times G_k$ by $\rho_1 \boxplus \cdots \boxplus \rho_k$ for simplicity. We call a triplet $(G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)$ “of separated type” when $(G, \rho) = (G_1 \times \cdots \times G_r, \rho_1 \boxplus \cdots \boxplus \rho_r)$ and $(G, \sigma) = (G_{r+1} \times \cdots \times G_{r+s}, \sigma_1 \boxplus \cdots \boxplus \sigma_s)$, where each $\rho_i$ (resp., $\sigma_j$) is an irreducible representation of a reductive algebraic group $G_i(i = 1, ..., r)$ (resp., $G_{r+j}$ ($j = 1, ..., s$)). If each irreducible component $(G_i \times GL(n), \rho_i \otimes \Lambda_1)$ or $(G_{r+j} \times GL(n), \sigma_j \otimes \Lambda_1^*)$ has the least dimension among its $GL(n)$-castling equivalence class (see Definition 3.2), we call this triplet reduced.

Irreducible reduced prehomogeneous vector spaces of type $(G \times GL(n), \rho \otimes \Lambda_1) \cong (G \times GL(n), \rho \otimes \Lambda_1^*) (n \geq 2)$ with $(G, \rho) \not\cong (SL(m), \Lambda_1)$, $(Sp(m/2), \Lambda_1) (m = \text{even})$, $(SO(m), \Lambda_1)$ exist only for $n = 2, 3$ with one exception (see [12]), and they are important PVs, which are investigated in detail. So, we want to classify the wider class of PVs of this type.

In this paper, we shall classify all the reductive PVs $(G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*) (n = 2, 3)$ of separated type with full scalars. Hence, we may assume that $GL(1) \cdot I_{m_i} \subset \rho_i(G_i)$ and $GL(1) \cdot I_{n_j} \subset \sigma_j(G_{r+j})$ with
$m_i = \deg \rho_i$ and $n_j = \deg \sigma_j$. We remark that there is some research for PVs with bigger $n$ (see [10]).

In Section 2, we give the main results of this paper, namely, we give the list of reductive reduced PVs $(G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)$ ($n = 2, 3$) of separated type with full scalars.

In Section 3, we give the preliminaries. The essential tools for this classification are Propositions 3.6 and 3.8.

In Section 4, we classify all irreducible PVs $(G \times GL(n), \rho \otimes \Lambda_1)$ ($n \geq 2$), where the $GL(n)$-part of a generic isotropy subgroup is $GL(n)$ itself. This implies that for any $\sigma : H \to GL(W)$, a triplet $(H \times GL(n), \sigma \otimes \Lambda_1, W \otimes V(n))$ is a PV, if and only if $(H \times G \times GL(n), \sigma \otimes 1 \otimes \Lambda_1 + 1 \otimes \rho \otimes \Lambda_1^*, W \otimes V(n) + V \otimes V(n)^*)$ is a PV.

In Section 5, we deal with the case $n = 2$. Since $\Lambda_1 = \Lambda_1^*$ for $SL(2)$, it is enough to consider the type $(G \times GL(2), \rho \otimes \Lambda_1)$ with full scalars for $n = 2$. Note that this does not hold without the condition of full scalars. For example, $(SL(1) \times SL(1) \times GL(2), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a PV, while $(SL(1) \times SL(1) \times GL(2), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*)$ is a non PV. However, we have $(GL(1) \times GL(1) \times GL(2), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1) \cong (GL(1) \times GL(1) \times GL(2), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*)$.

In Section 6, we deal with the case $n = 3$. Even for $n = 3$, $(G \times GL(3), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)$ is castling equivalent to the form $(G \times GL(3), \tau \otimes \Lambda_1)$ (see Proposition 3.9). However, we cannot assume that $(G \times GL(3), \tau \otimes \Lambda_1)$ is reduced. If we assume that it is reduced, we need the form $(G \times GL(3), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)$ in general.

In Section 7, we check their quasi-irreducibility and completely quasi-
reducibility. This notion was introduced by Rubenthaler ([11]).

Notation

We denote by $P(e_1, \ldots, e_r)$ the standard parabolic subgroup of $GL(n)$ with $n = e_1 + \cdots + e_r$. We denote by $\Lambda_1$ the standard representation of $GL(n)$ on $\mathbb{C}^n$. For a subgroup $H$ of $GL(n)$, the restriction $\Lambda_1|_H$ is also simply denoted by $\Lambda_1$. More generally, $\Lambda_k(k = 1, \ldots, r)$ denotes the fundamental irreducible representation of a simple algebraic group of rank $r$. In general, we denote by $\rho^*$ the dual representation of a rational representation $\rho$. The symbol $\Lambda_1^{(e)}$ implies $\Lambda_1$ or $\Lambda_1^*$. We denote by $(G_2)$ the exceptional simple algebraic group of rank 2 to distinguish from the second group $G_2$.

Since $\oplus$ and $\otimes$ (resp., $\oplus$ and $\boxtimes$) are hard to distinguish, in this paper, we use $+$ instead of $\oplus$.

We denote by $V(n)$ an $n$-dimensional vector space in general. If we use $V(n)^*$ with $V(n)$, then $V(n)^*$ denotes the dual vector space of $V(n)$.

2. Main Results

First, we give the main result for $n = 2$. Since $\Lambda_1 = \Lambda_1^*$ for $SL(2)$ and we assume the full scalars, we have $\left(G \times GL(2), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*\right) \cong \left(G \times GL(2), (\rho + \sigma) \otimes \Lambda_1\right)$.

**Theorem 2.1.** $(n = 2)$. Let $(G_i', \rho_i', V_i')$ be one of

1. $(GL(k) \times H, \Lambda_1 \otimes \sigma, V(k) \otimes V(h))$ with $k \geq 2h$ and $h \geq 1$, where $\sigma$ is an irreducible representation of any semisimple algebraic group $H$;
2. $(GL(2m + 1), \Lambda_2, V(m(2m + 1)))(m \geq 2)$;
3. $(GSp(m), \Lambda_1, V(2m))$;
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(4) \((GL(1) \times Spin(10), \Lambda_1 \otimes a \text{ half-spin rep.}, V(1) \otimes V(16))\);

and let \((G_i, \rho_i, V_i)\) be different from these 4 triplets satisfying \(GL(1) \cdot I_{V_j} \subset \rho_j(G_j)\). Then \((G_1 \times \cdots \times G_r \times G'_1 \times \cdots \times G'_s \times GL(2), (\rho_1 \boxplus \cdots \boxplus \rho_r \boxplus \rho'_1 \boxplus \cdots \boxplus \rho'_s) \otimes \Lambda_1)\) is a PV, if and only if \((G_1 \times \cdots \times G_r \times GL(2), (\rho_1 \boxplus \cdots \boxplus \rho_r) \otimes \Lambda_1)\) is a PV. Such PVs are given as follows:

1. \((GO(n) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(2)).\)

2. \((G Spin(7) \times GL(2), the \text{ spin rep.} \otimes \Lambda_1, V(8) \otimes V(2)).\)

3. \(((GL(1) \times (G_2)) \times GL(2), (\Lambda_1 \otimes \Lambda_2) \otimes \Lambda_1, V(7) \otimes V(2)).\)

4. \((GL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2)).\)

5. \((GL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2)).\)

6. \((SL(3) \times GL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2)).\)

7. \(((GL(1) \times E_6) \times GL(2), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, V(27) \otimes V(2)).\)

8. \((\underbrace{GL(1) \times \cdots \times GL(1)}_{k} \times GL(2), \underbrace{(\Lambda_1 \boxplus \cdots \boxplus \Lambda_1)}_{k} \otimes \Lambda_1, V(k) \otimes V(2)),\)

\((1 \leq k \leq 3).\)

9. \((GL(1) \times GO(n) \times GL(2), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1, (V(1) + V(n)) \otimes V(2)).\)

10. \((GL(1) \times G Spin(7) \times GL(2), (\Lambda_1 \boxplus \text{ the spin rep.}) \otimes \Lambda_1, (V(1) + V(8)) \otimes V(2)).\)

11. \((GL(1) \times (GL(1) \times (G_2)) \times GL(2), (\Lambda_1 \boxplus (\Lambda_1 \otimes \Lambda_2)) \otimes \Lambda_1, (V(1) + V(7)) \otimes V(2)).\)

Now, we give the main result for \(n = 3\). Note that \((G \times GL(3), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*) \cong (G \times GL(3), \rho \otimes \Lambda_1^* + \sigma \otimes \Lambda_1).\) It is castling equivalent to the
form \((G \times GL(3), \tau \otimes \Lambda_1)\), which is not reduced in general. For example, 
\[(GSp(m_1) \times GSp(m_2) \times GSp(m_3) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1) \oplus 1 \oplus 1 \oplus \Lambda_1 + 1 \oplus 1 \otimes (\Lambda_1 \oplus \Lambda_1) \oplus \Lambda_1^*) \equiv (GSp(m_1) \times GSp(m_2) \times GL(6m_3 - 1) \times Sp(m_3) \times GL(2) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus (\Lambda_1 \otimes \Lambda_1) \oplus \Lambda_1) \oplus \Lambda_1).\]

**Theorem 2.2.** \((n = 3)\). (1) Let \(\tau : \tilde{H} \to GL(h)\) be any representation of any semisimple algebraic group \(\tilde{H}\). Then \((G \times \tilde{H} \times GL(k) \times GL(3), (\rho \oplus (\tau \otimes \Lambda_1) \otimes \Lambda_1 + \sigma \otimes (1 \otimes 1) \otimes \Lambda_1^*)) (3h \leq k)\) is a PV, if and only if \((G \times GL(3), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)\) is a PV.

(2) We can obtain all PVs \((G \times GL(3), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)\) of separated type with full scalars from the following list by this PV-equivalence or castling transformations:

(1) \((\tilde{H} \times GL(k) \times GL(3), \sigma \otimes \Lambda_1 \otimes \Lambda_1, V(h) \otimes V(k) \otimes V(3))\) with \(3h \leq k\), where \(\sigma : \tilde{H} \to GL(h)\) is any representation of any semisimple algebraic group \(\tilde{H}\).

(2) \((GSp(m) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(2m) \otimes V(3))(m \geq 2)\).

(3) \((GL(1) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(3))\).

(4) \((GO(m) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(3))(m \geq 3)\).

Note that \((SL(2), 2\Lambda_1) = (SO(3), \Lambda_1), (SL(2) \times SL(2), \Lambda_1 \otimes \Lambda_1) = (SO(4), \Lambda_1), (Sp(5), \Lambda_2) = (SO(5), \Lambda_1), (SL(4), \Lambda_2) = (SO(6), \Lambda_1)\).

(5) \((GL(5) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(3))\).

(6) \((G Spin(7)) \times GL(3), the spin rep. \otimes \Lambda_1, V(8) \otimes V(3))\).

(7) \((G Spin(10)) \times GL(3), a half-spin rep. \otimes \Lambda_1, V(16) \otimes V(3))\).

(8) \((SL(2) \times GL(3) \times GL(3), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(3) \otimes V(3))\).
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(9) \((GL(2) \times GL(3), 3\Lambda_1 \otimes \Lambda_1)\).

(10) \((SL(2) \times GL(3) \times GSp(m) \times GL(3), \Lambda_1 \otimes \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 2)\).

(11) \((SL(2) \times GL(3) \times GL(1) \times GL(3), \Lambda_1 \otimes \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

(12) \((GL(5) \times GSp(m) \times GL(3), \Lambda_2 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 2)\).

(13) \((GO(m_1) \times GSp(m_2) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m_1 \geq 3, m_2 \geq 2)\).

(14) \((G Spin(7) \times GSp(m) \times GL(3), \text{the spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 2)\).

(15) \((G Spin(10) \times GSp(m) \times GL(3), \text{a half-spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 2)\).

(16) \((GL(5) \times GL(1) \times GL(3), \Lambda_2 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

(17) \((GO(m) \times GL(1) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 3)\).

(18) \((G Spin(7) \times GL(1) \times GL(3), \text{the spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

(19) \((G Spin(10) \times GL(1) \times GL(3), \text{a half-spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

(20) \((GSp(m_1) \times GSp(m_2) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m_1 \geq 2, m_2 \geq 2)\).

(21) \((GSp(m) \times GL(1) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 2)\).
(22) \((GL(1) \times GL(1) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*)\).

(23) \((GSp(m_1) \times GSp(m_2) \times GSp(m_3) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\((\Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m_1 \geq 2, m_2 \geq 2, m_3 \geq 2).

(24) \((GSp(m_1) \times GSp(m_2) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\(\Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m_1 \geq 2, m_2 \geq 2).

(25) \((GSp(m) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\(\Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m \geq 2).

(26) \((GL(1) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\(\Lambda_1 \oplus \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m_1 \geq 2, m_2 \geq 2, m_3 \geq 2, m_4 \geq 2).

(27) \((GSp(m_1 \times GSp(m_2) \times GSp(m_3) \times GSp(m_4) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\((\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m_1 \geq 2, m_2 \geq 2, m_3 \geq 2).

(28) \((GSp(m_1 \times GSp(m_2) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\((\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m \geq 2, m_2 \geq 2, m_3 \geq 2).

(29) \((GSp(m_1 \times GSp(m_2) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\((\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m \geq 2, m_2 \geq 2).

(30) \((GSp(m_1 \times GL(1) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1)\) (resp.,

\((\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*) (m_1 \geq 2, m_2 \geq 2).\)
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\( \otimes \Lambda_1 + 1 \otimes 1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^* (m_1 \geq 2). \)

(31) \((GL(1) \times GL(1) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)\)

(resp., \((\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*).\)

3. Preliminaries

A non-zero rational function \(f(v)\) on \(V\) is called an absolute invariant of \((G, \rho, V)\), if \(f(p(g)v) = f(v)\) for any \(g \in G\) as rational functions.

**Proposition 3.1** ([7, Proposition 2.4]). A triplet \((G, \rho, V)\) is a non PV, if and only if there exists a non-constant absolute invariant.

Let \((G, \rho, V)\) be a PV and \(O\) its Zariski-dense \(G\)-orbit. The isotropy subgroup \(G_v = \{g \in G| \rho(g)v = v\}\) at \(v \in O\) is called a generic isotropy subgroup of \((G, \rho, V)\). Since \(G_{p(g)}v = gG_vg^{-1}\) for \(g \in G\), any generic isotropy subgroup is conjugate of each other.

In general, for a subgroup \(K\) of a group \(G \times H\), the image of \(K\) by the second projection \(G \times H \rightarrow H\) will be called the \(H\)-part of \(K\).

Let \((G, \rho, V)\) be a PV and \(\sigma : G \rightarrow H\) a homomorphism. Applying \(\sigma\) to \(G_{p(g)}v = gG_vg^{-1}\), we have \(\sigma(G_{p(g)}v) = \sigma(g)\sigma(G_v)\sigma(g)^{-1}\) for \(g \in G\) and \(v \in V\). In particular, the \(GL(n)\)-part of any generic isotropy subgroup of \((G \times GL(n), \rho \otimes \Lambda_1)\) is conjugate of each other.

**Definition 3.2.** Let \(\rho : G \rightarrow GL(V)\) be a rational representation and \(H\) a connected algebraic subgroup of \(GL(n)\). We say that \((G, \rho, V)\) is of \(H\)-isotropy type, if \((G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))\) is a PV and the connected component of the \(GL(n)\)-part of its generic isotropy subgroup is conjugate to \(H\). On the other hand, \((G, \rho, V)\) is called of \(H\)-prehomogeneous type, if \((G \times H, \rho \otimes \Lambda_1, V \otimes V(n))\) is a PV. If an irreducible triplet \((G, \rho, V)\) is of \(H\)-isotropy type or \(H\)-prehomogeneous
type with $H \subset GL(n)$ satisfying $\deg \rho > n$, then $(GL(1) \times G, \Lambda_1 \otimes \Lambda_1)$ must be an irreducible PV by [8, Proposition 3.2].

Recall that $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))(m > n)$ is a PV, if and only if $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a PV and their generic isotropy subgroups are isomorphic. They are called a castling transform of each other, and $(G, \rho, V)$ and $(G', \rho', V')$ are castling equivalent if by a finite number of castling transformations, they are transformed to each other.

We say that $(G, \rho, V)$ and $(G', \rho', V')$ are $GL(n)$-castling equivalent, if $(G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ and $(G' \times GL(n), \rho' \otimes \Lambda_1, V' \otimes V(n))$ are castling equivalent.

We say $(G, \rho, V)$ is PV-equivalent to $(H, \sigma, W)$ when $(G, \rho, V)$ is a PV, if and only if $(H, \sigma, W)$ is a PV. For example, castling equivalence is one of PV-equivalence.

The following assertion is clear:

**Lemma 3.3.** Let $H_1, H_2$ be closed subgroups of $GL(n)$. Then the isotropy subgroup of $H_1 \times H_2$ at $g \in GL(n)(\subset M(n))$ of a triplet $(H_1 \times H_2, \Lambda_1 \otimes \Lambda^*_1, M(n))$ is $(H_1 \times H_2)_g = \{(h_1, h_2) \in H_1 \times H_2 | h_1gh_2^{-1}$

$= g\} \cong H_1 \cap gH_2g^{-1} \cong H_2 \cap g^{-1}H_1g$. In particular, the $H_1$-part and the $H_2$-part of the isotropy subgroup are conjugate and isomorphic to the isotropy subgroup itself.

**Definition 3.4.** For a PV of the type $(G_1 \times \cdots \times G_k \times GL(n), \rho_1 \otimes 1$ 
$
\otimes \cdots \otimes 1 \otimes \Lambda_1^{(*)} + \cdots + 1 \otimes \cdots \otimes \rho_k \otimes \Lambda_1^{(*)}, V_1 \otimes V(n)^{(s)} + \cdots + V_k \otimes V(n)^{(s)}),$ we denote the connected component of $GL(n)$-part of a generic isotropy subgroup by $H(1^{(*)}, \ldots, k^{(*)})$, where $i^{(*)} = i$ or $i^*(1 \leq i \leq k)$ according to $\Lambda_1^{(*)} = \Lambda_1$ or $\Lambda_1^*$. 

Proposition 3.5. Assume that \((G_i \times GL(n), \rho_i \otimes \Lambda_1^{(i)})\) is a PV with \(H(i^{(i)})\) for \(i = 1, 2\). Then the following assertions are equivalent:

1. \((G_1 \times G_2 \times GL(n), \rho_1 \otimes 1 \otimes \Lambda_1^{(1)} + 1 \otimes \rho_2 \otimes \Lambda_1^{(2)})\) is a PV with \(H(1^{(1)}, 2^{(2)})\).

2. \((H(1^{(i)})) \times H(2^{(i)}), \Lambda_1 \otimes \Lambda_1^*, M(n))\ is a PV and the connected component of its generic isotropy subgroup is isomorphic to \(H(1^{(i)}, 2^{(i)})\). In particular, each \(H(i^{(i)})\)-part is conjugate to \(H(1^{(i)}, 2^{(i)})\).

Proof. Since the generic isotropy subgroup of \((GL(n) \times GL(n), \Lambda_1 \otimes \Lambda_1^*, M(n))\) at the identity matrix \(I_n\) is given by \(\{(A, B) \in GL(n) \times GL(n) \mid AI_nB^{-1} = I_n\}\), we see that \((G_1 \times G_2 \times GL(n), \rho_1 \otimes 1 \otimes \Lambda_1^{(1)} + 1 \otimes \rho_2 \otimes \Lambda_1^{(2)})\) is a PV, if and only if \(((G_1 \times GL(n)) \times (G_2 \times GL(n)), (\rho_1 \otimes \Lambda_1^{(1)}) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (\rho_2 \otimes \Lambda_1^{(2)}) + 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)\) is a PV and the connected component of the \(GL(n)\)-part of its generic isotropy subgroup is \(H(1^{(1)}, 2^{(2)})\). Since \((G_i \times GL(n), \rho_i \otimes \Lambda_1^{(i)})\) is a PV with \(H(i^{(i)})\) for \(i = 1, 2\), it is equivalent that \((H(1^{(i)}) \times H(2^{(i)}), \Lambda_1 \otimes \Lambda_1^*, M(n))\ is a PV, where the connected component of a generic isotropy subgroup is \(H(1^{(i)}, 2^{(i)})\) (cf. Lemma 3.3).

For a subgroup \(H\) of \(GL(n)\), let \(\text{codim} H = \dim GL(n) - \dim H\).

Proposition 3.6. We use the notation in Definition 3.4. For \((G_1 \times \cdots \times G_k \times GL(n), \rho_1 \otimes 1 \otimes \cdots \otimes 1 \otimes \Lambda_1^{(i)} + \cdots + 1 \otimes \cdots \otimes 1 \otimes \rho_k \otimes \Lambda_1^{(i)})\), we have \(\text{codim} H(1^{(i)}, \ldots, k^{(i)}) = \text{codim} H(1^{(i)}, \ldots, (k-1)^{(i)}) + \text{codim} H(k^{(i)}) = \sum_{i=1}^{k} \text{codim} H(i^{(i)}) \leq n^2\).
Proof. It is enough to prove when \( k = 2 \). By Proposition 3.5, \( H(1^{(i)}, 2^{(i)}) \) is a generic isotropy subgroup of \( (H(1^{(i)}) \times H(2^{(i)}), \Lambda_1 \otimes \Lambda_1^*, M(n)) \) and hence, we have \( \dim H(1^{(i)}) + \dim H(2^{(i)}) - n^2 = \dim H(1^{(i)}, 2^{(i)}) \). This implies that \( \text{codim } H(1^{(i)}, 2^{(i)}) = \text{codim } H(1^{(i)}) + \text{codim } H(2^{(i)}) \).

Proposition 3.7. Let \((G_i \times GL(n), \rho_i \otimes \Lambda_i)\) be a PV with \( H(i^{(i)}) \subset SL(n) \) for \( i = 1, 2 \). Then \((G_1 \times G_2 \times GL(n), \rho_1 \otimes 1 \otimes \Lambda_1^{(i)} + 1 \otimes \rho_2 \otimes \Lambda_1^{(i)})\) is a non PV.

Proof. \((H(1^{(i)}) \times H(2^{(i)}), \Lambda_1 \otimes \Lambda_1^*, M(n))\) is a non PV by Proposition 3.1, since \( \det x (x \in M(n)) \) is a non-constant absolute invariant. Hence, by Proposition 3.5, we have our result.

Proposition 3.8 (cf. [9, Proposition 1.8] for finite PVs). Let \( \rho \) be a rational representation of any algebraic group \( H \). Then, the following assertions hold and the GL(3)-part of a generic isotropy subgroups are conjugate:

1. (a) \(((H \times GSp(m)) \times GL(3), (\rho \boxplus \Lambda_1) \otimes \Lambda_1)\) is a PV. ⇔
   (b) \(((H \times GL(2)) \times GL(3), (\rho \boxplus \Lambda_1) \otimes \Lambda_1)\) is a PV. ⇔
   (c) \((H \times GL(1) \times GL(3), \rho \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*)\) is a PV.

2. (d) \(((H \times GSp(m) \times GL(3), \rho \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*)\) is a PV. ⇔
   (e) \(((H \times GL(1)) \times GL(3), (\rho \boxplus \Lambda_1) \otimes \Lambda_1)\) is a PV. ⇔
   (f) \(((H \times GL(2) \times GL(3), \rho \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^*)\) is a PV.

Proof. The GL(3)-part of a generic isotropy subgroup of \((GSp(m) \times GL(3), \Lambda_1 \otimes \Lambda_1, M(2m, 3))\) is conjugate to the parabolic subgroup \( P(2, 1) \) (see p. 102 in [12], for \( m \geq 3 \). Note that it holds similarly for \( m = 2 \)). Hence (a) is PV-equivalent to \((H \times P(2, 1), \rho \otimes \Lambda_1)\).
However, the $GL(3)$-part of the generic isotropy subgroup at $(I_2|O) \in M(2, 3)$ of $(GL(2) \times GL(3), \Lambda_1 \otimes \Lambda_1, M(2, 3))$ is also $P(2, 1)$. Hence (a) is PV-equivalent to (b). Since (c) is a castling transform of (b) at $GL(2)$, we have (1). Since, we have $\tilde{I}_3^{-1}\Lambda_1^*(P(2, 1))\tilde{I}_3 = P(1, 2)$ for $\tilde{I}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, the $GL(3)$-part of a generic isotropy subgroup of $(GSp(m) \times GL(3), \Lambda_1 \otimes \Lambda_1^*)$ is conjugate to $P(1, 2)$. Hence (d) is PV-equivalent to $(H \times P(1, 2), \rho \otimes \Lambda_1)$. On the other hand, the $GL(3)$-part of the generic isotropy subgroup at $(1, 0, 0) \in M(1, 3)$ of $(GL(1) \times GL(3), \Lambda_1 \otimes \Lambda_1, M(1, 3))$ is $P(1, 2)$. Hence (d) and (e) are PV-equivalent. Note that (f) is a castling transform of (e) at $GL(1)$.

**Proposition 3.9.** Any PV of the form
\[
((G_1 \times \cdots \times G_r) \times (H_1 \times \cdots \times H_s) \times GL(n), (\rho_1 \boxplus \cdots \boxplus \rho_r) \otimes (1 \boxplus \cdots \boxplus 1) \otimes \Lambda_1 + (1 \boxplus \cdots \boxplus 1) \otimes (\sigma_1 \boxplus \cdots \boxplus \sigma_s) \otimes \Lambda_1^*)
\]
with full scalars is castling equivalent to the form
\[
((G_1 \times \cdots \times G_r \times H_1' \times \cdots \times H_s') \times GL(n), (\rho_1 \boxplus \cdots \boxplus \rho_r \boxplus \sigma_1' \boxplus \cdots \boxplus \sigma_s') \otimes \Lambda_1).
\]
However, for $n \geq 3$, if we assume that it is of the least dimension among the castling-equivalence class, it must be of the former form in general. For $n = 2$, it is enough to consider only the latter form.

**Proof.** For example, if we take $(H_j^*, \sigma_j^*) = (GL(m_j - 1) \times H_j, \Lambda_1 \otimes \sigma_j^*)$ with $m_j = \deg \sigma_j(j = 1, \ldots, s)$, then our assertion will be satisfied. Since we assume the full scalars, we obtain our assertion for $n = 2$ by the fact $\Lambda_1 = \Lambda_1^*$ for $SL(2)$.

**Proposition 3.10.** For any natural numbers $n_1, n_2, n_3, n_4, n$, the following holds:

1. $((GL(n_1) \times GL(n_2) \times GL(n_3)) \times GL(n), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)$ is a PV.
$$(2) \ ( (GL(n_1) \times GL(n_2) \times GL(n_3) \times GL(n_4)) \times GL(n), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1 \) \ is \ a \ PV, \ if \ and \ only \ if \ n_1 + n_2 + n_3 + n_4 \neq 2n.$$

**Proof.** (1) is a part of the famous Gabriel’s theorem about finite quivers ([1]). For (2), see Theorem 9.10 in p. 243 in [4].

\[ \square \]

### 4. Irreducible Triplets of $GL(n)$-Isotropy Type

In this section, we shall classify all irreducible PVs $$(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$$ with $n \geq 2$, where the $GL(n)$-part of its generic isotropy subgroup is $GL(n)$. Namely, we classify all irreducible triplets of $GL(n)$-isotropy type. This implies that $$(G \times \{I_n\}, \rho \otimes \Lambda_1, V(m) \otimes V(n))$$ is a PV. Since $\rho(G) \subset GL(m)$, we have $m^2 \geq \dim \rho(G) \geq mn$ and hence we obtain $m \geq n$.

If $m = n$, then the isotropy subgroup at $\{I_n\}$ is $\{(g, \rho(g)^{-1}) | g \in G\}$ and its $GL(n)$-part is $GL(n) = \rho^*(G)(\subset GL(n))$. Hence $G, \rho, V(m)$ must be $$(GL(n), \Lambda_1, V(n))$$ and we have $$(GL(n) \times GL(n), \Lambda_1 \otimes \Lambda_1, M(n)).$$

Now, we shall consider the case $m > n$. Assume that $$(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$$ is castling equivalent to a reduced irreducible PV $$(H_0, \rho_0, V_0).$$ Then its generic isotropy subgroup contains $GL(n)$ since generic isotropy subgroups are invariant up to isomorphism under castling transformations. Since we have the list of reduced irreducible PVs in [12], we should be able to find a reduced irreducible PV such that a generic isotropy subgroup contains $GL(n)$, and it is castling equivalent to a PV of a form $$(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n)).$$ For example, the $GL(5)$-part of a generic isotropy subgroup of $$(GL(8) \times GL(5), \Lambda_1 \otimes \Lambda_1)$$ is $GL(5)$, but it is castling-equivalent to $$(GL(8) \times GL(3), \Lambda_1 \otimes \Lambda_1).$$
First we deal with the case, which is not castling equivalent to a trivial PV (see [12, p. 43]). If the $GL(n)$-part of a generic isotropy subgroup of $(H \times GL(n), \rho \otimes \Lambda_1)$ is $SL(n)$, then that of $(GL(1) \times H \times GL(n), \Lambda_1 \otimes \rho \otimes \Lambda_1)$ is $GL(n)$. For the regular PVs, there exists a table of 29 reduced irreducible regular PVs in [7, Section 2.4] or [12, Section 7], and we use the numbers there.

A generic isotropy subgroup of (2), (3), (4), (6), (8), (11), (12), (13) with $m \geq 2$, (15), (16), (19), (24), (27), (28), (29) does not contain any $SL(n)$ with $n \geq 2$.

A generic isotropy subgroup of (10), (18), (21) (resp., (5), (7), (14), (17), (25); (22); (23)) contains $SL(n)$ with $n = 2$ (resp., 3; 5; 6), but it is not castling equivalent to the form $(G \times GL(n), \rho \otimes \Lambda_1)$ with $n = 2$ (resp., 3; 5; 6).

A generic isotropy subgroup of (9) $(SL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2))$ is locally isomorphic to $SL(2) \times SL(2) \times SL(2)$, but $GL(2)$-part of a generic isotropy subgroup is a finite group. A generic isotropy subgroup of (26) $((G_2) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(7) \otimes V(2))$ is $GL(2)$, but the $GL(2)$-part of a generic isotropy subgroup is $GL(1)$ (or rather we should say $SO(2)$).

The $GL(2)$-part of a generic isotropy subgroup of (12) $(Sp(n) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2))$ and (20) $(Spin(10) \times GL(2)$, a half-spin rep. $\otimes \Lambda_1, V(16) \otimes V(2))$ is $SL(2)$. Hence, the $GL(2)$-part of a generic isotropy subgroup of $((GL(1) \times Sp(n)) \times GL(2), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, (V(1) \otimes V(2n)) \otimes V(2))$ and $((GL(1) \times Spin(10)) \times GL(2), (\Lambda_1 \otimes \Lambda_1, (V(1) \otimes V(16)) \otimes V(2))$ is $GL(2)$.

For the case of non-regular non-trivial reduced PVs (see [12, Section 7]), the $GL(2)$-part of only $(SL(m + 1) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(m(2m + 1)) \otimes V(2))$ is $GL(2)$.

Next we deal the case, which is castling equivalent to a trivial PV.
If \((G' \times GL(n), \rho' \otimes \Lambda_1, V(m') \otimes V(n))\) is castling equivalent to a trivial PV and the \(GL(n)\)-part of a generic isotropy subgroup is \(GL(n)\), by doing castling transformations on \(G'\)-part, we have \((G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))\), which has the least dimension among them. The \(GL(n)\)-part of its generic isotropy subgroup is also \(GL(n)\).

Lemma 4.1. Let \(G\) be a simple algebraic group and \(\rho : G \to GL(m)\) be an irreducible representation. Then, except \((G, \rho) = (SL(m), \Lambda_1)\), \((SL(m), \Lambda_{m-1}(= \Lambda_1^*))\), \((Sp(m'), \Lambda_1)\) with \(2m' = m\), we have \(\dim G < m^2 / 2\).

Proof. This is a refinement of [12, Proposition 42, Section 1]. Assume that \(G = SL(r)\) and let \(\Lambda\) be its irreducible representation different from \(\Lambda_1, \Lambda_{r-1}\). If \(r \geq 4\), we have \(d(\Lambda) \geq d(\Lambda_2) = d(\Lambda_{r-2}) = r(r-1)/2 = m\) and \(\dim G = r^2 - 1 < m^2 / 2\). If \(G = SL(2)\) (resp., \(SL(3)\)), we have \(d(\Lambda) \geq d(2\Lambda_1) = 3\) (resp., 6), and we have \(\dim G = 3\) (resp., 8) \(< m^2 / 2 = 9 / 2\) (resp., 18). Next, assume that \(G = Sp(r)(r \geq 2)\). Let \(\Lambda\) be its irreducible representation different from \(\Lambda_1\). Then, we have \(d(\Lambda) \geq d(\Lambda_2) = (r-1)(2r+1)\) (see [12, p. 15]), and we have \(\dim G = r(2r+1) < (r-1)^2(2r+1)^2 / 2 \leq m^2 / 2\). If \(G = Spin(r)\) or \(SO(r)\), we have \(d(\Lambda) \geq d(\Lambda_1) = r(= m)\), we have \(\dim G = r(r-1)/2 < m^2 / 2\). If \(G\) is of type \((G_2)\) (resp., \(F_4, E_6, E_7, E_8\)), the degree of the least dimension is 7 (resp., 26, 27, 56, 248) and \(\dim G = 14\) (resp., 52, 78, 133, 248). Hence, we have \(\dim G < m^2 / 2\).
Proposition 4.2. Let \((G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))\) be a reductive (not necessarily irreducible) PV with \(2n > m > n \geq 2\). If the \(GL(n)\)-part of its generic isotropy subgroup is \(GL(n)\), then we have \((G, \rho) = (GL(m), \Lambda_1)\) or \((SL(m), \Lambda_1)\).

Proof. Note that our condition implies the prehomogeneity of \((G, \rho \otimes \cdots \otimes \rho(n \text{ copies}))\) and hence, we have \(\dim G \geq mn > m^2/2\). First, we show that \(\rho\) must be irreducible. If not, we can express \(\rho = \rho_1 \oplus \rho_2\) with \(\deg \rho_i = m_i (i = 1, 2)\) satisfying \(m_1 \leq m_2\), where each \(\rho_i\) is not necessarily irreducible. Then we have \(\rho_1(G) \subset GL(m_1)\). The prehomogeneity of \((G, \rho \otimes \cdots \otimes \rho(n \text{ copies}))\) implies the prehomogeneity of \((GL(m_1), \Lambda_1 \otimes \cdots \otimes \Lambda_1(n \text{ copies}))\). However, since \(m_1 \leq m/2 < n\), the dimension of the group is less than that of the space and this cannot be a PV, a contradiction. Next assume that \(G = G_1 \times G_2\). Since \(\rho\) is irreducible, we have \(\rho = \sigma_1 \otimes \sigma_2\), where \(n_i = \deg \sigma_i \geq 2(i = 1, 2)\) and \(n_1 \geq n_2\). Then \(\rho(G) \subset GL(n_1) \otimes GL(n_2)\) and we have \(n_1^2 + n_2^2 \geq \dim G > m^2/2 = (n_1n_2)^2/2 \geq 2n_1^2\), a contradiction. Hence \(G\) must be a simple algebraic group \(H\) or \(GL(1) \times H\). Then, by Lemma 4.1, \((H, \rho)\) must be \((SL(m), \Lambda_1)\) (\(\equiv (SL(m), \Lambda_{n-1})\)) or \((Sp(m'), \Lambda_1)\) with \(m = 2m'\). However, the \(GL(n)\)-part of a generic isotropy subgroup of \(((GL(1) \times Sp(m')) \times GL(n), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, V(m) \otimes V(n))\) with \(2n > m = 2m' > n \geq 2\) does not contain \(SL(n)\). Hence, we obtain our assertion.

From above, we obtain the following theorem:

Theorem 4.3. Let an irreducible triplet \((G, \rho, V)\) with \(GL(1) \cdot I_V \subset \rho(G)\) be of \(GL(n)\)-isotropy type. Then \((G, \rho, V)\) is \(GL(n)\)-castling equivalent to one of the following triplets:

1. \((n \geq 2)(GL(k) \times H, \Lambda_1 \otimes \sigma, V(k) \otimes V(h))\) with \(k \geq nh\) and \(h \geq 1\), where \(\sigma\) is an irreducible representation of any semisimple algebraic group \(H\).
(2) \((n = 2)(GL(2m + 1), \Lambda_2, V(m(2m + 1)) (m \geq 2).\)

(3) \((n = 2)(GL(1) \times Sp(m), \Lambda_1 \otimes \Lambda_1, C \otimes V(2m)).\)

(4) \((n = 2)(GL(1) \times Spin(10), \Lambda_1 \otimes \text{a half-spin rep.}, C \otimes V(16)).\)

**Proof.** Only we have to remark that the case of \((GL(m) \times GL(n), \Lambda_1 \otimes \Lambda_1)(\text{resp., } (SL(m) \times GL(n), \Lambda_1 \otimes \Lambda_1))\) with \(2n > m > n \geq 2\), is contained in the case 1 (resp., the case 2).

**Remark 4.4.** Although we do not deal with the following case, we obtain the following results. Let an irreducible triplet \((G, \rho, V)\) with a semisimple algebraic group \(G\) be of \(GL(n)\)-isotropy type. Then \((G, \rho, V)\) is \(GL(n)\)-castling equivalent to one of the following triplets:

(1) \((n \geq 2)(SL(k) \times H, \Lambda_1 \otimes \sigma, V(k) \otimes V(h))\) with \(k > nh\) and \(h \geq 1\), where \(\sigma\) is an irreducible representation of any semisimple algebraic group \(H\).

(2) \((n = 2)(SL(2m + 1), \Lambda_2, V(m(2m + 1))\) with \(m \geq 2\).

**Theorem 4.5.** The following assertions are equivalent:

(1) \((G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)\) is a PV.

(2) \(((G \times G_1 \times \cdots \times G_r) \times GL(n), (\rho \oplus \rho_1 \oplus \cdots \oplus \rho_r) \otimes \Lambda_1 + (\sigma \oplus \rho_r \oplus \cdots \oplus \rho_k) \otimes \Lambda_1^*)\) is a PV, where each \((G_i, \rho_i)\) is any irreducible triplet of \(GL(n)\)-isotropy type \((i = 1, \ldots, k)\).

**Proof.** It is clear by definition.

By this Theorem 4.5, from now on, we shall consider everything up to irreducible triplets of \(GL(n)\)-isotropy type, i.e., irreducible triplets, which are \(GL(n)\)-castling equivalent to triplets in Theorem 4.3.
5. A Classification of Reductive PVs

\((G \times GL(2), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)\) of Separated Type

Since we assume the full scalars, by Proposition 3.9, it is enough to consider only the form \(((G_1 \times \cdots \times G_k) \times GL(2), (\rho_1 \oplus \cdots \oplus \rho_k) \otimes \Lambda_1)\), where each \(\rho_i\) is an irreducible representation of a reductive algebraic group \(G_i\) satisfying \(GL(1) \cdot I \subset \rho_i(G_i)\). We may assume that each component \((G_i \times GL(2), \rho_i \otimes \Lambda_1)\) has the least dimension among the castling equivalent class. Let \(H(1, \ldots, k)\) be the \(GL(2)\)-part of a generic isotropy subgroup (see Definition 3.4). We classify them according to \(\text{codim } H(1, \ldots, k)(= 4 - \text{dim } H(1, \ldots, k))\) in view of Proposition 3.6. First, we deal with the irreducible case, i.e., \(k = 1\). Then, by using Proposition 3.6, we deal with the non-irreducible case.

5.1. The case \(k = 1\)

Irreducible triplets with \(\text{codim } H(1) = 0\) are given in Theorem 4.3. By [12], we have the following proposition:

**Proposition 5.1** (Irreducible case). The irreducible reduced PVs 
\((G \times GL(2), \rho \otimes \Lambda_1, V \otimes V(2))\) with \(GL(1) \cdot I_V \subset \rho(G)\) and \(\text{codim } H(1) \geq 1\) are given as follows:

1. \((GL(1) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2)), H(1) = P(1, 1), \text{codim } H(1) = 1.\)
2. \((GO(n) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(2)), H(1) = GL(1)^{2^2} = \{\text{diag}(\alpha, \beta) \in GL(2) | \alpha, \beta \in GL(1)\}, \text{codim } H(1) = 2.\)
3. \((G Spin(7) \times GL(2), \text{the spin rep. } \otimes \Lambda_1, V(8) \otimes V(2)), H(1) = GL(1)^{2^2} = \{\text{diag}(\alpha, \beta) \in GL(2) | \alpha, \beta \in GL(1)\}, \text{codim } H(1) = 2.\)
4. \(((GL(1) \times (G_2)) \times GL(2), (\Lambda_1 \otimes \Lambda_2) \otimes \Lambda_1, V(7) \otimes V(2)),\)
\( H(1) = GL(1)^2 = \{ \text{diag}(\alpha, \beta) \in GL(2) | \alpha, \beta \in GL(1) \} \), \( \text{codim} \, H(1) = 2 \).

(5) \( (GL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2)) \),

\[ H(1) = GL(1) \cdot I_2, \text{codim} \, H(1) = 3. \]

(6) \( (GL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2)) \),

\[ H(1) = GL(1) \cdot I_2, \text{codim} \, H(1) = 3. \]

(7) \( (SL(3) \times GL(3) \times GL(2), \Lambda_2 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2)) \),

\[ H(1) = GL(1) \cdot I_2, \text{codim} \, H(1) = 3. \]

(8) \( ((GL(1) \times E_6) \times GL(2), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, V(27) \otimes V(2)) \),

\[ H(1) = GL(1) \cdot I_2, \text{codim} \, H(1) = 3. \]

5.2. The case \( k = 2 \)

In this subsection, we shall classify \( ((G_1 \times G_2) \times GL(2), (\rho_1 \oplus \rho_2) \otimes \Lambda_1) \), which is reduced. By Proposition 3.6, we have \( \text{codim} \, H(1) + \text{codim} \, H(2) \leq 4 \). Hence, if we assume \( 1 \leq \text{codim} \, H(1) \leq \text{codim} \, H(2) \), the possibility of \( (\text{codim} \, H(1), \text{codim} \, H(2)) \) are \( (1, 1) \), \( (1, 2) \), \( (1, 3) \), and \( (2, 2) \) by Proposition 5.1.

**Proposition 5.2.** For \( (1, 1) \), the triplet \( ((GL(1) \times GL(1)) \times GL(2), (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1, M(2)) \) is a regular trivial PV, and \( H(1, 2) = \{ \text{diag}(\alpha, \beta) \in GL(2) | \alpha, \beta \in GL(1) \} \), \( \text{codim} \, H(1, 2) = 2 \).

**Proposition 5.3.** For \( (1, 2) \), the following triplets are PVs with \( H(1, 2) = GL(1) \cdot I_2 : \)

1. \( (GL(1) \times GO(n) \times GL(2), (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1, (V(1) + V(n)) \otimes V(2)) \).

2. \( (GL(1) \times G \, Spin(7) \times GL(2), (\Lambda_1 \oplus \text{the spin rep.}) \otimes \Lambda_1, (V(1) + V(8)) \otimes V(2)) \).
(3) \((GL(1) \times (GL(1) \times (G_2)) \times GL(2), (\Lambda_1 \boxplus (\Lambda_1 \otimes \Lambda_2)) \otimes \Lambda_1, (V(1) + V(7)) \otimes V(2))\).

**Proof.** In this case, we have \(H(2) = \{\text{diag}(\alpha, \beta) \in GL(2) | \alpha, \beta \in GL(2)\}\) by Proposition 5.1. Hence, they are PVs since \((GL(1) \times H(2), \Lambda_1 \otimes \Lambda_1)\) is a PV and we have \(H(1, 2) = GL(1) \cdot I_2 \). 

**Proposition 5.4.** For \((1, 3)\), the following triplets are non PVs:

1. \(((GL(1) \times GL(3)) \times GL(2), (\Lambda_1 \boxplus 2\Lambda_1) \otimes \Lambda_1, (V(1) + V(6)) \otimes V(2))\).
2. \(((GL(1) \times GL(6)) \times GL(2), (\Lambda_1 \boxplus \Lambda_2) \otimes \Lambda_1, (V(1) + V(15)) \otimes V(2))\).
3. \(((GL(1) \times (SL(3) \times GL(3))) \times GL(2), (\Lambda_1 \boxplus (\Lambda_1 \otimes \Lambda_1)) \otimes \Lambda_1, (V(1) + (V(3) \otimes V(3)) \otimes V(2))\).
4. \(((GL(1) \times (GL(1) \times E_6)) \times GL(2), (\Lambda_1 \boxplus (\Lambda_1 \otimes \Lambda_1)) \otimes \Lambda_1, (V(1) + V(27)) \otimes V(2))\).

**Proof.** In this case, we have \(H(2) = GL(1) \cdot I_2\) by Proposition 5.1. Since \((GL(1) \times H(2), \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2))\) is clearly a non PV, we have our result.

**Proposition 5.5.** For \((2, 2)\), the following triplets are non PVs:

1. \((GO(m) \times GO(n) \times GL(2), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1, (V(m) + V(n)) \otimes V(2))\).
2. \((GO(m) \times G \text{Spin}(7) \times GL(2), (\Lambda_1 \boxplus \text{the spin rep.}) \otimes \Lambda_1, (V(m) + V(8)) \otimes V(2))\).
3. \((GO(m) \times (GL(1) \times (G_2)) \times GL(2), (\Lambda_1 \boxplus (\Lambda_1 \otimes \Lambda_2)) \otimes \Lambda_1, (V(m) + V(7)) \otimes V(2))\).
4. \((G \text{Spin}(7) \times G \text{Spin}(7) \times GL(2), (\text{the spin rep.} \boxplus \text{the spin rep.}) \otimes \Lambda_1, (V(8) + V(8)) \otimes V(2))\).
(5) \((GL(1) \times (G_2)) \times G \Spin(7) \times GL(2), (\Lambda_1 \otimes \Lambda_2) \oplus \text{the spin rep.}) \otimes \Lambda_1, (V(7) + V(8)) \otimes V(2)).\)

(6) \(((GL(1) \times (G_2)) \times (GL(1) \times (G_2)) \times GL(2), ((\Lambda_1 \otimes \Lambda_2) \oplus (\Lambda_1 \otimes \Lambda_2)) \otimes \Lambda_1, (V(7) + V(7)) \otimes V(2)).\)

**Proof.** First, note that \((G \Spin(7), \text{the spin rep.}) \subset (GO(8), \Lambda_1)\) and \((GL(1) \times (G_2), \Lambda_1 \otimes \Lambda_2) \subset (GO(7), \Lambda_1)\). Hence, it is enough to prove that \((GO(m) \times GO(n) \times GL(2), (\Lambda_1 \oplus \Lambda_2) \otimes \Lambda_1, (V(m) + V(n)) \otimes V(2))\) is a non PV. In this case, we have \(H(2) = \{\text{diag}(\alpha, \beta) \in GL(2) | \alpha, \beta \in GL(2)\}\) by Proposition 5.1. Therefore, it is PV-equivalent to \((GL(1)^2 \times GO(m), \Lambda_1 + \Lambda_1, V(m) + V(m))\), which is a non PV by \([3]\).\)

**5.3. The case \(k = 3\)**

In this subsection, we shall classify \(((G_1 \times G_2 \times G_3) \times GL(2), (\rho_1 \oplus \rho_2 \oplus \rho_3) \otimes \Lambda_1)\), which is reduced. By Proposition 3.6, we have codim \(H(1) + \text{codim } H(2) + \text{codim } H(3) \leq 4\). Hence, if we assume \(1 \leq \text{codim } H(1) \leq \text{codim } H(2) \leq \text{codim } H(3)\), the possibility of \((\text{codim } H(1), \text{codim } H(2), \text{codim } H(3))\) are \((1, 1, 1)\) and \((1, 1, 2)\) by Proposition 5.1.

**Proposition 5.6.** For \((1, 1, 1)\), the triplet \(((GL(1) \times GL(1)) \times GL(2), (\Lambda_1 \oplus \Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, M(3, 2))\) is a regular PV, and \(H(1, 2) = GL(1) \cdot I_2\).

**Proof.** See \([3]\).\)

**Proposition 5.7.** For \((1, 1, 2)\), the following triplets are non PVs:

1. \((GL(1) \times GL(1) \times GO(n) \times GL(2), (\Lambda_1 \oplus \Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, (V(1) + V(1) + V(n)) \otimes V(2)).\)
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(2) \((GL(1) \times GL(1) \times G \text{Spin}(7) \times GL(2), (\Lambda_1 \boxplus \Lambda_1 \boxplus \text{the spin rep.}) \otimes \Lambda_1, (V(1) + V(1) + V(8)) \otimes V(2))\).

(3) \((GL(1) \times GL(1) \times (GL(1) \times (G_2)) \times GL(2), (\Lambda_1 \boxplus \Lambda_1 \boxplus (\Lambda_1 \otimes \Lambda_2)) \otimes \Lambda_1, (V(1) + V(1) + V(7)) \otimes V(2))\).

**Proof.** In this case, we have \(H(2, 3) = GL(1) \cdot I_1\) by Proposition 5.3. Hence, they are non PVs since \((GL(1) \times H(2, 3), \Lambda_1 \otimes \Lambda_1)\) is a non PV.  

5.4. The case \(k = 4\)

In this subsection, we shall classify \(((G_1 \times G_2 \times G_3 \times G_4) \times GL(2), (\rho_1 \boxplus \rho_2 \boxplus \rho_3 \boxplus \rho_4) \otimes \Lambda_1)\), which is reduced. By Proposition 3.6, we have\(\text{codim } H(1) + \text{codim } H(2) + \text{codim } H(3) + \text{codim } H(4) \leq 4\). Hence, we have \((\text{codim } H(1), \text{codim } H(2), \text{codim } H(3), \text{codim } H(4)) = (1, 1, 1, 1)\) by Proposition 5.1. However, the triplet \(((GL(1) \times GL(1) \times GL(1)) \times GL(2), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1, M(4, 2))\) is a non PV since the dimension of the image of the group is 7, while the dimension of the space is 8.

6. A Classification of Reductive PVs

\((G \times GL(3), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*)\) of Separated Type

In this section, we shall classify the PVs \(((G_1 \times \cdots \times G_r) \times (G_{r+1} \times \cdots \times G_k) \times GL(3), (\rho_1 \boxplus \cdots \boxplus \rho_r) \otimes 1 \otimes \Lambda_1 + 1 \otimes (\rho_{r+1} \boxplus \cdots \boxplus \rho_k) \otimes \Lambda_1^*) (r \geq 1)\), \(\text{where each irreducible component } (G_i \times GL(3), \rho_i \otimes \Lambda_1^{(*)})\) has the least dimension among its \(GL(3)\)-castling equivalence class, i.e., the reduced triplets.

Let \(H\) be the \(GL(3)\)-part of a generic isotropy subgroup. We classify them according to \(\text{codim } H(= 9 - \dim H)\) in view of Proposition 3.6. First, we deal with the irreducible case, i.e., \(k = 1\). Then by using Proposition 3.6, we deal with the non-irreducible case.
6.1. The case \( k = 1 \)

From [12], we have the following proposition:

**Proposition 6.1** (Irreducible case). The irreducible reduced PVs \((G \times GL(3), \rho \otimes \Lambda_1^{(\ast)}, V \otimes V(3)^{(\ast)})\) with \( GL(1) \cdot I_V \subset \rho(G) \) are given as follows. Here \( H(1^{(\ast)}) \) is the \( GL(3) \)-part of a generic isotropy subgroup (see Definition 3.4).

1. \((\tilde{H} \times GL(k) \times GL(3), \sigma \otimes \Lambda_1 \otimes \Lambda_{1}^{(\ast)}, V(h) \otimes V(k) \otimes V(3)^{(\ast)}\) with \(3h \leq k\), where \( \sigma : \tilde{H} \to GL(h) \) is any representation of any semisimple algebraic group \( \tilde{H} \).

\[ H(1^{(\ast)}) = GL(3), \text{ codim } H(1^{(\ast)}) = 0. \]

2. \((GL(1) \times GL(3), \Lambda_1 \otimes \Lambda_{1}^{(\ast)}, V(1) \otimes V(3)^{(\ast)}\),

\[ H(1) = P(2, 1), H(1^{\ast}) = P(1, 2), \text{ codim } H(1^{(\ast)}) = 2. \]

3. \((GL(1) \times GL(3), \Lambda_1 \otimes \Lambda_{1}^{(\ast)}, V(1) \otimes V(3)^{(\ast)}\),

\[ H(1) = P(1, 2), H(1^{\ast}) = P(2, 1), \text{ codim } H(1^{(\ast)}) = 2. \]

4. \((GO(n) \times GL(3), \Lambda_1 \otimes \Lambda_{1}^{(\ast)}, V(n) \otimes V(3)^{(\ast)}\),

\[ H(1^{(\ast)}) = GO(3), \text{ codim } H(1^{(\ast)}) = 5. \]

Note that \((SL(2), 2 \Lambda_1) = (SO(3), \Lambda_1), (SL(2) \times SL(2), \Lambda_1 \otimes \Lambda_1) = (SO(4), \Lambda_1), (Sp(5), \Lambda_2) = (SO(5), \Lambda_1), (SL(4), \Lambda_2) = (SO(6), \Lambda_1).\)

5. \((GL(5) \times GL(3), \Lambda_2 \otimes \Lambda_{1}^{(\ast)}, V(10) \otimes V(3)^{(\ast)}\),

\[ H(1^{(\ast)}) = GO(3), \text{ codim } H(1^{(\ast)}) = 5. \]

6. \((Spin(7) \times GL(3), \text{ a half-spin rep. } \otimes \Lambda_{1}^{(\ast)}, V(8) \otimes V(3)^{(\ast)}\),

\[ H(1^{(\ast)}) = GO(3), \text{ codim } H(1^{(\ast)}) = 5. \]
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(7) \((G \text{ Spin}(10) \times GL(3), \text{ a half-spin rep. } \otimes \Lambda_1^{(*)}, V(16) \otimes V(3)^{(s)})\),
\[ H(1^{(s)}) = GO(3), \text{ codim } H(1^{(s)}) = 5. \]

(8) \((SL(2) \times GL(3) \times GL(3), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)}, V(2) \otimes V(3) \otimes V(3)^{(s)})\),
\[ H(1^{(s)}) = \{\text{diag}(\alpha, \beta, \gamma) \in GL(3)|\alpha, \beta, \gamma \in GL(1)\}, \text{ codim } H(1^{(s)}) = 6. \]

(9) \((GL(2) \times GL(3), 3\Lambda_1 \otimes \Lambda_1^{(*)}, V(4) \otimes V(3)^{(s)})\),
\[ H(1^{(s)}) = GL(1) \cdot I_3, \text{ codim } H(1^{(s)}) = 8. \]

6.2. The case \( k = 2 \)

In this subsection, we shall classify \( (G_1 \times G_2 \times GL(3), \rho_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \rho_2 \otimes \Lambda_1^{(*)}) \), which is reduced. By Proposition 3.6, we have \( \text{codim } H(1) + \text{codim } H(2^{(*)}) \leq 9 \). Hence, the possibility of \( (\text{codim } H(1), \text{codim } H(2^{(*)})) = (6, 2), (5, 2), (2, 2) \) by Proposition 6.1.

**Proposition 6.2.** For the case \( (\text{codim } H(1), \text{codim } H(2^{(*)})) = (6, 2) \), the followings are PVs with \( H(1, 2^{(*)}) = \{\alpha I_3 | \alpha \in GL(1)\} \) \( (\text{codim } H(1, 2^{(*)}) = 8) \).

(1) \((SL(2) \times GL(3) \times GSp(m) \times GL(3), \Lambda_1 \otimes \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)}) \)
\((m \geq 2)\).

(2) \((SL(2) \times GL(3) \times GL(1) \times GL(3), \Lambda_1 \otimes \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)}) \).

**Proof.** (1) is a PV-equivalent to (2) by Proposition 3.8. But (2) is PV-equivalent to \((GL(1) \times H(1^{(*)}), \Lambda_1 \otimes \Lambda_1)\) with \( H(1^{(*)}) = \{\text{diag}(\alpha, \beta, \gamma) \in GL(3)|\alpha, \beta, \gamma \in GL(1)\}, \) which is clearly a PV and the generic isotropy subgroup is \( \{\alpha I_3 | \alpha \in GL(1)\} \).

\[ \blacksquare \]
**Proposition 6.3.** For the case \((\text{codim } H(1), \text{codim } H(2^{(*)})) = (5, 2)\), the followings are PVs with \(H(1, 2^{(*)}) = \{\text{diag } (\alpha^2, \alpha\beta, \beta^2) \in GL(3) | \alpha, \beta \in GL(1)\} \)
\((\text{codim } H(1, 2^{(*)}) = 7)\):

1. \((GL(5) \times GSp(m) \times GL(3), \Lambda_2 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 2)\).

2. \((GO(m_1) \times GSp(m_2) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m_1 \geq 3, m_2 \geq 2)\).

3. \((G \text{ Spin}(7) \times GSp(m) \times GL(3), \text{ the spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)}) \)
\((m \geq 2)\).

4. \((G \text{ Spin}(10) \times GSp(m) \times GL(3), \text{ a half-spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)}) \)
\((m \geq 2)\).

5. \((GL(5) \times GL(1) \times GL(3), \Lambda_2 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

6. \((GO(m) \times GL(1) \times GL(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})(m \geq 3)\).

7. \((G \text{ Spin}(7) \times GL(1) \times GL(3), \text{ the spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

8. \((G \text{ Spin}(10) \times GL(1) \times GL(3), \text{ a half-spin rep. } \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^{(*)})\).

**Proof.** (1) ~ (4) are PV-equivalent to \((GSp(m) \times GO(3), \Lambda_1 \otimes \Lambda_1)\), which is a PV (see p. 104 in [12]). The GO(3)-part of a generic isotropy subgroup is \(\{\text{diag } (\alpha^2, \alpha\beta, \beta^2) \in GL(3) | \alpha, \beta \in GL(1)\}\). (5) ~ (8) are PV-equivalent to \((GL(1) \times GO(3), \Lambda_1 \otimes \Lambda_1)\), which is a PV (see p. 109 in [12]). The GO(3)-part of a generic isotropy subgroup is \(\{\text{diag } (\alpha^2, \alpha\beta, \beta^2) \in GL(3) | \alpha, \beta \in GL(1)\}\). \(\blacksquare\)
Proposition 6.4. For the case \((\text{codim } H(1), \text{codim } H(2^{(e)})) = (2, 2)\),
the followings are PVs with \(H(1, 2^{(e)}) = \begin{pmatrix} a & 0 & d \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix} | a, b, c \in GL(1)\),
d, e \in \mathbb{C}\} for \((1) \sim (3)\) and \(H(1, 2^{(e)}) = \begin{pmatrix} GL(1) \\ O \\ GL(2) \end{pmatrix}\) for \((4) \sim (6)\) \((\text{codim } H(1, 2^{(e)}) = 4)\):

1. \((GSp(m_1) \times GSp(m_2) \times GL(3), \Lambda_1 \otimes 1 \otimes 1_1 + 1_1 \otimes \Lambda_1 \otimes 1_1 + 1_1 \otimes 1_1 \otimes 1_1vised subgroup is \(\begin{pmatrix} a & 0 & d \\ 0 & b & e \\ 0 & 0 & e \end{pmatrix} | a, b, e \in GL(1)\). By Proposition 3.8 and castling transformations, \((4) \sim (6)\) are PV-equivalent to \((GL(1) \times GL(2) \times GL(3), (\Lambda_1 \oplus 1_1) \otimes 1_1 \otimes 1_1)\), which is a regular trivial PV.

Thus, we obtain \((10) \sim (22)\) in Theorem 2.2.
6.3. The case \( k = 3 \)

In this subsection, we shall classify \((G_1 \times G_2 \times G_3 \times GL(3), \rho_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \rho_2 \otimes 1 \otimes \Lambda_1^{(s)} + 1 \otimes 1 \otimes \rho_3 \otimes \Lambda_1^{(s)}),\) which is reduced. By Proposition 3.6, we have \( \text{codim } H(1, 2^{(s)}) + \text{codim } H(3^{(s)}) \leq 9, \) Hence, the possibility of \((\text{codim } H(1, 2^{(s)}), \text{codim } H(3^{(s)})) = (7, 2), (4, 5), (4, 2)\) by Propositions 6.2 ~ 6.4.

**Proposition 6.5.** For the case \((\text{codim } H(1, 2^{(s)}), \text{codim } H(3^{(s)})) = (7, 2), (4, 5),\) the following triplets are non PVs:

1. \((GL(5) \times GSp(m_1) \times GSp(m_2) \times GL(3), (\Lambda_2 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp.}, \Lambda_2 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^*; \Lambda_2 \otimes \Lambda_1) \otimes 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*(m_1 \geq 2, m_2 \geq 2).\)

2. \((GO(m_1) \times GSp(m_2) \times GSp(m_3) \times GL(3), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp.}, \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^*; \Lambda_1 \otimes \Lambda_1) \otimes 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*(m_1 \geq 3, m_2 \geq 2, m_3 \geq 2).\)

3. \((G\text{ Spin}(7) \times GSp(m_1) \times GSp(m_2) \times GL(3), (\text{the spin rep.} \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp.}, \text{the spin rep.} \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^*; \text{the spin rep.} \boxplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*(m_1 \geq 2, m_2 \geq 2).\)

4. \((G\text{ Spin}(10) \times GSp(m_1) \times GSp(m_2) \times GL(3), (\text{a half-spin rep.} \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp., a half-spin rep.} \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^*; \text{a half-spin rep.} \boxplus \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*(m_1 \geq 2, m_2 \geq 2).\)

5. \((GL(5) \times GSp(m) \times GL(1) \times GL(3), (\Lambda_2 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp.,} \Lambda_2 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1^*(m_1 \geq 2, m_2 \geq 2).\)
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\[ 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_1 + 1 \otimes \Lambda_1 \]
\[ \otimes 1 \otimes \Lambda_1^*; (\Lambda_2 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^* (m \geq 2). \]

(6) \((GO(m) \times GSp(m_2) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (\text{resp., } \Lambda_1) \)
\[ \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_1 + 1 \otimes \Lambda_1 \]
\[ \otimes 1 \otimes \Lambda_1^*; (\Lambda_2 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^* (m \geq 3, m_2 \geq 2). \]

(7) \((G \text{Spin}(7) \times GSp(m) \times GL(1) \times GL(3), (\text{the spin rep.} \oplus \Lambda_1) \otimes \Lambda_1) (\text{resp., } \Lambda_1) \)
\[ \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\text{the spin rep.} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes \Lambda_1^*; (\text{the spin rep.} \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^* (m \geq 2). \]

(8) \((G \text{Spin}(10) \times GSp(m) \times GL(1) \times GL(3), (\text{a half-spin rep.} \oplus \Lambda_1) \otimes \Lambda_1) (\text{resp., } \Lambda_1) \)
\[ \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\text{a half-spin \rep.} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes \Lambda_1^*; (\text{a half-spin \rep.} \oplus \Lambda_1) \]
\[ \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^* (m \geq 2). \]

(9) \((GL(5) \times GL(1) \times GL(1) \times GL(3), (\Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (\text{resp., } \Lambda_2 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_2 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^* \).

(10) \((GO(m) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (\text{resp., } \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\Lambda_1 \oplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^* (m \geq 3). \)

(11) \((G \text{Spin}(7) \times GL(1) \times GL(1) \times GL(3), (\text{the spin rep.} \oplus \Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1) (\text{resp., } \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \oplus \Lambda_1) \otimes \Lambda_1^*; (\text{the spin rep.} \oplus \Lambda_1) \)
\[ \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*. \]
(12) \((G \text{ Spin}(10) \times GL(1) \times GL(1) \times GL(3)), (a \text{ half-spin rep. } \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes \Lambda_1)(\text{resp., a half-spin rep. } \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \mathbb{1}_1) \otimes \Lambda_1^\ast; (a \text{ half-spin rep. } \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^\ast)(m \geq 2)\).

**Proof.** By Proposition 3.8, all these triplets are PV-equivalent to \((G_1 \times GL(1) \times GL(1) \times GL(3), \rho_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes 1 \otimes \Lambda_1^\ast + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^\ast),\) where the \(GL(3)\)-part of a generic isotropy subgroup of \((G_1 \times GL(3), \rho_1 \otimes \Lambda_1)\) is \(GO(3)\). Since \(\Lambda_1 = \Lambda_1^\ast\) for \(SO(3)\), they are PV-equivalent to \((GL(1) \times GL(1) \times GO(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1) \equiv (GL(1) \times GL(1) \times SO(3), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1),\) which is a non PV by the dimension reason, i.e., \(\dim(GL(1) \times GL(1) \times SO(3)) = 5 < 6 = \dim V\).  

**Proposition 6.6.** For the case \((\text{codim } H(1, 2^{(\ast)}), \text{codim } H(3^{(\ast)})) = (4, 2),\) for following triplets are PVs:

1. \((GSp(m_1) \times GSp(m_2) \times GSp(m_3) \times GL(3), (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes \Lambda_1)(\text{resp., } (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^\ast)(m_1 \geq 2, m_2 \geq 2, m_3 \geq 2).\)

2. \((GSp(m_1) \times GSp(m_2) \times GL(1) \times GL(3), (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes \Lambda_1)(\text{resp., } \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \mathbb{1}_1) \otimes \Lambda_1^\ast; (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^\ast)(m_1 \geq 2, m_2 \geq 2).\)

3. \((GSp(m) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes \Lambda_1)(\text{resp., } \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \mathbb{1}_1) \otimes \Lambda_1^\ast; (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^\ast)(m \geq 2).\)

4. \((GL(1) \times GL(1) \times GL(1) \times GL(3), (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1 \mathbb{1}_1) \otimes \Lambda_1)(\text{resp., } (\Lambda_1 \mathbb{1}_1 \mathbb{1}_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^\ast).\)
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**Proof.** By Proposition 3.8, it is PV-equivalent to \((GL(n_1) \times GL(n_2) \times GL(n_3)) \times GL(3), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1\), where \(n_1, n_2, n_3 = 1, 2\). By (1) of Proposition 3.10, this is a PV. 

6.4. The case \(k = 4\)

In this subsection, we shall classify \((G_1 \times G_2 \times G_3 \times G_4 \times GL(3), \rho_1 \otimes 1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \rho_2 \otimes 1 \otimes 1 \otimes \Lambda_1^* + 1 \otimes 1 \otimes \rho_3 \otimes 1 \otimes \Lambda_1^* + 1 \otimes 1 \otimes 1 \otimes \rho_4 \otimes \Lambda_1^*)\), which is reduced. By Proposition 3.6, we have

\[
\text{codim } H(1, 2^*) + \text{codim } H(3^*) + \text{codim } H(4^*) \leq 9.
\]

Hence, the possibility of \((\text{codim } H(1, 2^*), \text{codim } H(3^*), \text{codim } H(4^*)) = (4, 2, 2)\) by Propositions 6.2 ~ 6.4. Since this is the maximal, we know that there does not exist the case \(k \geq 5\).

**Proposition 6.7.** For the case \((\text{codim } H(1, 2^*), \text{codim } H(3^*), \text{codim } H(4^*)) = (4, 2, 2)\), the following triplets in I are PVs, while triplets in II are non PVs.

I (a) \((GSp(m_1) \times GSp(m_2) \times GSp(m_3) \times GSp(m_4) \times GL(3), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp.}, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*)(m_1 \geq 2, m_2 \geq 2, m_3 \geq 2, m_4 \geq 2)\).

(b) \((GSp(m_1) \times GSp(m_2) \times GSp(m_3) \times GL(1) \times GL(3), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1)(\text{resp.}, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1^*; (\Lambda_1 \boxplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^*)(m_1 \geq 2, m_2 \geq 2, m_3 \geq 2)\).

(c) \((GSp(m_1) \times GSp(m_2) \times GL(1) \times GL(1) \times GL(3), \Lambda_1 \otimes 1 \otimes 1 \otimes 1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1^*)(\text{resp.}, (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes 1 \otimes \Lambda_1\)

...
Proof. By Proposition 3.8, they are PV-equivalent to

\[ ((GL(n_1) \times GL(n_2)) \times GL(n_3) \times GL(n_4)) \times GL(3), (\Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1 ), \]

where \( n_1, n_2, n_3, n_4 \) are 1 or 2. By (2) of Proposition 3.10, it is a PV, if and only if \( n_1 + n_2 + n_3 + n_4 \neq 6 \). Hence, we have our result. \( \blacksquare \)
7. Quasi-Irreducibility

Rubenthaler defined the notion of quasi-irreducibility and completely quasi-reducibility of PVs ([11]).

**Theorem 7.1.** Non-irreducible quasi-irreducible PVs in Theorems 2.1 and 2.2 are given as follows. Here N denotes the number of the basic relative invariants.

1. \[(GL(1) \times GL(1)) \times GL(2), (A_1 \oplus A_1) \otimes A_1, (V(1) + V(1)) \otimes V(2)), \quad N = 1.\]

2. \[(GL(1) \times GL(1) \times GL(3), A_1 \otimes 1 \otimes A_1 + 1 \otimes A_1 \otimes A_1^*, V(1) \otimes V(1) \otimes V(3) + V(1) \otimes V(1) \otimes V(3)^*), \quad N = 1.\]

3. \[(GL(1) \times GL(1) \times GL(3), (A_1 \oplus A_1 \oplus A_1) \otimes A_1, (V(1) + V(1) + V(1)) \otimes V(3)), \quad N = 1.\]

**Theorem 7.2.** There is no completely quasi-reducibility of PVs in Theorems 2.1 and 2.2, which is not quasi-irreducible.

**References**


[9] T. Kimura, T. Kamiyoshi, N. Maki, M. Ouchi and M. Takano, A classification of representations $\rho \otimes \Lambda_1$ of reducible algebraic groups $G \times SL_n(n \geq 2)$ with finitely many orbits, Algebras Groups and Geometries 25 (2008), 115-160.

